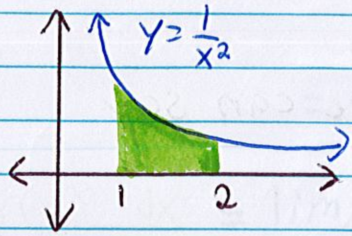
Lesson 11: Improper Integrals

Suppose we want to find the area under the curve $y = \frac{1}{x^2}$ with bounds $x = 1$ and $x = 2$.



$$A = \int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

How about I change the bounds to be $x = 1$ and $x = 3$?

$$A = \int_1^3 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big]_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$$

How about I change the bounds to be $x = 1$ and $x = 4$?

$$A = \int_1^4 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

So note A gets closer and closer to 1 as the top bound gets larger. So we can say

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

This integral on the left is what we can **improper integral**.

Definition: An improper integral is a definite integral $\int_a^b f(x)\,dx$ such that the integrand $f(x)$ is defined on (a,b) but not necessarily at a or b.

ex. $\int_a^\infty f(x)dx$; $\int_{-\infty}^b f(x)dx$; $\int_{-\infty}^\infty f(x)dx$

To compute such integrals, we need to bring back the notion of limits.

Definition: If $f(x)$ approaches L as x approaches c, we say that the limit of $f(x)$ as x approaches c is L.

i.e. $\lim\limits_{x \to c} f(x) = L$.

Now with the definition of limits, we can say

$$\int_a^\infty f(x)\,dx = \lim_{N \to \infty} \int_a^N f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{N \to -\infty} \int_N^b f(x)\,dx$$

If the value of an improper integral is finite #, we say that the integral converges, and if not the integral diverges.

Example 1: Evaluate

ⓐ $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N \to \infty} \int_1^N \frac{1}{x^2}\,dx = \lim_{N \to \infty} \left(\frac{-1}{x} \right]_1^N \Bigg)$$

$$= \lim_{N \to \infty} \left(\frac{-1}{N} + 1 \right) = 1$$

ⓑ $$\int_0^\infty 5xe^{-x}\,dx = \lim_{N \to \infty} \int_0^N 5xe^{-x}\,dx \qquad \begin{array}{ll} u = 5x & dv = e^{-x}\,dx \\ \hline du = 5\,dx & v = -e^{-x} \end{array}$$

$$= \lim_{N \to \infty} \left(-5xe^{-x} \Big]_0^N - \int_0^N 5(-e^{-x})\,dx \right)$$

$$= \lim_{N \to \infty} \left(-5xe^{-x} \Big]_0^N + \int_0^N 5e^{-x}\,dx \right)$$

$$= \lim_{N \to \infty} \left(-5xe^{-x} \Big]_0^N - 5e^{-x} \Big]_0^N \right)$$

$$= \lim_{N \to \infty} \left(-5Ne^{-N} + 0 - 5e^{-N} + 5 \right)$$

$$= \lim_{N \to \infty} \left(\underbrace{\frac{-5N}{e^N}}_{0} \underbrace{- 5e^{-N}}_{0} + 5 \right) = 5$$

ⓒ $\int_2^\infty \frac{dx}{x\sqrt{\ln x}} = \lim_{N\to\infty} \int_2^N \frac{dx}{x\sqrt{\ln x}}$ $\quad \frac{u=\ln x}{du=\frac{1}{x}dx}$ $\quad \lim_{N\to\infty} \int \frac{du}{u^{1/2}}$

$= \lim_{N\to\infty} \int u^{-1/2}\,du = \lim_{N\to\infty} \frac{2}{1} u^{1/2}$

$= \lim_{N\to\infty} \left(2\sqrt{\ln x} \Big]_2^N \right) = \lim_{N\to\infty} \left(2\sqrt{\ln N} - 2\sqrt{\ln 2} \right)$

$= \infty \Rightarrow$ diverges

ⓓ $\int_2^\infty \frac{dx}{x} = \lim_{N\to\infty} \int_2^N \frac{dx}{x} = \lim_{N\to\infty} \left(\ln|x| \Big]_2^N \right)$

$= \lim_{N\to\infty} \left(\ln|N| - \ln(2) \right)$

$= \infty \Rightarrow$ diverges

Now there is one more integral that is also considered improper.

- If f is continuous on $[a,b)$ and discontinuous on b, then

$$\int_a^b f(x)\,dx = \lim_{N\to b^-} \int_a^N f(x)\,dx$$

- If f is continuous on $(a,b]$ and discontinuous on a, then

$$\int_a^b f(x)\,dx = \lim_{N\to a^+} \int_N^b f(x)\,dx$$

Example 2: Evaluate

ⓐ $\int_0^{\pi/2} \tan x\,dx$ $\quad$ Note that $\tan(x)$ @ $x=\pi/2$ is undefined.

$= \lim_{N\to \pi/2^-} \int_0^N \frac{\sin x}{\cos x}\,dx$ $\quad \frac{u=\cos x}{du=-\sin x}$

$= \lim_{N\to \pi/2^-} \int -\frac{du}{u} = \lim_{N\to\pi/2^-} \left(-\ln|u| \right)$

$= \lim_{N\to \pi/2^-} \left(-\ln|\cos x| \Big]_0^N \right) = \lim_{N\to\pi/2^-} \left(-\ln|\cos N| + \ln|\cos(0)| \right)$.

$$= \lim_{N \to \pi/2^-} \left(-\ln|\cos(x)| + \ln|1| \right)$$

$$= -\ln\left|\cos\left(\frac{\pi}{2}\right)\right| + 0 = -\ln|0| = -\infty \Rightarrow \text{diverges}$$

(b) $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$ Note that $\dfrac{1}{\sqrt[4]{x+2}}$ is undefined at $x=-2$

$$= \lim_{N \to -2^+} \int_N^{14} (x+2)^{-1/4}\,dx = \lim_{N \to -2^+} \left(\frac{4}{3}(x+2)^{3/4} \Big]_N^{14} \right)$$

$$= \lim_{N \to -2^+} \left(\frac{4}{3}(14+2)^{3/4} - \frac{4}{3}(N+2)^{3/4} \right)$$

$$= \lim_{N \to -2^+} \left(\frac{32}{3} - \frac{4}{3}(N+2)^{3/4} \right) = \frac{32}{3} - \frac{4}{3}(0)^{3/4} = \frac{32}{3}$$