Lesson 4: Integration By Substitution II

Last time, when $\int h(x)dx = \int f(u(x)) \cdot u'(x)dx$

① Choose $u(x)$

② Then differentiate to get $du = u'(x)dx$

To produce a new integral

$$\int h(x)\,dx = \int f(u)\,du$$

Then integrate and plug back $u(x)$ to get our answer.

Example 1: Compute the following integrals:

ⓐ $\int 5\sin(3t)\cos^8(3t)dt = \int 5\sin(3t) \cdot (\cos(3t))^8 dt$

$u = \cos(3t)$

$du = -\sin(3t) \cdot 3dt$

$\frac{du}{-3\sin(3t)} = dt$

$$\int 5\cancel{\sin(3t)} \cdot u^8 \cdot \frac{du}{-3\cancel{\sin(3t)}}$$

$$= \frac{-5}{3}\int u^8 du = \frac{-5}{3} \cdot \frac{u^9}{9} + C$$

$$= \frac{-5}{27}\cos^9(3t) + C$$

ⓑ $\int 8e^{7x}\csc^2(e^{7x})dx$

$u = e^{7x}$

$du = 7e^{7x}dx$

$\frac{du}{7e^{7x}} = dx$

$$\int 8\cancel{e^{7x}}\csc^2(u) \cdot \frac{du}{7\cancel{e^{7x}}}$$

$$= \frac{8}{7}\int \csc^2(u)du = \frac{-8}{7}\cot(u) + C$$

$$= \frac{-8}{7}\cot(e^{7x}) + C$$

Sometimes we need to do extra work with u, as shown in the next example(s).

Example 2: Compute the following integrals:

ⓐ $\int x\sqrt{x-5}\,dx \;\xrightarrow[du=dx]{u=x-5}\; \int x\cdot\sqrt{u}\,du$

Issue: Our integral has x and u!
Can I rewrite x into some function of u? Yes

$u=x-5 \iff u+5=x$

$= \int (u+5)u^{1/2}\,du = \int (u^{3/2}+5u^{1/2})\,du$

$= \frac{2}{5}u^{5/2} + 5\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x-5)^{5/2} + \frac{10}{3}(x-5)^{3/2} + C$

ⓑ $\int \frac{2x}{(x+7)^{3/2}}\,dx \;\xrightarrow[du=dx]{u=x+7 \iff x=u-7}\; \int \frac{2(u-7)}{u^{3/2}}\,du$

$= \int (2u^{-1/2} - 14u^{-3/2})\,du$

$= 2\cdot\frac{2}{1}u^{1/2} - 14\cdot\frac{2}{-1}u^{-1/2} + C$

$= 4(x+7)^{1/2} + 28(x+7)^{-1/2} + C$

Now let's do integration by substitution for definite integrals.

Example 3: Compute

$\int_2^4 x\sin(x^2)\,dx \;\xrightarrow[du=2x\,dx]{u=x^2}\; \int \cancel{x}\sin(u)\frac{du}{2\cancel{x}} = \int \frac{1}{2}\sin(u)\,du$

$\frac{du}{2x} = dx$

Issue: What do I do with the limit values? i.e. $\int_2^4$

Well $\int_2^4 x\sin(x^2)\,dx = \int_{x=2}^{x=4} x\sin(x^2)\,dx$

Method 1: Changing $x=2$ and $x=4$ using $u=x^2$

$$x=2 \downarrow u=4 \qquad x=4 \downarrow u=16$$

$$\int_{x=2}^{x=4} x\sin(x^2)\,dx \quad \frac{u=x^2}{\frac{du}{2x}=dx} \quad \int_{u=4}^{u=16} \sin(u)\frac{du}{2}$$

$$= -\frac{\cos(u)}{2}\Big]_{u=4}^{u=16} = -\frac{1}{2}\left(\cos(16)-\cos(4)\right)$$

Method 2: To treat the integral after the u-sub as indefinite and when you plug u back evaluate with original bounds.

$$\int_{x=2}^{x=4} x\sin(x^2)\,dx = \frac{u=x^2}{\frac{du}{2x}=dx} \quad \int \frac{\sin(u)}{2}du$$

$$= -\frac{\cos(u)}{2} = -\frac{\cos(x^2)}{2}\Big]_{x=2}^{x=4}$$

$$= -\frac{1}{2}\left(\cos(16)-\cos(4)\right)$$

Example 4: If the area of the region under the curve $y=\frac{1}{\sqrt{9x+4}}$ over the interval $0\le x\le a$ is 6, then what is a?

i.e. Solve $\int_0^a \frac{1}{\sqrt{9x+4}}dx = 6$ for a.

$$6 = \int_0^a \frac{1}{\sqrt{9x+4}}dx \quad \frac{u=9x+4}{du=9dx} \quad \int \frac{1}{\sqrt{u}}\cdot\frac{du}{9} = \frac{1}{9}\int u^{-1/2}du$$

$$\frac{du}{9}=dx$$

$$= \frac{1}{9}\cdot\frac{2}{1}u^{1/2} = \frac{2}{9}(9x+4)^{1/2}\Big]_0^a = \frac{2}{9}(9a+4)^{1/2} - \frac{2}{9}(4)^{1/2}$$

$$= \frac{2}{9}(9a+4)^{1/2} - \frac{4}{9}$$

$$\frac{58}{9} = \frac{2}{9}(9a+4)^{1/2}$$

$$\frac{9}{2} \cdot \frac{58}{9} = \frac{9}{2} \cdot \frac{2}{9}(9a+4)^{1/2}$$

$$29 = (9a+4)^{1/2}$$

$$(29)^2 = 9a+4$$

$$841 = 9a+4$$

$$837 = 9a$$

$$a = \frac{837}{9} = 93$$