Lesson 6: Partial Fractions I

Recall the concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{2}+\frac{1}{3}=\frac{1}{2}\cdot\frac{3}{3}+\frac{1}{3}\cdot\frac{2}{2}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

So we can say that a partial fraction decomposition for 5/6 is

$$\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$$

This concept can be used with functions of x.

Example 1: Combine the following fractions

$$\frac{1}{x-2}+\frac{3}{x-5}=\frac{1}{(x-2)}\cdot\frac{(x-5)}{(x-5)}+\frac{3}{(x-5)}\cdot\frac{(x-2)}{(x-2)}$$

$$=\frac{(x-5)+3(x-2)}{(x-2)(x-5)}$$

$$=\frac{x-5+3x-6}{x^2-2x-5x+10}$$

$$=\frac{4x-11}{x^2-7x+10}$$

Why do we care about partial fraction decomposition?
It's because u-sub isn't enough.

Example 2: Evaluate $\displaystyle\int\frac{4x-11}{x^2-7x+10}\,dx$

Let's first try a u-sub.

$$\frac{u=x^2-7x+10}{du=(2x-7)dx}\qquad\int\frac{4x-11}{u}\cdot\frac{du}{2x-7}$$

As you can see there is no way to eliminate the x's.

Now let's try partial fraction decomposition. Using Ex1,

$$\int\frac{4x-11}{x^2-7x+10}\,dx=\int\frac{1}{x-2}\,dx+\int\frac{3}{x-5}\,dx$$

Now we know how to integrate these functions:

- $\int \frac{1}{x-2}\,dx \quad \begin{matrix} u=x-2 \\ du=dx \end{matrix} \quad \int \frac{1}{u}\,du = \ln|u| = \ln|x-2|$

- $\int \frac{3}{x-5}\,dx \quad \begin{matrix} u=x-5 \\ du=dx \end{matrix} \quad \int \frac{3}{u}\,du = 3\ln|u| = 3\ln|x-5|$

So $\int \frac{4x-11}{x^2-7x+10}\,dx = \ln|x-2| + 3\ln|x-5| + C$

Method of Decomposing into Partial Fractions.

Given a rational function $\frac{N(x)}{D(x)}$

① Factor the denominator as much as possible.

② Write the fraction into decomposition form.

ⓐ Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

ⓑ Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

ⓒ Distinct irreducible quadratic terms like x^2+a^2 decompose to

$$\frac{Ax+B}{x^2+a^2}$$

ⓓ Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

③ Combine your decomposition from ② as 1 fraction.

④ Set the original numerator, $N(x)$, equal to the numerator from ③.

⑤ Equate the coefficients of the terms, to yields a system of equations. Then solve the constants. i.e. A, B, C.

⑥ Plug the values found in (5) in (2).

In today's class, we will be focussing on this method.

Example 3: Determine the partial fraction decomposition without explicitly solving for the constants.

i.e. Do Steps 1+2 of the Method of Decomposing into Partial Fractions. Handout

ⓐ $f(x) = \dfrac{6x+10}{x^2+5x}$

① Factor the denominator, x^2+5x, completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.
Note x and $x+5$ are distinct linear terms. So by (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{6x+10}{x^2+5x} = \frac{6x+10}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}$$

ⓑ $f(x) = \dfrac{40}{x^2-16}$

① Factor the denominator, x^2-16, completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form.
Note $x-4$ and $x+4$ are distinct linear terms. So by (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{40}{x^2-16} = \frac{40}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4}$$

ⓒ $f(x) = \dfrac{x^2+2}{x^3+3x^2+2x}$

① Factor the denominator, x^3+3x^2+2x, completely.

$$x^3+3x^2+2x = x(x^2+3x+2) = x(x+1)(x+2)$$

② Write the fraction into decomposition form.
Note x, $x+1$, and $x+2$ are distinct linear term
So by (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{x^2+2}{x^3+3x^2+2x} = \frac{x^2+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

ⓓ $f(x) = \dfrac{4x^2-4}{x^3-2x^2}$

① Factor the denominator, x^3-2x^2, completely.

$$x^3-2x^2 = x^2(x-2) = (x-0)^2(x-2)$$

② Write the fraction into decomposition form.
Note $(x-0)^2$ is a repeated linear term, and
$x-2$ is a distinct linear term.
So by (2b) and (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{4x^2-4}{x^3-2x^2} = \frac{4x^2-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

ⓔ $f(x) = \dfrac{31}{(x^2-81)^2}$

① Factor the denominator, $(x^2-81)^2$, completely.

$$(x^2-81)^2 = [(x-9)(x+9)]^2 = (x-9)^2(x+9)^2$$

② Write the fraction into decomposition form.
Note both $(x-9)^2$ and $(x+9)^2$ are repeated linear terms. So by (2b) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{31}{(x^2-81)^2} = \frac{31}{(x-9)^2(x+9)^2}$$

$$= \frac{A}{x-9} + \frac{B}{(x-9)^2} + \frac{C}{x+9} + \frac{D}{(x+9)^2}$$

(f) $f(x) = \dfrac{5x^2+9}{x^3+3x}$

① Factor the denominator, x^3+3x, completely.

$$x^3+3x = x(x^2+3)$$

② Write the fraction into decomposition form.

Note x is a distinct linear term, and

x^2+3 is a distinct irreducible quadratic term,

So by (2a) and (2c) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{5x^2+9}{x^3+3x} = \frac{5x^2+9}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

(g) $f(x) = \dfrac{27x-4}{(x^2-25)(x^2+64)}$

① Factor the denominator, $(x^2-25)(x^2+64)$, completely.

$$\underbrace{(x^2-25)}_{\text{Factorable}}\underbrace{(x^2+64)}_{\text{Not}} = (x-5)(x+5)(x^2+64)$$

② Write the fraction into decomposition form.

Note $x-5$ and $x+5$ are distinct linear terms, and

x^2+64 is a distinct irreducible quadratic term.

So by (2a) and (2c) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{27x-4}{(x^2-25)(x^2+64)} = \frac{27x-4}{(x-5)(x+5)(x^2+64)}$$

$$= \frac{A}{x-5} + \frac{B}{x+5} + \frac{Cx+D}{x^2+64}$$

(h) $f(x) = \dfrac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

① Factor the denominator, $(x^2+x+1)^2$, completely.

Done!

② Write the fraction into decomposition form.

Note $(x^2+x+1)^2$ is a repeated irreducible quadratic term

So by (2d) in the Method of Decomposing Into Partial

Fractions Handout

$$f(x)=\frac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}$$

Now let's find those constants.

Example 4: Let $f(x) = \frac{6x+10}{x^2+5x}$

Determine the partial fraction decomposition of $f(x)$.

① Factor x^2+5x completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+5}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+5} = \frac{A}{x} \cdot \frac{(x+5)}{(x+5)} + \frac{B}{x+5} \cdot \frac{x}{x}$$

$$= \frac{A(x+5) + Bx}{x(x+5)} = \frac{Ax+5A+Bx}{x(x+5)} = \frac{(A+B)x+5A}{x(x+5)}$$

④ Set the old numerater = new numerator.

$$6x+10 = (A+B)x + 5A$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=6 & (i) \\ 5A = 10 & (ii) \end{cases}$$

From (ii), we find $A=2$.

Plug $A=2$ into (i)

$$A+B=6$$
$$2+B=6$$
$$B=4$$

⑥ Plug $A=2$ and $B=4$ into ②.

$$\frac{2}{x} + \frac{4}{x+5}$$

Example 5: Let $f(x) = \dfrac{40}{x^2-16}$

Determine the partial fraction decomposition of $f(x)$.

① Factor x^2-16 completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form.

$$\frac{A}{x-4} + \frac{B}{x+4}$$

③ Combine the fractions in ②.

$$\frac{A}{x-4} + \frac{B}{x+4} = \frac{A}{x-4}\cdot\frac{(x+4)}{(x+4)} + \frac{B}{x+4}\cdot\frac{(x-4)}{(x-4)}$$

$$= \frac{A(x+4)+B(x-4)}{(x-4)(x+4)}$$

$$= \frac{Ax+4A+Bx-4B}{(x-4)(x+4)} = \frac{(A+B)x+(4A-4B)}{(x-4)(x+4)}$$

④ Set the old numerator = new numerator

$$0x+40 = (A+B)x + (4A-4B)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=0 \\ 4A-4B=40 \end{cases} \Rightarrow \begin{cases} B=-A & (i) \\ A-B=10 & (ii) \end{cases}$$

Plug (i) into (ii).

$$A-B=10$$
$$A-(-A)=10$$
$$A+A=10$$
$$2A=10$$
$$A=5$$

Plug $A=5$ into (i)

$$B=-A$$
$$=-5$$

⑥ Plug $A=5$ and $B=-5$ into ②.

$$\frac{5}{x-4} + \frac{-5}{x+4}$$

Example 6: Let $f(x) = \dfrac{x^2+2}{x^3+3x^2+2x}$

Determine the partial fraction decomposition of $f(x)$.

① Factor x^3+3x^2+2x completely.

$$x^3+3x^2+2x = x(x^2+3x+2)$$
$$= x(x+1)(x+2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A}{x} \cdot \frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{B}{x+1} \cdot \frac{x(x+2)}{x(x+2)} + \frac{C}{x+2} \cdot \frac{x(x+1)}{x(x+1)}$$
$$= \frac{A(x^2+3x+2) + B(x^2+2x) + C(x^2+x)}{x(x+1)(x+2)}$$
$$= \frac{(A+B+C)x^2 + (3A+2B+C)x + (2A)}{x(x+1)(x+2)}$$

④ Set the old numerator = new numerater

$$x^2+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$
$$x^2+0x+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B+C = 1 & (i) \\ 3A+2B+C = 0 & (ii) \\ 2A = 2 & (iii) \end{cases}$$

From (iii), we find $A=1$. So we can plug that into (i) and (ii) yielding

$$\begin{cases} 1+B+C = 1 \\ 3+2B+C = 0 \end{cases} \Rightarrow \begin{cases} B+C = 0 \\ 2B+C = -3 \end{cases} \Rightarrow \begin{cases} C = -B & (i') \\ 2B+C = -3 & (ii') \end{cases}$$

Plug (i') into (ii').

$$2B+C = -3$$
$$2B-B = -3$$
$$B = -3$$

Plug $B=-3$ into (i')

$$C = -B$$
$$= -(-3)$$
$$= 3$$

⑥ Plug $A=1$, $B=-3$, $C=3$ into ②.

$$\frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$$

Example 7: Let $f(x) = \dfrac{4x^2-4}{x^3-2x^2}$

Determine the partial fraction decomposition of $f(x)$.

① Factor x^3-2x^2 completely.

$$x^3-2x^2 = x^2(x-2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

③ Combine the fractions in ②.

Note the common denominator is $x^2(x-2)$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$

$$= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)}$$

$$= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^3-2x^2}$$

④ Set the old numerator = new numerator

$$4x^2 - 4 = (A+C)x^2 + (B-2A)x - 2B$$

$$4x^2 + 0x - 4 = (A+C)x^2 + (B-2A)x - 2B$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+C=4 & \text{(i)} \\ B-2A=0 & \text{(ii)} \\ -2B=-4 & \text{(iii)} \end{cases}$$

From (iii) $-2B=-4 \Rightarrow B=2$

Plug $B=2$ into (ii)

$$B-2A=0$$
$$2-2A=0$$
$$2=2A \Rightarrow A=1$$

Plug $A=1$ into (i).

$$A+C=4$$
$$1+C=4$$
$$C=3$$

⑥ Plug $A=1, B=2, C=3$ into ②.

$$\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x-2}$$

Example 8: Let $f(x)=\dfrac{5x^2+9}{x^3+3x}$

Determine the partial decomposition of $f(x)$.

① Factor x^3+3x completely.

$$x^3+3x=x(x^2+3)$$

② Write the fraction into decomposition form.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

③ Combine the fractions in ②.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3}=\frac{A(x^2+3)+(Bx+C)x}{x(x^2+3)}$$

$$=\frac{Ax^2+3A+Bx^2+Cx}{x^3+3x}$$

$$=\frac{(A+B)x^2+Cx+3A}{x^3+3x}$$

④ Set the old numerator = new numerator

$$5x^2+9=(A+B)x^2+Cx+3A$$

$$5x^2+0x+9=(A+B)x^2+Cx+3A$$

⑤ Create a system of equations from ④ and solve.

$$\begin{cases} 5=A+B & (i) \\ 0=C & (ii) \\ 9=3A & (iii) \end{cases}$$

Note (ii) already gives us $C=0$.
From (iii) $9=3A \Rightarrow A=3$

Plug $A=3$ into (i).
$5=A+B$
$5=3+B$
$2=B$

⑥ Plug $A=3$, $B=2$, $C=0$ into ②.

$$\frac{3}{x}+\frac{2x+0}{x^2+3}$$

Example 9: Let $f(x)=\dfrac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

Determine the partial fraction decomposition of $f(x)$.

① Factor $(x^2+x+1)^2$ completely.

Done!

② Write the fraction into decomposition form.

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}$$

③ Combine the fractions in ②.

Note the common denominator is $(x^2+x+1)^2$

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}=\frac{(Ax+B)(x^2+x+1)+Cx+D}{(x^2+x+1)^2}$$

$$=\frac{Ax^3+Bx^2+Ax^2+Bx+Ax+B+Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)}{(x^2+x+1)^2}$$

④ Set the old numerator = new numerator

$$2x^3 + 3x^2 + 5x + 2 = Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A = 2 & \text{(i)} \\ A+B = 3 & \text{(ii)} \\ A+B+C = 5 & \text{(iii)} \\ B+D = 2 & \text{(iv)} \end{cases}$$

From (i), we have $A=2$.

Plug $A=2$ into (ii)

$A+B=3$

$2+B=3$

$B=1$

Plug $A=2, B=1$ into (iii)

$A+B+C=5$

$2+1+C=5$

$3+C=5$

$C=2$

Plug $B=1$ into (iv)

$B+D=2$

$1+D=2$

$D=1$

⑥ Plug $A=2$, $B=1$, $C=2$, $D=1$ into ②.

$$\frac{2x+1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$$

Lesson 7: Partial Fractions II

Recall the concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{2}+\frac{1}{3}=\frac{1}{2}\cdot\frac{3}{3}+\frac{1}{3}\cdot\frac{2}{2}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

So we can say that a partial fraction decomposition for 5/6 is

$$\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$$

This concept can be used with functions of x.

Example 1: Combine the following fractions

$$\frac{1}{x-2}+\frac{3}{x-5}=\frac{1}{(x-2)}\cdot\frac{(x-5)}{(x-5)}+\frac{3}{(x-5)}\cdot\frac{(x-2)}{(x-2)}$$

$$=\frac{(x-5)+3(x-2)}{(x-2)(x-5)}$$

$$=\frac{x-5+3x-6}{x^2-2x-5x+10}$$

$$=\frac{4x-11}{x^2-7x+10}$$

Why do we care about partial fraction decomposition? It's because u-sub isn't enough.

Example 2: Evaluate $\int \frac{4x-11}{x^2-7x+10}\,dx$

Let's first try a u-sub.

$$\frac{u=x^2-7x+10}{du=(2x-7)dx} \qquad \int \frac{4x-11}{u}\cdot\frac{du}{2x-7}$$

As you can see there is no way to eliminate the x's.

Now let's try partial fraction decomposition. Using Ex1,

$$\int\frac{4x-11}{x^2-7x+10}\,dx=\int\frac{1}{x-2}\,dx+\int\frac{3}{x-5}\,dx$$

Now we know how to integrate these functions:

- $\int \frac{1}{x-2}\,dx \;\; \frac{u=x-2}{du=dx} \;\; \int \frac{1}{u}\,du = \ln|u| = \ln|x-2|$

- $\int \frac{3}{x-5}\,dx \;\; \frac{u=x-5}{du=dx} \;\; \int \frac{3}{u}\,du = 3\ln|u| = 3\ln|x-5|$

So $\int \frac{4x-11}{x^2-7x+10}\,dx = \ln|x-2| + 3\ln|x-5| + C$

<u>Method of Decomposing into Partial Fractions</u>

Given a rational function $\frac{N(x)}{D(x)}$

① Factor the denominator as much as possible.

② Write the fraction into decomposition form.

ⓐ Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

ⓑ Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

ⓒ Distinct irreducible quadratic terms like x^2+a^2 decompose to

$$\frac{Ax+B}{x^2+a^2}$$

ⓓ Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

③ Combine your decomposition from ② as 1 fraction.

④ Set the original numerator, $N(x)$, equal to the numerator from ③.

⑤ Equate the coefficients of the terms, to yields a system of equations. Then solve the constants. i.e. A, B, C.

⑥ Plug the values found in (5) in (2).

Example 3: Let $f(x) = \dfrac{6x+10}{x^2+5x}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^2+5x completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+5}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+5} = \frac{A}{x}\cdot\frac{(x+5)}{(x+5)} + \frac{B}{x+5}\cdot\frac{x}{x}$$

$$= \frac{A(x+5)+Bx}{x(x+5)} = \frac{Ax+5A+Bx}{x(x+5)} = \frac{(A+B)x+5A}{x(x+5)}$$

④ Set the old numerater = new numerator.

$$6x+10 = (A+B)x + 5A$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=6 & \text{(i)} \\ 5A=10 & \text{(ii)} \end{cases}$$

From (ii), we find $A=2$.

Plug $A=2$ into (i)

$$A+B=6$$
$$2+B=6$$
$$B=4$$

⑥ Plug $A=2$ and $B=4$ into ②.

$$\frac{2}{x} + \frac{4}{x+5}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{6x+10}{x^2+5x}dx = \int \frac{2}{x}dx + \int \frac{4}{x+5}dx$$

$$= 2\ln|x| + 4\ln|x+5| + C$$

Example 4: Let $f(x) = \dfrac{40}{x^2-16}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^2-16 completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form.

$$\frac{A}{x-4} + \frac{B}{x+4}$$

③ Combine the fractions in ②.

$$\frac{A}{x-4} + \frac{B}{x+4} = \frac{A}{x-4} \cdot \frac{(x+4)}{(x+4)} + \frac{B}{x+4} \cdot \frac{(x-4)}{(x-4)}$$

$$= \frac{A(x+4)+B(x-4)}{(x-4)(x+4)}$$

$$= \frac{Ax+4A+Bx-4B}{(x-4)(x+4)} = \frac{(A+B)x+(4A-4B)}{(x-4)(x+4)}$$

④ Set the old numerator = new numerator

$$0x+40 = (A+B)x + (4A-4B)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=0 \\ 4A-4B=40 \end{cases} \Rightarrow \begin{cases} B=-A & (i) \\ A-B=10 & (ii) \end{cases}$$

Plug (i) into (ii).

$$A-B=10$$
$$A-(-A)=10$$
$$A+A=10$$
$$2A=10$$
$$A=5$$

Plug $A=5$ into (i)

$$B=-A$$
$$=-5$$

⑥ Plug $A=5$ and $B=-5$ into ②.

$$\frac{5}{x-4} + \frac{-5}{x+4}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int \frac{40}{x^2-16}\,dx = \int \frac{5}{x-4}\,dx + \int \frac{-5}{x+4}\,dx$$

$$= 5\ln|x-4| - 5\ln|x+4| + C$$

Example 5: Let $f(x) = \dfrac{x^2+2}{x^3+3x^2+2x}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^3+3x^2+2x completely.

$$x^3+3x^2+2x = x(x^2+3x+2)$$
$$= x(x+1)(x+2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A}{x}\cdot\frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{B}{x+1}\cdot\frac{x(x+2)}{x(x+2)} + \frac{C}{x+2}\cdot\frac{x(x+1)}{x(x+1)}$$
$$= \frac{A(x^2+3x+2) + B(x^2+2x) + C(x^2+x)}{x(x+1)(x+2)}$$
$$= \frac{(A+B+C)x^2 + (3A+2B+C)x + (2A)}{x(x+1)(x+2)}$$

④ Set the old numerator = new numerater

$$x^2+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$
$$x^2+0x+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B+C=1 & (i) \\ 3A+2B+C=0 & (ii) \\ 2A=2 & (iii) \end{cases}$$

From (iii), we find $A=1$. So we can plug that into (i) and (ii) yielding

$$\begin{cases} 1+B+C=1 \\ 3+2B+C=0 \end{cases} \Rightarrow \begin{cases} B+C=0 \\ 2B+C=-3 \end{cases} \Rightarrow \begin{cases} C=-B & (i') \\ 2B+C=-3 & (ii') \end{cases}$$

Plug (i') into (ii').

$$2B+C=-3$$
$$2B-B=-3$$
$$B=-3$$

Plug $B=-3$ into (i')

$$C=-B$$
$$=-(-3)$$
$$=3$$

⑥ Plug $A=1$, $B=-3$, $C=3$ into ②.

$$\frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$

$$\int \frac{x^2+2}{x^3+3x^2+2x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-3}{x+1}\,dx + \int \frac{3}{x+2}\,dx$$

$$= \ln|x| - 3\ln|x+1| + 3\ln|x+2| + C$$

Example 6: Let $f(x) = \dfrac{4x^2-4}{x^3-2x^2}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^3-2x^2 completely.

$$x^3-2x^2 = x^2(x-2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

③ Combine the fractions in ②.

Note the common denominator is $x^2(x-2)$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$

$$= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)}$$

$$= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^3-2x^2}$$

④ Set the old numerator = new numerator

$$4x^2 - 4 = (A+C)x^2 + (B-2A)x - 2B$$

$$4x^2 + 0x - 4 = (A+C)x^2 + (B-2A)x - 2B$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+C=4 & \text{(i)} \\ B-2A=0 & \text{(ii)} \\ -2B=-4 & \text{(iii)} \end{cases}$$

From (iii) $-2B=-4 \Rightarrow B=2$

Plug $B=2$ into (ii)

$$B-2A=0$$
$$2-2A=0$$
$$2=2A \Rightarrow A=1$$

Plug $A=1$ into (i).

$$A+C=4$$
$$1+C=4$$
$$C=3$$

⑥ Plug $A=1$, $B=2$, $C=3$ into ②.

$$\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x-2}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{4x^2-4}{x^3-2x^2}dx = \int \frac{1}{x}dx + \int \frac{2}{x^2}dx + \int \frac{3}{x-2}dx$$

$$= \int \frac{1}{x}dx + \int 2x^{-2}dx + \int \frac{3}{x-2}dx$$

$$= \ln|x| + \frac{2x^{-1}}{-1} + 3\ln|x-2| + C$$

$$= \ln|x| - \frac{2}{x} + 3\ln|x-2| + C$$

Example 7: Let $f(x) = \dfrac{5x^2+9}{x^3+3x}$

ⓐ Determine the partial decomposition of $f(x)$.

① Factor x^3+3x completely.

$$x^3+3x = x(x^2+3)$$

② Write the fraction into decomposition form.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

③ Combine the fractions in ②.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3} = \frac{A(x^2+3)+(Bx+C)x}{x(x^2+3)}$$

$$= \frac{Ax^2+3A+Bx^2+Cx}{x^3+3x}$$

$$= \frac{(A+B)x^2+Cx+3A}{x^3+3x}$$

④ Set the old numerator = new numerator

$$5x^2+9 = (A+B)x^2+Cx+3A$$

$$5x^2+0x+9 = (A+B)x^2+Cx+3A$$

⑤ Create a system of equations from ④ and solve.

$$\begin{cases} 5 = A+B & \text{(i)} \\ 0 = C & \text{(ii)} \\ 9 = 3A & \text{(iii)} \end{cases}$$

Note (ii) already gives us $C=0$.
From (iii) $9=3A \Rightarrow A=3$

Plug $A=3$ into (i).
$5=A+B$
$5=3+B$
$2=B$

(6) Plug $A=3$, $B=2$, $C=0$ into (2).
$$\frac{3}{x}+\frac{2x+0}{x^2+3}$$

(b) Using (a), evaluate $\int f(x)dx$.
$$\int \frac{5x^2+9}{x^3+3x}dx = \int \frac{3}{x}dx + \underbrace{\int \frac{2x}{x^2+3}dx}_{u\text{-sub}}$$
$u=x^2+3$
$du=2x\,dx$
$$=\int \frac{3}{x}dx+\int \frac{du}{u}$$
$$=3\ln|x|+\ln|u|+C$$
$$=3\ln|x|+\ln|x^2+3|+C$$

Example 8: Let $f(x)=\dfrac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

(a) Determine the partial fraction decomposition of $f(x)$.
(1) Factor $(x^2+x+1)^2$ completely.
Done!
(2) Write the fraction into decomposition form.
$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}$$
(3) Combine the fractions in (2).
Note the common denominator is $(x^2+x+1)^2$
$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}=\frac{(Ax+B)(x^2+x+1)+Cx+D}{(x^2+x+1)^2}$$
$$=\frac{Ax^3+Bx^2+Ax^2+Bx+Ax+B+Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)}{(x^2+x+1)^2}$$

④ Set the old numerator = new numerator

$$2x^3 + 3x^2 + 5x + 2 = Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A = 2 & (i) \\ A + B = 3 & (ii) \\ A + B + C = 5 & (iii) \\ B + D = 2 & (iv) \end{cases}$$

From (i), we have $A = 2$.

Plug $A = 2$ into (ii)

$A + B = 3$

$2 + B = 3$

$B = 1$

Plug $A = 2$, $B = 1$ into (iii)

$A + B + C = 5$

$2 + 1 + C = 5$

$3 + C = 5$

$C = 2$

Plug $B = 1$ into (iv)

$B + D = 2$

$1 + D = 2$

$D = 1$

⑥ Plug $A = 2$, $B = 1$, $C = 2$, $D = 1$ into ②.

$$\frac{2x+1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int \frac{2x^3 + 3x^2 + 5x + 2}{(x^2+x+1)^2}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{2x+1}{(x^2+x+1)^2}$$

Note both integrals have a u-sub.
This time only it's the same u.

$u = x^2 + x + 1$
$du = (2x+1)\,dx$

$$\int \frac{du}{u} + \int \frac{du}{u^2} = \int \frac{du}{u} + \int u^{-2}\,du$$

$$= \ln|u| - u^{-1} + C = \ln|u| - \frac{1}{u} + C$$

$$= \ln|x^2+x+1| - \frac{1}{x^2+x+1} + C$$