MATH 16020 | PRACTICE EXAM 3 | WED., NOV. 8, 2023

> Please show **all** your work! Answers without supporting work will not be given credit.
> Write answers in spaces provided.

Name: Solutions

1. The rate of change of the population $n(t)$ of a sample of bacteria is directly proportional to the number of bacteria present, so $N'(t) = kN$, where time t is measured in minutes. Initially, there are 210 bacteria present. If the number of bacteria after 5 hours is 360, find the growth rate k in terms of minutes. Round to four decimal places.

Recall $N' = kN \rightarrow N = Ce^{kt}$

$\underline{N(0)=210}$: $210 = Ce^{k\cdot 0}$

$210 = C \rightarrow N = 210e^{kt}$

$\underline{N(5)=360}$: $360 = 210e^{k\cdot 5}$

$\frac{12}{7} = e^{5k}$

$\ln(12/7) = 5k$

$k = \boxed{\frac{1}{5}\ln\left(\frac{12}{7}\right)}$

2. Let y denote the mass of a radioactive substance at time t. Suppose this substance obeys the equation

$$y' = -18y$$

Assume that initially, the mass of the substance is $y(0) = 20$ grams. At what time t in hours does half the original mass remain? Round your answer to 3 decimal places.

$y' = -18y \Rightarrow y = Ce^{-18t}$

$y(0) = 20 \Rightarrow 20 = Ce^{-18(0)}$

$20 = C \Rightarrow y = 20e^{-18t}$

We want solve $\frac{1}{2}(20) = y(t)$ for t.

$10 = 20e^{-18t}$

$1/2 = e^{-18t}$

$\ln(1/2) = -18t$

$\frac{\ln(1/2)}{-18} = t$

$t = \boxed{0.039}$

3. Find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Rewrite: $y\,dy = 3x^2\,dx$

$\int y\,dy = \int 3x^2\,dx$

$\frac{y^2}{2} = x^3 + c$

$y^2 = 2x^3 + c$

$y = \pm\sqrt{2x^3+c}$

$y = \boxed{\pm\sqrt{2x^3+c}}$

4. Find the general solution to the differential equation:

$$\frac{dy}{dx} = 5y$$

Rewrite $dy = 5y\,dx$

$\frac{dy}{y} = 5dx$

$\int \frac{dy}{y} = \int 5dx$

$\ln|y| = 5x + c$

$|y| = e^{5x+c}$

$\pm y = e^c e^{5x}$

$y = \underbrace{\pm e^c}_{=c} e^{5x}$

$y = ce^{5x}$

or memorize

$\frac{dy}{dx} = ky \Rightarrow y = Ce^{kx}$

$y = \boxed{Ce^{5x}}$

5. Find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Rewrite: $y\,dy = -x\,dx$

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = \frac{-x^2}{2} + c$

$y^2 = -x^2 + c$

$y = \pm\sqrt{c - x^2}$

$y = \boxed{\pm\sqrt{c-x^2}}$

6. Find the general solution to the given differential question. Use C as an arbitrary constant.

$$\frac{dy}{dt} - 15y = 0$$

Note there are 2 ways to do this problem.

1. Separation of Variables
2. First-Order Linear Eqn

By method 1,

$\frac{dy}{dt} = 15y$

$\frac{dy}{y} = 15\,dt$

$\int \frac{dy}{y} = \int 15\,dt$

$\ln|y| = 15t + c$

$y = e^{15t + c}$

$y = e^{c} e^{15t}$

$y = Ce^{15t}$

$y = \boxed{Ce^{15t}}$

7. Find the general solution to the given differential question. Use C as an arbitrary constant.

$$\frac{dy}{dx} = \frac{3}{y}$$

$y\,dy = 3\,dx$

$\int y\,dy = \int 3\,dx$

$\frac{y^2}{2} = 3x + C$

$y^2 = 6x + 2C$

$y^2 = 6x + C$

$y = \pm\sqrt{6x + C}$

$y =$ $\boxed{\pm\sqrt{6x+C}}$

8. Find the general solution to the given differential question. Use C as an arbitrary constant.

$$\frac{dy}{dx} = 3x^2 y$$

$\frac{dy}{y} = 3x^2\,dx$

$\int \frac{dy}{y} = \int 3x^2\,dx$

$\ln|y| = x^3 + C$

$y = e^{x^3 + C}$

$y = e^C e^{x^3}$

$y = Ce^{x^3}$

$y =$ $\boxed{Ce^{x^3}}$

9. Find the general solution to the given differential question. Use C as an arbitrary constant.

$$\frac{dy}{dt} = 8e^{-4t-y}$$

$dy = 8e^{-4t}e^{-y}\,dt$

$e^{y}\,dy = 8e^{-4t}\,dt$

$\int e^{y}\,dy = \int 8e^{-4t}\,dt$

$e^{y} = \frac{8}{-4}e^{-4t} + C$

$e^{y} = -2e^{-4t} + C$

$y = \ln(-2e^{-4t} + C)$

$y = \boxed{\ln(-2e^{-4t} + C)}$

10. Find the particular solution to the differential equation.

$$\frac{dy}{dx} = \frac{3x+2}{2y} \quad \text{and} \quad y(0) = 4$$

$2y\,dy = (3x+2)\,dx$

$\int 2y\,dy = \int (3x+2)\,dx$

$y^2 = \frac{3x^2}{2} + 2x + C$

when $y(0) = 4$

$4^2 = 0 + 0 + C$

$16 = C$

So $y^2 = \frac{3x^2}{2} + 2x + 16$

$y = \pm\sqrt{\frac{3x^2}{2} + 2x + 16}$

$y = \boxed{\pm\sqrt{\frac{3x^2}{2} + 2x + 16}}$

11. Find the particular solution to the differential equation.

$$\frac{dy}{dx} = \frac{5y}{6x+3} \quad \text{and} \quad y(0) = 1$$

$$\frac{dy}{y} = \frac{5}{6x+3}dx$$

$$\int \frac{dy}{y} = \int \frac{5}{6x+3}dx$$

$$\ln|y| = \frac{5}{6}\ln|6x+3| + C$$

$$y = \exp\left[\frac{5}{6}\ln|6x+3| + C\right]$$

$$y = e^C \exp\left[\ln|6x+3|^{5/6}\right]$$

$$y = C\cdot|6x+3|^{5/6}$$

When $y(0)=1$

$$1 = C\cdot|6(0)+3|^{5/6}$$

$$1 = C\cdot 3^{5/6}$$

$$C = 3^{-5/6}$$

$y =$ $\boxed{3^{-5/6}\cdot|6x+3|^{5/6}}$

12. Consider the following IVP:

$$\frac{dy}{dx} = 11x^2e^{-x^3} \quad \text{where} \quad y = 10 \quad \text{when} \quad x = 2$$

Find the value of the integration constant, C.

$$dy = 11x^2e^{-x^3}dx$$

$$\int dy = \underbrace{\int 11x^2e^{-x^3}dx}$$

$$u = -x^3$$

$$du = -3x^2dx$$

$$y = \int -\frac{11}{3}e^u du$$

$$y = -\frac{11}{3}e^{-x^3} + C$$

When $y=10$ and $x=2$

$$10 = -\frac{11}{3}e^{-2^3} + C$$

$$10 = -\frac{11}{3}e^{-8} + C$$

$$C = 10 + \frac{11}{3}e^{-8}$$

$C =$ $\boxed{10 + \frac{11}{3}e^{-8}}$

13. Calculate the constant of integration, C, for the given differential equation.

$$\frac{dy}{dx} = \frac{7x^3}{6y}, \qquad y(1) = 2$$

Rewrite $6y\,dy = 7x^3\,dx$

$\int 6y\,dy = \int 7x^3\,dx$

$3y^2 = \frac{7x^4}{4} + C$

Note we want C when $y(1)=2$

$3(2)^2 = \frac{7(1)^4}{4} + C$

$12 = \frac{7}{4} + C$

$C = 41/4$

$C=$ $\boxed{\frac{41}{4}}$

14. The volume of an object $V(t)$ in cubic millimeter at any time t in seconds changes according to the model

$$\frac{dV}{dt} = \cos\left(\frac{t}{10}\right),$$

where $V(0) = 5$. Find the volume of the object at $t = 3$ seconds. Round to 4 decimal places.

Rewrite $dV = \cos\left(\frac{t}{10}\right)dt$

$\int dV = \int \cos\left(\frac{t}{10}\right)dt$

$V = 10\sin\left(\frac{t}{10}\right) + C$

Find C w/ $V(0)=5$

$5 = 10\sin\left(\frac{0}{10}\right) + C$

$C = 5$

So $V = 10\sin\left(\frac{t}{10}\right) + 5$

$V(3) = 10\sin\left(\frac{3}{10}\right) + 5$

≈ 7.9552

$V(3) =$ $\boxed{7.9552}$

15. What is the **integrating factor** of the following differential equation?

$$y' + \left(\frac{2x+3}{x}\right) y = 10\ln(x)$$

$P(x) = \frac{2x+3}{x}$ $Q(x) = 10\ln(x)$

$u(x) = \exp\left[\int P(x)\,dx\right]$

$= \exp\left[\int \frac{2x+3}{x}\,dx\right]$

$= \exp\left[\int 2 + \frac{3}{x}\,dx\right]$

$= \exp\left[2 + 3\ln x\right]$

$= e^{2+3\ln x}$

$= e^2 e^{3\ln x}$

$= e^2 e^{\ln x^3}$

$= x^3 e^2$

$u(x) =$ $\boxed{x^3 e^2}$

16. What is the **integrating factor** of the following differential equation?

$$\frac{2y' + \left(\frac{6}{x}\right) y}{2} = \frac{10\ln(x)}{2}$$

$y' + \frac{3}{x} y = 5\ln x$

$P(x) = \frac{3}{x}$ $Q(x) = 5\ln x$

$u(x) = \exp\left[\int \frac{3}{x}\,dx\right]$

$= \exp\left[3\ln x\right]$

$= \exp\left[\ln x^3\right]$

$= x^3$

$u(x) =$ $\boxed{x^3}$

17. What is the **integrating factor** of the following differential equation?

$$\frac{x^8y' - 14x^7y}{x^8} = \frac{32e^{7x}}{x^8}$$

$$y' + \underbrace{\left(\frac{-14}{x}\right)}_{P} y = \underbrace{\frac{32e^{7x}}{x^8}}_{Q}$$

$$u(x) = \exp\left[\int P(x)\,dx\right]$$
$$= \exp\left[\int -\frac{14}{x}\,dx\right]$$
$$= \exp\left[-14\ln x\right]$$
$$= \exp\left[\ln x^{-14}\right]$$
$$= x^{-14}$$
$$= \frac{1}{x^{14}}$$

$u(x) =$ $\boxed{1/x^{14}}$

18. What is the **integrating factor** of the following differential equation?

$$\frac{(x+1)\frac{dy}{dx} - 2(x^2+x)y}{(x+1)} = \frac{(x+1)e^{x^2}}{(x+1)}$$

$$\frac{dy}{dx} - \frac{2x(x+1)}{(x+1)}y = e^{x^2}$$

$$\frac{dy}{dx} + (-2x)\cdot y = e^{x^2}$$

$$u(x) = \exp\left[\int P(x)\,dx\right]$$
$$= \exp\left[\int -2x\,dx\right]$$
$$= \exp\left[-x^2\right]$$

$u(x) =$ $\boxed{e^{-x^2}}$

19. What is the **integrating factor** of the following differential equation?

$$y' + \cot(x) \cdot y = \sin^2(x)$$

$u(x) = \exp\left[\int P(x)\,dx\right]$
$= \exp\left[\int \cot x\,dx\right]$
$= \exp\left[\int \frac{\cos x}{\sin x}\,dx\right]$

$u = \sin x$
$du = \cos x\,dx$

$= \exp\left[\int \frac{du}{u}\right]$
$= \exp[\ln u]$
$u(x) = \exp[\ln \sin x]$
$= \sin x$

$u(x) =$ $\boxed{\sin x}$

20. What is the **integrating factor** of the following differential equation?

$$y' + \tan(x) \cdot y = \sec(x)$$

$u(x) = \exp\left[\int P(x)\,dx\right]$
$= \exp\left[\int \tan x\,dx\right]$
$= \exp\left[\int \frac{\sin x}{\cos x}\,dx\right]$

$u = \cos x$
$du = -\sin x\,dx$

$= \exp\left[-\int \frac{du}{u}\right]$
$= \exp[-\ln u]$
$u(x) = \exp[-\ln(\cos x)]$
$= \exp[\ln(\cos x)^{-1}]$
$= (\cos x)^{-1} = \sec x$

$u(x) =$ $\boxed{\sec(x)}$

Note there are 2 ways to do this problem.
① Separation of Variables
② First-Order Linear Eqn

21. Find the general solution of the following differential equation.

$$\frac{dy}{dx} + (4x-1)y = 8x - 2$$

$P(x) = 4x - 1 \qquad Q(x) = 8x - 2$

$u(x) = \exp\left[\int (4x-1)\,dx\right]$

$= \exp\left[2x^2 - x\right]$

$= e^{2x^2 - x}$

$y\,u(x) = \int Q(x)u(x)\,dx + C$

$y e^{2x^2-x} = \underbrace{\int (8x-2)e^{2x^2-x}\,dx}_{\substack{u = 2x^2 - x \\ du = 4x-1\,dx}} + C$

$y e^{2x^2-x} = \int \frac{8x-2}{4x-1} e^u\,du + C$

$y e^{2x^2-x} = \int \frac{2\cancel{(4x-1)}}{\cancel{4x-1}} e^u\,du + C$

$y e^{2x^2-x} = \int 2e^u\,du + C$

$y e^{2x^2-x} = 2e^u + C$

$y e^{2x^2-x} = 2e^{2x^2-x} + C$

$y = \frac{2e^{2x^2-x} + C}{e^{2x^2-x}}$

$y = 2 + Ce^{-(2x^2-x)}$

$= 2 + Ce^{x-2x^2}$

$y = \boxed{2 + Ce^{x-2x^2}}$

22. Find the particular solution to the differential equation.

$$\frac{dy}{dx} = 6x^2(y+4) \text{ and } y(0) = 3$$

$y' = 6x^2y + 24x^2$

$y' - 6x^2y = 24x^2$

$P(x) = -6x^2 \quad Q(x) = 24x^2$

$u(x) = \exp\left[\int -6x^2\,dx\right]$

$= \exp\left[-2x^3\right]$

$= e^{-2x^3}$

$y \cdot u(x) = \int Q(x)u(x)\,dx + C$

$ye^{-2x^3} = \underbrace{\int 24x^2 e^{-2x^3}\,dx}_{}+C$

$u = -2x^3$

$du = -6x^2\,dx$

$ye^{-2x^3} = \int -4e^u\,du + C$

$ye^{-2x^3} = -4e^u + C$

$ye^{-2x^3} = -4e^{-2x^3} + C$

$y = -4 + Ce^{2x^3}$

With $y(0) = 3$

$3 = -4 + Ce^{2\cdot 0^3}$

$3 = -4 + C$

$7 = C$

So $y = -4 + 7e^{2x^3}$

$y =$ $\boxed{-4 + 7e^{2x^3}}$

23. Solve the initial value problem.

$$x^4 y' + 4x^3 \cdot y = 10x^9 \quad \text{with} \quad f(1) = 23$$

$$\frac{x^4 y' + 4x^3 y}{x^4} = \frac{10x^9}{x^4}$$

$$y' + \frac{4}{x} \cdot y = 10x^5$$

$$P(x) = \frac{4}{x} \quad Q(x) = 10x^5$$

$$u(x) = \exp\left[\int P(x)\,dx\right]$$
$$= \exp\left[\int \frac{4}{x}\,dx\right]$$
$$= \exp[4 \ln x]$$
$$= \exp[\ln x^4]$$
$$= x^4$$

$$y \cdot u(x) = \int Q(x) u(x)\,dx + C$$
$$y \cdot x^4 = \int 10x^5 x^4\,dx + C$$
$$y \cdot x^4 = \int 10x^9\,dx + C$$
$$y \cdot x^4 = x^{10} + C$$
$$y = \frac{x^{10}}{x^4} + \frac{C}{x^4}$$
$$y = x^6 + \frac{C}{x^4}$$

$$23 = 1 + \frac{C}{1}$$
$$22 = C$$
$$y = x^6 + \frac{22}{x^4}$$

$y =$ $\boxed{x^6 + \frac{22}{x^4}}$

24. (a) Use summation notation to write the series in compact form.

$$1 - 0.6 + 0.36 - 0.216 + \ldots$$

$$= 1 - \frac{6}{10} + \frac{36}{100} - \frac{216}{1000} + \ldots$$

$$= 1 - \frac{6}{10} + \left(\frac{6}{10}\right)^2 - \left(\frac{6}{10}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{6}{10}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{-6}{10}\right)^n$$

Answer: $\boxed{\sum_{n=0}^{\infty} \left(\frac{-6}{10}\right)^n}$

(b) Use the sum from (a) and compute the sum.

$$\sum_{n=0}^{\infty} \left(\frac{-6}{10}\right)^n = \frac{1}{1-(-6/10)} = \frac{1}{1+6/10} = \frac{1}{16/10} = \frac{10}{16} = \frac{5}{8}$$

Answer: $\boxed{5/8}$

25. If the given series converges, then find its sum. If not, state that it diverges.

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$$

Note $r = 3/2$ and

$\left|\frac{3}{2}\right| < 1$ is false

So the sum diverges

$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n =$ $\boxed{\text{diverges}}$

26. If the given series converges, then find its sum. If not, state that it diverges.

$$\sum_{n=0}^{\infty} 6\left(-\frac{1}{9}\right)^n$$

$$= \frac{6}{1-(-1/9)}$$

$$= \frac{6}{1+1/9}$$

$$= \frac{6}{10/9}$$

$$= 6 \cdot \frac{9}{10}$$

$$= 3 \cdot \frac{9}{5} = \frac{27}{5}$$

$$\sum_{n=0}^{\infty} 6\left(-\frac{1}{9}\right)^n = \boxed{27/5}$$

27. If the given series converges, then find its sum. If not, state that it diverges.

$$\sum_{n=0}^{\infty} \left(\frac{7}{4^n}\right)$$

$$= \sum_{n=0}^{\infty} 7\left(\frac{1}{4}\right)^n$$

$$= \frac{7}{1-1/4}$$

$$= \frac{7}{3/4}$$

$$= 7 \cdot \frac{4}{3} = \frac{28}{3}$$

$$\sum_{n=0}^{\infty} \left(\frac{7}{4^n}\right) = \boxed{28/3}$$

28. Compute

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n}$$

$$= \frac{5^3}{6} + \frac{5^4}{6^2} + \frac{5^5}{6^3} + \cdots$$

$$= \frac{5^3}{6}\left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right)$$

$$= \frac{125}{6} \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n = \frac{125}{6} \cdot \frac{1}{1-5/6}$$

$$= \frac{125}{6} \cdot \frac{1}{1/6} = \frac{125}{6} \cdot \frac{6}{1} = 125$$

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n} = \boxed{125}$$

29. Compute

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-2)^n}{3 \cdot 3^{2n}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3} \cdot \frac{(-2)^n}{(3^2)^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3} \left(-\frac{2}{9}\right)^n$$

$$= \frac{1/3}{1-(-2/9)}$$

$$= \frac{1/3}{1+2/9}$$

$$= \frac{1/3}{11/9}$$

$$= \frac{1}{3} \cdot \frac{9}{11}$$

$$= 3/11$$

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{2n+1}} = \boxed{3/11}$$

30. Evaluate the sum of the following infinite series.

$$\sum_{n=0}^{\infty}(-1)^n\frac{5^{n+1}}{3^{2n}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 5^n \cdot 5^1}{(3^2)^n}$$

$$= \sum_{n=0}^{\infty} 5\left(-\frac{5}{9}\right)^n$$

$$= \frac{5}{1-(-5/9)}$$

$$= \frac{5}{14/9}$$

$$= \frac{5}{1}\cdot\frac{9}{14}$$

$$= \frac{45}{14}$$

Answer: $\boxed{45/14}$

31. Evaluate the sum of the following infinite series.

$$\sum_{n=1}^{\infty}\left(\frac{3^{n-1}}{4^n}+\frac{(-1)^{n+1}}{9^n}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{3^{-1}}{1}\cdot\frac{3^n}{4^n} + \frac{(-1)^1}{1}\cdot\frac{(-1)^n}{9^n}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{3}\left(\frac{3}{4}\right)^n - \left(\frac{-1}{9}\right)^n\right)$$

$$= \frac{1}{3}\left(\frac{3}{4}\right)^1 - \left(\frac{-1}{9}\right)^1$$

$$+ \frac{1}{3}\left(\frac{3}{4}\right)^2 - \left(\frac{-1}{9}\right)^2$$

$$+ \frac{1}{3}\left(\frac{3}{4}\right)^3 - \left(\frac{-1}{9}\right)^3$$

$$+ \ldots$$

$$= \frac{1}{3}\left(\frac{3}{4}\right)\left[1+\left(\frac{3}{4}\right)+\left(\frac{3}{4}\right)^2+\ldots\right]$$

$$-\left(\frac{1}{-9}\right)\left[1+\left(-\frac{1}{9}\right)+\left(\frac{-1}{9}\right)^2\ldots\right]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n + \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{-1}{9}\right)^n$$

$$= \frac{1}{4}\cdot\frac{1}{1-3/4} + \frac{1}{9}\cdot\frac{1}{1-(-1/9)}$$

Answer: $\boxed{1.1}$

32. Find the radius of convergence for the power series shown below.

$$\sum_{n=0}^{\infty} 3(-2x)^n$$

Remember

$\sum_{n=0}^{\infty} \square^n = \frac{1}{1-\square}$ where $|\square| < 1$

$|-2x| < 1$

$|2x| < 1$

$2|x| < 1$

$|x| < 1/2 = R$

$R =$ $\boxed{1/2}$

33. Find the radius of convergence for the power series shown below.

$$\sum_{n=0}^{\infty} 3\left(7x^2\right)^n$$

Remember

$\sum_{n=0}^{\infty} \square^n = \frac{1}{1-\square}$ where $|\square| < 1$

$|7x^2| < 1$

$7|x^2| < 1$

$|x^2| < 1/7$

$-1/7 < x^2 < 1/7$

By algebra

$x^2 < 1/7$

$x < \pm\sqrt{1/7}$

$|x| < \sqrt{1/7}$

$R =$ $\boxed{\sqrt{1/7}}$

34. Use the first three terms of the powers series representation of the $f(x) = \dfrac{3x}{10+2x}$ to estimate $f(0.5)$. Round to 4 decimal places.

$$\frac{3x}{10\left(1+\frac{2}{10}x\right)} = \frac{3x}{16} \cdot \frac{1}{1-\left(-\frac{2}{10}x\right)}$$

$$\frac{1}{1-\left(-\frac{2}{10}x\right)} = \sum_{n=0}^{\infty} \left(-\frac{2}{10}x\right)^n$$

$$f(x) = \frac{3x}{10} \cdot \frac{1}{1-\left(-\frac{2}{10}x\right)} = \frac{3x}{10} \sum_{n=0}^{\infty} \left(-\frac{2}{10}x\right)^n$$

$$f(x) = \frac{3x}{10} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{10^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n \cdot 3^1 x^{n+1}}{10^{n+1}}$$

$$f(0.5) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n \cdot 3 \cdot (0.5)^{n+1}}{10^{n+1}}$$

$$= \frac{3(0.5)}{10} - \frac{2 \cdot 3(0.5)^2}{10^2} + \frac{2^2 \cdot 3(0.5)^3}{10^3}$$

$$\approx 0.1365$$

$f(0.5) \approx$ $\boxed{0.1365}$

35. Express $f(x) = \dfrac{3}{1+2x}$ as a power series and determine it's radius of converge.

$$\frac{3}{1+2x} = \frac{3}{1} \cdot \frac{1}{1+2x} = \frac{3}{1} \cdot \frac{1}{1-(-2x)}$$

$$\frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n \text{ where } |-2x| < 1$$

$$f(x) = \frac{3}{1-(-2x)} = 3\sum_{n=0}^{\infty} (-2x)^n \text{ where } 2|x|<1$$

$$= \sum_{n=0}^{\infty} 3(-1)^n 2^n x^n \text{ where } |x| < 1/2$$

$\dfrac{3}{1+2x} =$ $\boxed{\sum_{n=0}^{\infty} 3(-1)^n 2^n x^n}$

$R =$ $\boxed{1/2}$

36. Express $f(x) = \dfrac{5x}{3+2x^2}$ as a power series and determine it's radius of converge.

$$\frac{5x}{3(1+2x^2/3)} = \frac{5x}{3} \cdot \frac{1}{1-(-(2x^2/3))}$$

$$\frac{1}{1-(-2x^2/3)} = \sum_{n=0}^{\infty} \left(-\frac{2x^2}{3}\right)^n \text{ where } \left|-\frac{2x^2}{3}\right| < 1$$

$$\frac{2}{3}|x^2| < 1$$

$$|x^2| < \frac{3}{2}$$

$$-\frac{3}{2} < x^2 < \frac{3}{2}$$

By algebra

$$x^2 < \frac{3}{2}$$

$$-\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$$

$$|x| < \sqrt{\frac{3}{2}}$$

$$f(x) = \frac{5x}{3} \cdot \frac{1}{1-(-2x^2/3)} = \frac{5x}{3} \sum_{n=0}^{\infty} \left(\frac{-2x^2}{3}\right)^n$$

$$f(x) = \frac{5x}{3} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n}}{3^n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n \cdot 5 \cdot x^{2n+1}}{3^{n+1}}$$

$\dfrac{5x}{3+2x^2} =$ $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^n \cdot 5 \cdot x^{2n+1}}{3^{n+1}}}$

$R =$ $\boxed{\sqrt{3/2}}$

37. What are the first 3 non-zero terms of the Maclaurin series representation of the follow?

$$\int \sin(x^{3/2})\,dx$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin(x^{3/2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^{3/2})^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{3n+3/2}$$

$$\int \sin(x^{3/2})\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{3n+3/2}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{3n+3/2}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{3n+5/2}}{3n+5/2}$$

$$= \frac{x^{5/2}}{5/2} - \frac{x^{11/2}}{6\cdot(3+5/2)} + \frac{x^{17/5}}{5!\,(6+5/2)}$$

$\int \sin(x^{3/2})\,dx =$ $\boxed{\frac{x^{5/2}}{5/2} - \frac{x^{11/2}}{6\cdot(3+5/2)} + \frac{x^{17/5}}{5!\,(6+5/2)}}$

38. Evaluate the following indefinite integral as a series. [Leave your answer as a sum.]

$$\int 5e^{5x^3}\,dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{5x^3} = \sum_{n=0}^{\infty} \frac{(5x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n x^{3n}}{n!}$$

$$5e^{5x^3} = 5\sum_{n=0}^{\infty} \frac{5^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{5^{n+1}}{n!} x^{3n}$$

$$\int 5e^{5x^3}\,dx = \int \sum_{n=0}^{\infty} \frac{5^{n+1}}{n!} x^{3n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{5^{n+1}}{n!} \int x^{3n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{5^{n+1}}{n!} \cdot \frac{x^{3n+1}}{(3n+1)}$$

$\int 5e^{5x^3}\,dx =$ $\boxed{\sum_{n=0}^{\infty} \frac{5^{n+1}}{n!} \cdot \frac{x^{3n+1}}{(3n+1)}}$

39. Use a power series to approximate the definite integral using the first 3 terms of the series.

$$\int_0^{0.24} \frac{x}{5+x^6}\,dx$$

$$\frac{x}{5+x^6} = \frac{x}{5-(-x^6)} = \frac{x}{5[1-(-x^6/5)]} = \frac{x}{5}\cdot\frac{1}{1-(-x^6/5)}$$

$$\frac{1}{1-(-x^6/5)} = \sum_{n=0}^{\infty}\left(\frac{-x^6}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{5^n}$$

$$\frac{x}{5}\cdot\frac{1}{1-(-x^6/5)} = \frac{x}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+1}}{5^{n+1}}$$

$$\int_0^{0.24}\frac{x}{5+x^6}\,dx = \int_0^{0.24}\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+1}}{5^{n+1}}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}\int_0^{0.24} x^{6n+1}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}\cdot\left.\frac{x^{6n+2}}{(6n+2)}\right]_0^{0.24}$$

$$= \left.\left(\frac{1}{5}\cdot\frac{x^2}{2} - \frac{1}{5^2}\cdot\frac{x^8}{8} + \frac{1}{5^3}\cdot\frac{x^{14}}{14}\right)\right]_0^{0.24}$$

$$\approx 0.00576$$

$\int_0^{0.24}\frac{x}{5+x^6}\,dx \approx$ $\boxed{0.00576}$

40. Use a power series to approximate the definite integral using the first 4 terms of the series. Round to 5 decimal places.

$$\int_0^{0.11} \frac{1}{1+x^4}\,dx$$

$$\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

$$\int_0^{0.11} \frac{1}{1+x^4}\,dx = \int_0^{0.11} \sum_{n=0}^{\infty} (-1)^n x^{4n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{0.11} x^{4n}\,dx$$

$$= \sum_{n=0}^{\infty} \left.\frac{(-1)^n x^{4n+1}}{4n+1}\right]_0^{0.11}$$

$$= \left(x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13}\right)\Big]_0^{0.11}$$

$\int_0^{0.11} \frac{1}{1+x^4}\,dx \approx$ $\boxed{0.11000}$

41. Use a power series to approximate the definite integral using the first 3 terms of the series. Round to 5 decimal places.

$$\int_0^{0.23} e^{-x^2}\,dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

$$\int_0^{0.23} e^{-x^2}\,dx = \int_0^{0.23} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{0.23} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \left.\frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\right]_0^{0.23}$$

$$= \left(\frac{x}{0!} - \frac{x^3}{1!(3)} + \frac{x^5}{2!(5)}\right)\Big]_0^{0.23}$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{10}\right)\Big]_0^{0.23}$$

$\int_0^{0.23} e^{-x^2}\,dx \approx$ $\boxed{0.226}$

42. Use a power series to approximate the definite integral using the first 4 terms of the series. Round to 5 decimal places.

$$\int_0^{0.45} 4x\cos(\sqrt{x})\,dx$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^{1/2})^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$$

$$f(x) = 4x\cos(\sqrt{x}) = 4x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 4x^{n+1}$$

$$\int_0^{0.45} 4x\cos(\sqrt{x})\,dx = \int_0^{0.45} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 4x^{n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 4\int_0^{0.45} x^{n+1}\,dx$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 4\frac{x^{n+2}}{n+2}\Bigg]_0^{0.45}$$

$$= \left(\frac{4x^2}{0!(2)} - \frac{4x^3}{2!(3)} + \frac{4x^4}{4!(4)} - \frac{4x^5}{6!(5)}\right)\Bigg]_0^{0.45}$$

$$= \left(2x^2 - \frac{2x^3}{3} + \frac{x^4}{6} - \frac{x^5}{900}\right)\Bigg]_0^{0.45}$$

$\int_0^{0.45} 4x\cos(\sqrt{x})\,dx \approx$ $\boxed{0.35106}$

43. Use the first 3 terms of the Macluarin series for $f(x) = \ln(1+x)$ to evaluate $ln(1.56)$. Round to 5 decimal places.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Note $1.56 = 1 + 0.56$

$$\ln(1+0.56) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (0.56)^n$$

$$= 0.56 - \frac{(0.56)^2}{2} + \frac{(0.56)^3}{3}$$

$\ln(1.56) \approx$ $\boxed{0.46174}$

44. Use the first 4 terms of the Macluarin series for $f(x) = \sin(x)$ to evaluate $\sin(0.75)$. Round to 5 decimal places.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(0.75) = \sum_{n=0}^{\infty} \frac{(-1)^n (0.75)^{2n+1}}{(2n+1)!} = \frac{0.75}{1!} - \frac{(0.75)^3}{3!} + \frac{(0.75)^5}{5!} - \frac{(0.75)^7}{7!}$$

$\sin(0.75) \approx$ $\boxed{0.74631}$

45. Find the domain of

$$f(x,y) = \frac{\sqrt{x+y-1}}{\ln(y-11)-9}$$

$\sqrt{?} \to ? \geq 0$

$\sqrt{x+y-1} \to x+y-1 \geq 0$

$x+y \geq 1$

$\ln(?) \to ? > 0$

$\ln(y-11) \to y-11 > 0$

$y > 11$

$\frac{1}{?} \to ? \neq 0$

$\ln(y-11)-9 \neq 0$

$\ln(y-11) \neq 9$

$y-11 \neq e^9$

$y \neq e^9+11$

Domain = $\boxed{\{(x,y) \mid x+y \geq 1,\ y > 11,\ y \neq 11+e^9\}}$

46. Find the domain of

$$f(x,y) = \frac{\ln(x^2-y+3)}{\sqrt{x-6}}$$

$\ln(?) \to ? > 0$

$\ln(x^2-y+3) \to x^2-y+3 > 0$

$x^2+3 > y$

$\frac{1}{\sqrt{?}} \to ? > 0$

$\frac{1}{\sqrt{x-6}} \to x-6 > 0$

$x > 6$

Domain = $\boxed{\{(x,y) \mid x > 6,\ x^2+3 > y\}}$

47. Describe the indicated level curves $f(x, y) = C$

$$f(x, y) = \ln(x^2 + y^2) \;\; C = \ln(36)$$

$\ln(x^2+y^2)=\ln(36)$
$x^2+y^2=36$
$x^2+y^2=6^2$

(a) Parabola with vertices at $(0, 0)$
(b) Circle with center at $(0, \ln(36))$ and radius 6
(c) Parabola with vertices at $(0, \ln(36))$
(d) Circle with center at $(0, 0)$ and radius 6
(e) Increasing Logarithm Function

48. What do the level curves for the following function look like?

$$f(x, y) = \ln(y - e^{5x})$$

$\ln(y-e^{5x})=C$
$y-e^{5x}=e^C$
$y-e^{5x}=C$
$y=e^{5x}+C$

(a) Increasing exponential functions
(b) Rational Functions with x-axis symmetry
(c) Natural logarithm functions
(d) Decreasing exponential functions
(e) Rational Functions with y-axis symmetry

49. What do the level curves for the following function look like?

$$f(x, y) = \sqrt{x^2 + y^2}$$

$\sqrt{x^2+y^2}=C$
$x^2+y^2=C^2$

(a) Lines
(b) Parabolas
(c) Circles
(d) Point at the origin
(e) Ellipses
(f) Hyperbolas

50. What do the level curves for the following function look like?

$$f(x, y) = \cos(y + 4x^2)$$

$\cos(y+4x^2)=C$
$y+4x^2=\cos^{-1}(C)$
$y+4x^2=C$
$y=-4x^2+C$

(a) Lines
(b) Parabolas
(c) Circles
(d) Point at the origin
(e) Ellipses
(f) Hyperbolas

51. For the following function $f(x, y)$, evaluate $f_y(-2, -3)$.

$$f(x, y) = 8x^4y^5 + 3x^3 - 12y^2$$

$$f_y(x,y) = \frac{d}{dy}\left(8x^4y^5 + 3x^3 - 12y^2\right)$$

$$= 8x^4 \frac{d}{dy}(y^5) + 3x^3 \frac{d}{dy}(1) - \frac{d}{dy}(12y^2)$$

$$= (8x^4)(5y^4) + (3x^3)(0) - 24y$$

$$= 40x^4y^4 - 24y$$

$$f_y(-2,-3) = 40(-2)^4(-3)^4 - 24(-3)$$

$$= 51912$$

$f_y(-2, -3) =$ $\boxed{51912}$

52. Compute $f_x(6, 5)$ when

$$f(x, y) = \frac{(6x - 6y)^2}{\sqrt{y^2 - 1}}$$

$$f_x(x,y) = \frac{d}{dx}\left(\frac{(6x-6y)^2}{\sqrt{y^2-1}}\right)$$

$$= \frac{1}{\sqrt{y^2-1}} \frac{d}{dx}\left((6x-6y)^2\right)$$

$$= \frac{1}{\sqrt{y^2-1}} \cdot 2(6x-6y) \frac{d}{dx}(6x+6y)$$

$$= \frac{1}{\sqrt{y^2-1}} \cdot 2(6x-6y) \cdot 6$$

$$= \frac{72x - 72y}{\sqrt{y^2-1}}$$

$f_x(6, 5) =$ $\boxed{72/\sqrt{24}}$

53. Find the first order partial derivatives of

$$f(x,y)=\frac{3x^2y^3}{(y-1)^2}$$

$$f(x,y)=3x^2\cdot\frac{y^3}{(y-1)^2}$$

$$f_x(x,y)=\frac{d}{dx}\left(3x^2\cdot\frac{y^3}{(y-1)^2}\right)=\frac{y^3}{(y-1)^2}\cdot\frac{d}{dx}\left(3x^2\right)=\frac{y^3}{(y-1)^2}\cdot 6x$$

$$f_y(x,y)=\frac{d}{dy}\left(3x^2\cdot\frac{y^3}{(y-1)^2}\right)=3x^2\frac{d}{dy}\left(\frac{y^3}{(y-1)^2}\right)=3x^2\left(\frac{3y^2(y-1)^2-y^3\cdot 2(y-1)}{(y-1)^4}\right)$$

$$=3x^2\left(\frac{\cancel{(y-1)}\left[3y^2(y-1)-2y^3\right]}{(y-1)^{\cancel{4}\,3}}\right)=\frac{3x^2(3y^3-3y^2-2y^3)}{(y-1)^3}$$

$$=\frac{3x^2(y^3-3y^2)}{(y-1)^3}$$

$f_x(x,y)=$ $\boxed{6xy^3/(y-1)^2}$

$f_y(x,y)=$ $\boxed{\dfrac{3x^2(y^3-3y^2)}{(y-1)^3}}$

54. Find the first order partial derivatives of

$$f(x,y)=x\sin(xy)$$

$$f_x(x,y)=\frac{d}{dx}(x\sin(xy))=\frac{d}{dx}(x)\sin(xy)+x\frac{d}{dx}(\sin(xy))$$

$$=\sin(xy)+x\cos(xy)\frac{d}{dx}(xy)$$

$$=\sin(xy)+x\cdot y\cos(xy)$$

$$f_y(x,y)=\frac{d}{dy}(x\sin(xy))=x\frac{d}{dy}(\sin(xy))$$

$$=x\cos(xy)\frac{d}{dy}(xy)$$

$$=x^2\cos(xy)$$

$f_x(x,y)=$ $\boxed{\sin(xy)+xy\cos(xy)}$

$f_y(x,y)=$ $\boxed{x^2\cos(xy)}$

55. Find the first order partial derivatives of $f(x, y) = (xy - 1)^2$

$$f_x(x,y) = \frac{d}{dx}\left((xy-1)^2\right) = 2(xy-1)\frac{d}{dx}(xy-1)$$
$$= 2(xy-1)\,y$$
$$= 2xy^2 - 2y$$

$$f_y(x,y) = \frac{d}{dy}\left((xy-1)^2\right) = 2(xy-1)\frac{d}{dy}(xy-1)$$
$$= 2(xy-1)\,x$$
$$= 2x^2y - 2x$$

$f_x(x, y) =$ $\boxed{2xy^2 - 2y}$

$f_y(x, y) =$ $\boxed{2x^2y - 2x}$

56. Find the first order partial derivatives of $f(x, y) = xe^{x^2+xy+y^2}$

$$f_x(x,y) = \frac{d}{dx}(x)\,e^{x^2+xy+y^2} + x\,\frac{d}{dx}\left(e^{x^2+xy+y^2}\right)$$
$$= e^{x^2+xy+y^2} + x\left(e^{x^2+xy+y^2}\right)(2x+y)$$
$$= (1+2x^2+xy)e^{x^2+xy+y^2}$$

$$f_y(x,y) = x\,\frac{d}{dy}\left(e^{x^2+xy+y^2}\right) = x\left(e^{x^2+xy+y^2}\right)(x+2y)$$
$$= (x^2+2xy)e^{x^2+xy+y^2}$$

$f_x(x, y) =$ $\boxed{(1+2x^2+xy)e^{x^2+xy+y^2}}$

$f_y(x, y) =$ $\boxed{(x^2+2xy)e^{x^2+xy+y^2}}$

57. Find the first order partial derivatives of $f(x, y) = -7\tan(x^7y^8)$

$$f_x(x,y) = -7\frac{d}{dx}\left(\tan(x^7y^8)\right) = -7\sec^2(x^7y^8)\frac{d}{dx}(x^7y^8)$$
$$= -7\cdot 7x^6y^8\sec^2(x^7y^8) = -49x^6y^8\sec^2(x^7y^8)$$

$$f_y(x,y) = -7\frac{d}{dy}\left(\tan(x^7y^8)\right) = -7\sec^2(x^7y^8)\frac{d}{dy}(x^7y^8)$$
$$= -7\cdot 8x^7y^7\sec^2(x^7y^8)$$
$$= -56x^7y^7\sec^2(x^7y^8)$$

$f_x(x, y) =$ $\boxed{-49x^6y^8\sec^2(x^7y^8)}$

$f_y(x, y) =$ $\boxed{-56x^7y^7\sec^2(x^7y^8)}$

58. Find the first order partial derivatives of $f(x, y) = y\cos(x^2y)$

$$f_x(x,y) = y\frac{d}{dx}\left(\cos(x^2y)\right) = y(-\sin(x^2y))\frac{d}{dx}(x^2y) = -y\sin(x^2y)[2xy]$$
$$= -2xy^2\sin(x^2y)$$

$$f_y(x,y) = \frac{d}{dy}(y)\cos(x^2y) + y\frac{d}{dy}\left(\cos(x^2y)\right)$$
$$= \cos(x^2y) + y(-\sin(x^2y))\frac{d}{dy}(x^2y)$$
$$= \cos(x^2y) - y\sin(x^2y)[x^2]$$
$$= \cos(x^2y) - x^2y\sin(x^2y)$$

$f_x(x, y) =$ $\boxed{-2xy^2\sin(x^2y)}$

$f_y(x, y) =$ $\boxed{\cos(x^2y) - x^2y\sin(x^2y)}$