Lesson 10: Partial Fractions II

Last class, we introduced the method of decomposing a fraction into partial fraction, and how to integrate them

In today's lecture, we will be focussing on cases ⓑ - ⓓ as outlined on the Handout: Method of Decomposing Into Partial Fractions

Example 1: Let $f(x) = \dfrac{4x^2-4}{x^3-2x^2}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^3-2x^2 completely.

$$x^3-2x^2 = x^2(x-2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

③ Combine the fractions in ②.

Note the common denominator is $x^2(x-2)$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$

$$= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)}$$

$$= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^3-2x^2}$$

④ Set the old numerator = new numerator

$$4x^2 - 4 = (A+C)x^2 + (B-2A)x - 2B$$

$$4x^2 + 0x - 4 = (A+C)x^2 + (B-2A)x - 2B$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A + C = 4 & \text{(i)} \\ B - 2A = 0 & \text{(ii)} \\ -2B = -4 & \text{(iii)} \end{cases}$$

From (iii) $-2B = -4 \Rightarrow B = 2$

Plug $B=2$ into (ii)

$$B - 2A = 0$$
$$2 - 2A = 0$$
$$2 = 2A \Rightarrow A = 1$$

Plug $A=1$ into (i).

$$A + C = 4$$
$$1 + C = 4$$
$$C = 3$$

⑥ Plug $A=1$, $B=2$, $C=3$ into ②.

$$\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x-2}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int\frac{4x^2-4}{x^3-2x^2}dx=\int\frac{1}{x}dx+\int\frac{2}{x^2}dx+\int\frac{3}{x-2}dx$$

$$=\int\frac{1}{x}dx+\int 2x^{-2}dx+\int\frac{3}{x-2}dx$$

$$=\ln|x|+\frac{2x^{-1}}{-1}+3\ln|x-2|+C$$

$$=\ln|x|-\frac{2}{x}+3\ln|x-2|+C$$

Example 2: Let $f(x)=\dfrac{5x^2+9}{x^3+3x}$

ⓐ Determine the partial decomposition of $f(x)$.

① Factor x^3+3x completely.

$$x^3+3x=x(x^2+3)$$

② Write the fraction into decomposition form.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

③ Combine the fractions in ②.

$$\frac{A}{x}+\frac{Bx+C}{x^2+3}=\frac{A(x^2+3)+(Bx+C)x}{x(x^2+3)}$$

$$=\frac{Ax^2+3A+Bx^2+Cx}{x^3+3x}$$

$$=\frac{(A+B)x^2+Cx+3A}{x^3+3x}$$

④ Set the old numerator = new numerator

$$5x^2+9=(A+B)x^2+Cx+3A$$

$$5x^2+0x+9=(A+B)x^2+Cx+3A$$

⑤ Create a system of equations from ④ and solve.

$$\begin{cases}5=A+B & (i)\\ 0=C & (ii)\\ 9=3A & (iii)\end{cases}$$

Note (ii) already gives us $C=0$.
From (iii) $9=3A \Rightarrow A=3$

Plug $A=3$ into (i).
$5=A+B$
$5=3+B$
$2=B$

⑥ Plug $A=3$, $B=2$, $C=0$ into ②.

$$\frac{3}{x}+\frac{2x+0}{x^2+3}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{5x^2+9}{x^3+3x}dx = \int \frac{3}{x}dx + \underbrace{\int \frac{2x}{x^2+3}dx}_{u\text{-sub}}$$

$u = x^2+3$
$du = 2x\,dx$

$$= \int \frac{3}{x}dx + \int \frac{du}{u}$$
$$= 3\ln|x| + \ln|u| + C$$
$$= 3\ln|x| + \ln|x^2+3| + C$$

Example 3: Let $f(x) = \dfrac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

ⓐ Determine the partial fraction decomposition of $f(x)$.
① Factor $(x^2+x+1)^2$ completely.
Done!
② Write the fraction into decomposition form.

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2}$$

③ Combine the fractions in ②.
Note the common denominator is $(x^2+x+1)^2$

$$\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{(x^2+x+1)^2} = \frac{(Ax+B)(x^2+x+1)+Cx+D}{(x^2+x+1)^2}$$
$$= \frac{Ax^3+Bx^2+Ax^2+Bx+Ax+B+Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)}{(x^2+x+1)^2}$$

④ Set the old numerator = new numerator

$$2x^3 + 3x^2 + 5x + 2 = Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A = 2 & \text{(i)} \\ A+B = 3 & \text{(ii)} \\ A+B+C = 5 & \text{(iii)} \\ B+D = 2 & \text{(iv)} \end{cases}$$

From (i), we have $A=2$.

Plug $A=2$ into (ii)

$A+B=3$

$2+B=3$

$B=1$

Plug $A=2$, $B=1$ into (iii)

$A+B+C=5$

$2+1+C=5$

$3+C=5$

$C=2$

Plug $B=1$ into (iv)

$B+D=2$

$1+D=2$

$D=1$

⑥ Plug $A=2$, $B=1$, $C=2$, $D=1$ into ②.

$$\frac{2x+1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int \frac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{2x+1}{(x^2+x+1)^2}$$

Note both integrals have a u-sub.
This time only it's the same u.

$$\frac{u = x^2+x+1}{du = (2x+1)dx} \quad \int \frac{du}{u} + \int \frac{du}{u^2} = \int \frac{du}{u} + \int u^{-2}\,du$$

$$= \ln|u| - u^{-1} + C = \ln|u| - \frac{1}{u} + C$$

$$= \ln|x^2+x+1| - \frac{1}{x^2+x+1} + C$$