Lesson 9 + 10: Partial Fractions

Recall the concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{2}+\frac{1}{3}=\frac{1}{2}\cdot\frac{3}{3}+\frac{1}{3}\cdot\frac{2}{2}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

So we can say that a partial fraction decomposition for 5/6 is

$$\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$$

This concept can be used with functions of x.

Example 1: Combine the following fractions

$$\frac{1}{x-2}+\frac{3}{x-5}=\frac{1}{(x-2)}\cdot\frac{(x-5)}{(x-5)}+\frac{3}{(x-5)}\cdot\frac{(x-2)}{(x-2)}$$

$$=\frac{(x-5)+3(x-2)}{(x-2)(x-5)}$$

$$=\frac{x-5+3x-6}{x^2-2x-5x+10}$$

$$=\frac{4x-11}{x^2-7x+10}$$

Why do we care about partial fraction decomposition? It's because u-sub isn't enough.

Example 2: Evaluate $\displaystyle\int\frac{4x-11}{x^2-7x+10}\,dx$

Let's first try a u-sub.

$$\frac{u=x^2-7x+10}{du=(2x-7)dx}\qquad \int\frac{4x-11}{u}\cdot\frac{du}{2x-7}$$

As you can see there is no way to eliminate the x's.

Now let's try partial fraction decomposition. Using Ex1,

$$\int\frac{4x-11}{x^2-7x+10}\,dx=\int\frac{1}{x-2}\,dx+\int\frac{3}{x-5}\,dx$$

Now we know how to integrate these functions:

- $\int \frac{1}{x-2}\,dx \;\frac{u=x-2}{du=dx}\; \int \frac{1}{u}\,du = \ln|u| = \ln|x-2|$

- $\int \frac{3}{x-5}\,dx \;\frac{u=x-5}{du=dx}\; \int \frac{3}{u}\,du = 3\ln|u| = 3\ln|x-5|$

So $\int \frac{4x-11}{x^2-7x+10}\,dx = \ln|x-2| + 3\ln|x-5| + C$

Method of Decomposing into Partial Fractions

Given a rational function $\frac{N(x)}{D(x)}$

① Factor the denominator as much as possible.

② Write the fraction into decomposition form.

ⓐ Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

ⓑ Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

We will cover these next class

ⓒ Distinct irreducible quadratic terms like x^2+a^2 decompose to

$$\frac{Ax+B}{x^2+a^2}$$

ⓓ Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

③ Combine your decomposition from ② as 1 fraction.

④ Set the original numerator, $N(x)$, equal to the numerator from ③.

⑤ Equate the coefficients of the terms, to yields a system of equations. Then solve the constants. i.e. A, B, C.

⑥ Plug the values found in (5) in (2).

Example 3: Let $f(x) = \dfrac{6x+10}{x^2+5x}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^2+5x completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+5}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+5} = \frac{A}{x} \cdot \frac{(x+5)}{(x+5)} + \frac{B}{x+5} \cdot \frac{x}{x}$$

$$= \frac{A(x+5) + Bx}{x(x+5)} = \frac{Ax + 5A + Bx}{x(x+5)} = \frac{(A+B)x + 5A}{x(x+5)}$$

④ Set the old numerater = new numerator.

$$6x + 10 = (A+B)x + 5A$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A + B = 6 & \text{(i)} \\ 5A = 10 & \text{(ii)} \end{cases}$$

From (ii), we find $A = 2$.

Plug $A = 2$ into (i)

$$A + B = 6$$
$$2 + B = 6$$
$$B = 4$$

⑥ Plug $A = 2$ and $B = 4$ into ②.

$$\frac{2}{x} + \frac{4}{x+5}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int \frac{6x+10}{x^2+5x}\,dx = \int \frac{2}{x}\,dx + \int \frac{4}{x+5}\,dx$$

$$= 2\ln|x| + 4\ln|x+5| + C$$

Example 4: Let $f(x) = \dfrac{40}{x^2-16}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^2-16 completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form.

$$\frac{A}{x-4} + \frac{B}{x+4}$$

③ Combine the fractions in ②.

$$\frac{A}{x-4} + \frac{B}{x+4} = \frac{A}{x-4} \cdot \frac{(x+4)}{(x+4)} + \frac{B}{x+4} \cdot \frac{(x-4)}{(x-4)}$$

$$= \frac{A(x+4)+B(x-4)}{(x-4)(x+4)}$$

$$= \frac{Ax+4A+Bx-4B}{(x-4)(x+4)} = \frac{(A+B)x+(4A-4B)}{(x-4)(x+4)}$$

④ Set the old numerator = new numerator

$$0x+40 = (A+B)x + (4A-4B)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=0 \\ 4A-4B=40 \end{cases} \Rightarrow \begin{cases} B=-A & (i) \\ A-B=10 & (ii) \end{cases}$$

Plug (i) into (ii).

$$A-B=10$$
$$A-(-A)=10$$
$$A+A=10$$
$$2A=10$$
$$A=5$$

Plug $A=5$ into (i)

$$B=-A$$
$$=-5$$

⑥ Plug $A=5$ and $B=-5$ into ②.

$$\frac{5}{x-4} + \frac{-5}{x+4}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int \frac{40}{x^2-16}\,dx = \int \frac{5}{x-4}\,dx + \int \frac{-5}{x+4}\,dx$$

$$= 5\ln|x-4| - 5\ln|x+4| + C$$

Example 5: Let $f(x) = \dfrac{x^2+2}{x^3+3x^2+2x}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^3+3x^2+2x completely.

$$\begin{aligned} x^3+3x^2+2x &= x(x^2+3x+2) \\ &= x(x+1)(x+2) \end{aligned}$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

③ Combine the fractions in ②.

$$\begin{aligned} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} &= \frac{A}{x} \cdot \frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{B}{x+1} \cdot \frac{x(x+2)}{x(x+2)} + \frac{C}{x+2} \cdot \frac{x(x+1)}{x(x+1)} \\ &= \frac{A(x^2+3x+2) + B(x^2+2x) + C(x^2+x)}{x(x+1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (3A+2B+C)x + (2A)}{x(x+1)(x+2)} \end{aligned}$$

④ Set the old numerator = new numerater

$$x^2+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$
$$x^2+0x+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B+C=1 & (i) \\ 3A+2B+C=0 & (ii) \\ 2A=2 & (iii) \end{cases}$$

From (iii), we find $A=1$. So we can plug that into (i) and (ii) yielding

$$\begin{cases} 1+B+C=1 \\ 3+2B+C=0 \end{cases} \Rightarrow \begin{cases} B+C=0 \\ 2B+C=-3 \end{cases} \Rightarrow \begin{cases} C=-B & (i') \\ 2B+C=-3 & (ii') \end{cases}$$

Plug (i') into (ii').

$$\begin{aligned} 2B+C &= -3 \\ 2B-B &= -3 \\ B &= -3 \end{aligned}$$

Plug $B=-3$ into (i')

$$\begin{aligned} C &= -B \\ &= -(-3) \\ &= 3 \end{aligned}$$

⑥ Plug $A=1$, $B=-3$, $C=3$ into ②.

$$\frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{x^2+2}{x^3+3x^2+2x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-3}{x+1}\,dx + \int \frac{3}{x+2}\,dx$$

$$= \ln|x| - 3\ln|x+1| + 3\ln|x+2| + C$$

ⓑ using ⓐ, evaluate $\int f(x)\,dx$

$$\int \frac{x^2+2}{x^3+3x^2+2x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-3}{x+1}\,dx + \int \frac{3}{x+2}\,dx$$

$$= \ln|x| - 3\ln|x+1| + 3\ln|x+2| + C$$

Example 6: Let $f(x) = \dfrac{4x^2-4}{x^3-2x^2}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^3-2x^2 completely.

$$x^3-2x^2 = x^2(x-2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

③ Combine the fractions in ②.

Note the common denominator is $x^2(x-2)$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$

$$= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)}$$

$$= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^3-2x^2}$$

④ Set the old numerator = new numerator

$$4x^2 - 4 = (A+C)x^2 + (B-2A)x - 2B$$

$$4x^2 + 0x - 4 = (A+C)x^2 + (B-2A)x - 2B$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A + C = 4 & \text{(i)} \\ B - 2A = 0 & \text{(ii)} \\ -2B = -4 & \text{(iii)} \end{cases}$$

From (iii) $-2B = -4 \Rightarrow B = 2$

Plug $B=2$ into (ii)

$$B - 2A = 0$$
$$2 - 2A = 0$$
$$2 = 2A \Rightarrow A = 1$$

Plug $A=1$ into (i).

$$A + C = 4$$
$$1 + C = 4$$
$$C = 3$$

⑥ Plug $A=1$, $B=2$, $C=3$ into ②.

$$\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x-2}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{4x^2-4}{x^3-2x^2}dx = \int \frac{1}{x}dx + \int \frac{2}{x^2}dx + \int \frac{3}{x-2}dx$$

$$= \int \frac{1}{x}dx + \int 2x^{-2}dx + \int \frac{3}{x-2}dx$$

$$= \ln|x| + \frac{2x^{-1}}{-1} + 3\ln|x-2| + C$$

$$= \ln|x| - \frac{2}{x} + 3\ln|x-2| + C$$

Example 7: Let $f(x) = \dfrac{5x^2+9}{x^3+3x}$

ⓐ Determine the partial decomposition of $f(x)$.

① Factor x^3+3x completely.

$$x^3+3x = x(x^2+3)$$

② Write the fraction into decomposition form.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

③ Combine the fractions in ②.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3} = \frac{A(x^2+3)+(Bx+C)x}{x(x^2+3)}$$

$$= \frac{Ax^2+3A+Bx^2+Cx}{x^3+3x}$$

$$= \frac{(A+B)x^2+Cx+3A}{x^3+3x}$$

④ Set the old numerator = new numerator

$$5x^2+9 = (A+B)x^2+Cx+3A$$

$$5x^2+0x+9 = (A+B)x^2+Cx+3A$$

⑤ Create a system of equations from ④ and solve.

$$\begin{cases} 5 = A+B & (i) \\ 0 = C & (ii) \\ 9 = 3A & (iii) \end{cases}$$

Note (ii) already gives us $C=0$.
From (iii) $9=3A \Rightarrow A=3$

Plug $A=3$ into (i).

$5=A+B$

$5=3+B$

$2=B$

⑥ Plug $A=3, B=2, C=0$ into ②.

$$\frac{3}{x}+\frac{2x+0}{x^2+3}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{5x^2+9}{x^3+3x}dx = \int \frac{3}{x}dx + \underbrace{\int \frac{2x}{x^2+3}dx}_{u\text{-sub}}$$

$u=x^2+3$
$du=2x\,dx$

$$= \int \frac{3}{x}dx + \int \frac{du}{u}$$

$$= 3\ln|x| + \ln|u| + C$$

$$= 3\ln|x| + \ln|x^2+3| + C$$

Example 8: Let $f(x)=\dfrac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor $(x^2+x+1)^2$ completely.

Done!

② Write the fraction into decomposition form.

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}$$

③ Combine the fractions in ②.

Note the common denominator is $(x^2+x+1)^2$

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2} = \frac{(Ax+B)(x^2+x+1)+Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3+Bx^2+Ax^2+Bx+Ax+B+Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)}{(x^2+x+1)^2}$$

④ Set the old numerator = new numerator

$$2x^3 + 3x^2 + 5x + 2 = Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A = 2 & (i) \\ A + B = 3 & (ii) \\ A + B + C = 5 & (iii) \\ B + D = 2 & (iv) \end{cases}$$

From (i), we have $A = 2$.

Plug $A = 2$ into (ii)

$$A + B = 3$$
$$2 + B = 3$$
$$B = 1$$

Plug $A = 2, B = 1$ into (iii)

$$A + B + C = 5$$
$$2 + 1 + C = 5$$
$$3 + C = 5$$
$$C = 2$$

Plug $B = 1$ into (iv)

$$B + D = 2$$
$$1 + D = 2$$
$$D = 1$$

⑥ Plug $A = 2, B = 1, C = 2, D = 1$ into ②.

$$\frac{2x+1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int \frac{2x^3 + 3x^2 + 5x + 2}{(x^2+x+1)^2}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{2x+1}{(x^2+x+1)^2}$$

Note both integrals have a u-sub.
This time only it's the same u.

$$\frac{u = x^2 + x + 1}{du = (2x+1)\,dx} \quad \int \frac{du}{u} + \int \frac{du}{u^2} = \int \frac{du}{u} + \int u^{-2}\,du$$

$$= \ln|u| - u^{-1} + C = \ln|u| - \frac{1}{u} + C$$

$$= \ln|x^2+x+1| - \frac{1}{x^2+x+1} + C$$