

Reminders

- NEXT WEDNESDAY QUIZ 6 on
 - Volume of Revolutions
 - Disks (Lesson 14)
 - Washers (Lesson 15)

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MA 16020: Lesson 15 Volume By Revolution Washer Method

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Last Time, we talked about...

- How Geometry gave us formulas for simple shapes and solids to find their area or volume, and
- How Integration can allow us to find area or volume of **ANYTHING!**
How?
 - We introduced this notion of cross-sections which can be of the form of
 - Disks (Lesson 14), or
 - Washers (Lesson 15)

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Geometry: How to Calculate The Area of a Shaded Region

Suppose we are asked to find the area of a rectangle with a triangle missing from the middle.

How do we calculate that area?



4

Geometry: How to Calculate The Area of a Shaded Region

First, we would find the area of the rectangle and the area of the triangle separately.



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Geometry: How to Calculate The Area of a Shaded Region

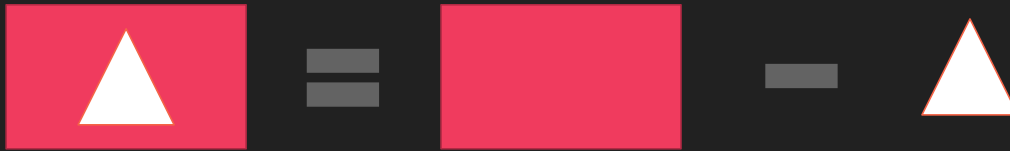
Then we would subtract these two values ...



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Geometry: How to Calculate The Area of a Shaded Region

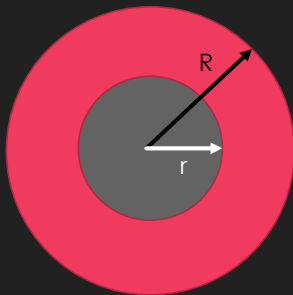
... to find the remaining area.



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What if we did this with disks?

Let's find the area of the red annulus.



The area of the red circle is πR^2 , and the area of the gray circle is πr^2 .

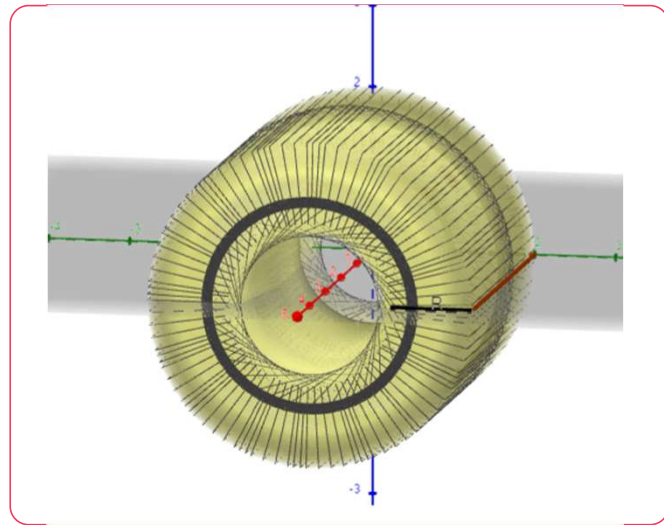
So if we subtract the two, we get

$$\pi R^2 - \pi r^2 = \pi (R^2 - r^2)$$

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Today's Lecture

- In this lesson, we are going to play with disks, but remove a portion of it.
- This method is called the washer method.



<https://www.geogebra.org/m/uym6dwyd>

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Washer Method Formula

Since we are just cutting out the middle of the solid, we choose dx or dy in the same way as the disk method.

- Rotating around x-axis \Rightarrow " dx " problem
- Rotating around y-axis \Rightarrow " dy " problem

$$V = \pi \int_a^b (R^2 - r^2) dx$$

where a and b are bounds of the region we are rotating.

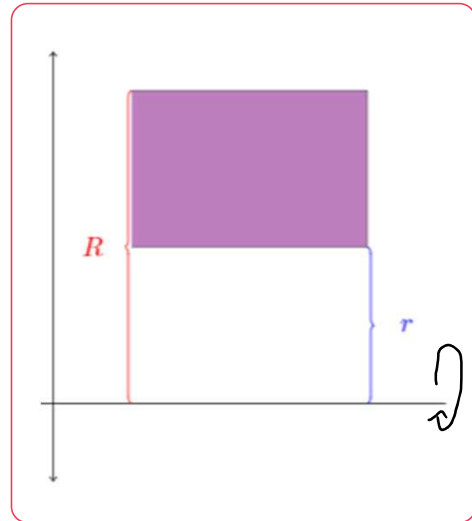
- R is the **farthest** from the axis rotation
- r is the **closest**

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Let's talk a bit more about R and r

- Recall Lessons 12+13 which were about finding the area between 2 curves.
- The same principle applies here.
- For rotation around the x-axis,
 - R is the "Top" Function
 - r is the "Bottom" Function
- Just remember the formula is

$$V = \pi \int_a^b (R^2 - r^2) dx$$

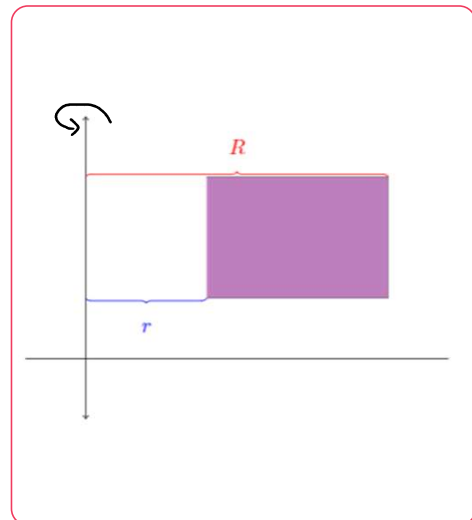


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Let's talk a bit more about R and r

- For rotation around the y-axis,
 - R is the "Right" Function
 - r is the "Left" Function
- Just remember the formula is

$$V = \pi \int_c^d (R^2 - r^2) dy$$



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How to Proceed with Washer Problems

1. Draw the region
2. Determine which axis you are rotating on
 - a. If x - axis: Determine Top and Bottom Function
 - i. R is Top
 - ii. r is Bottom
 - b. If y - axis: Determine Right and Left Function
 - i. R is Right
 - ii. r is Left
3. Finally, apply the washer formula

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Examples

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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \quad y = 3x, \quad \text{and} \quad x = 2$$

About the x-axis.

<https://www.geogebra.org/m/m2p2kdmp>

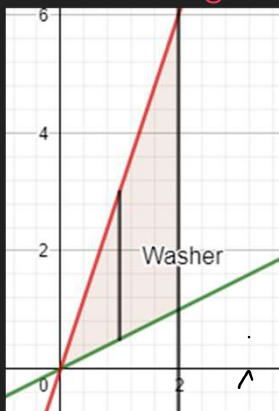
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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \quad y = 3x, \quad \text{and} \quad x = 2$$

About the x-axis.

First draw the region.



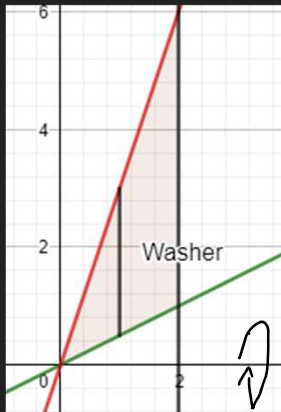
<https://www.geogebra.org/m/m2p2kdmp>

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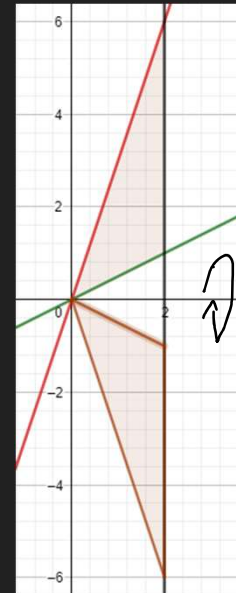
Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \quad y = 3x, \quad \text{and} \quad x = 2$$

About the x-axis.



Rotation about x-axis



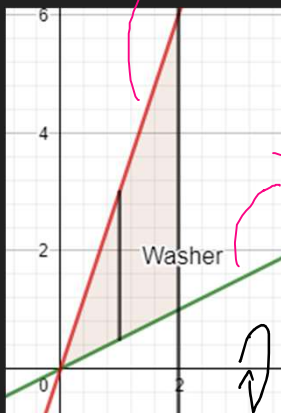
<https://www.geogebra.org/m/m2p2kdmp>

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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \quad y = 3x, \quad \text{and} \quad x = 2$$

About the x-axis.



$y = 3x \rightarrow \text{Top}$

$y = \frac{x}{2} \rightarrow \text{Bottom}$

$$V = \pi \int_0^2 (3x)^2 - \left(\frac{x}{2}\right)^2 dx$$

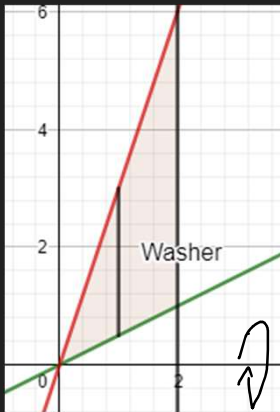
<https://www.geogebra.org/m/m2p2kdmp>

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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \quad y = 3x, \quad \text{and} \quad x = 2$$

About the x-axis.



$$\begin{aligned} V &= \pi \int_0^2 \left(9x^2 - \frac{x^2}{4} \right) dx \\ &= \pi \int_0^2 \frac{35}{4} x^2 dx \\ &= \frac{35\pi}{4} \left[\frac{x^3}{3} \right]_0^2 = \frac{70\pi}{3} \end{aligned}$$

<https://www.geogebra.org/m/m2p2kdmp>

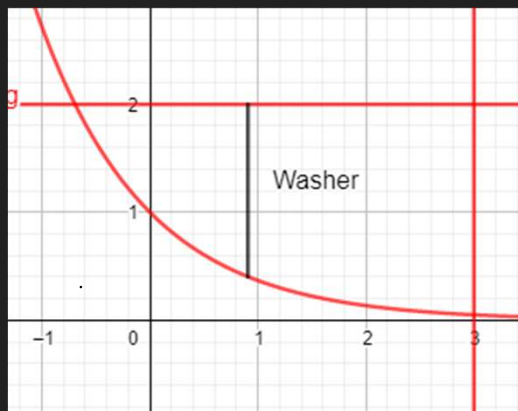
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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \quad \text{and} \quad x = 3$$

About the x-axis.

First draw the region.



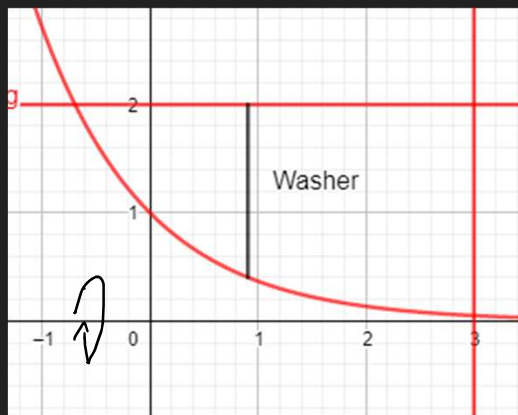
<https://www.geogebra.org/m/jfta4b52>

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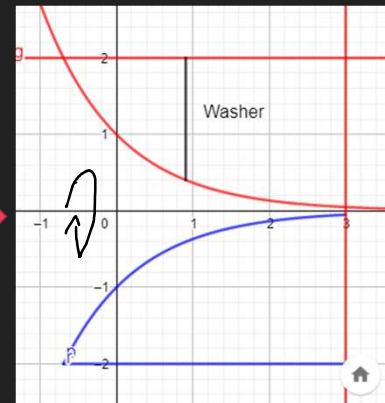
Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \quad \text{and} \quad x = 3$$

About the x-axis.



Rotation about x-axis



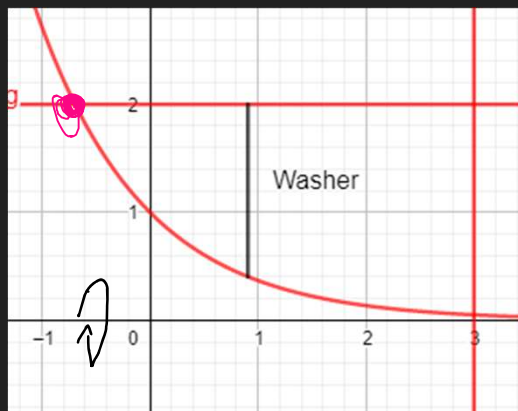
<https://www.geogebra.org/m/jfta4b52>

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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \quad \text{and} \quad x = 3$$

About the x-axis.



<https://www.geogebra.org/m/jfta4b52>

See $x=3$ is one bound, we need to find the other

$$2 = e^{-x}$$

$$\ln 2 = -x$$

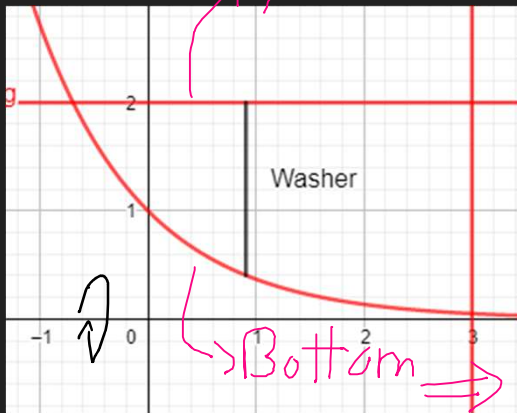
$$-\ln 2 = x$$

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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \quad \text{and} \quad x = 3$$

About the x-axis.



So $y = 2 \rightarrow$ Top

$$V = \pi \int_{-\ln 2}^3 (2^2 - (e^{-x})^2) dx$$

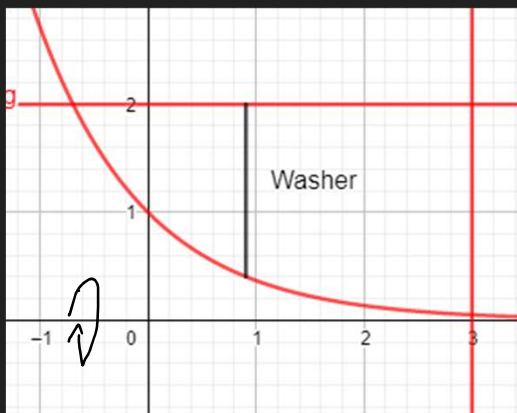
<https://www.geogebra.org/m/jfta4b52>

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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \quad \text{and} \quad x = 3$$

About the x-axis.



$$V = \pi \int_{-\ln 2}^3 (4 - e^{-2x}) dx$$

$$= \pi \left(4x - \frac{e^{-2x}}{-2} \right) \Big|_{-\ln 2}^3$$

$$= \pi \left(4x + \frac{e^{-2x}}{2} \right) \Big|_{-\ln 2}^3$$

$$\approx 40.13$$

<https://www.geogebra.org/m/jfta4b52>

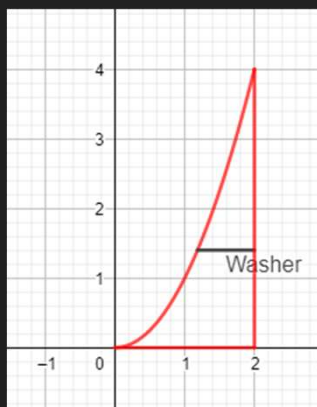
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Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \quad \text{and} \quad y = 0$$

About the y-axis.

First draw the region.



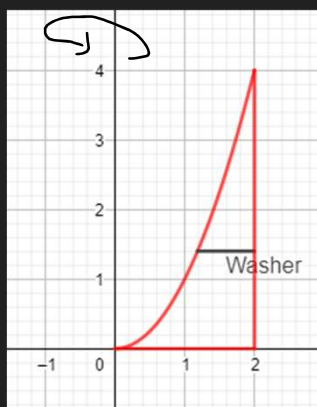
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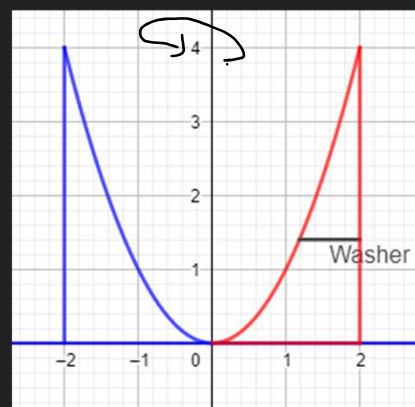
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \quad \text{and} \quad y = 0$$

About the y-axis.



Rotation about y-axis



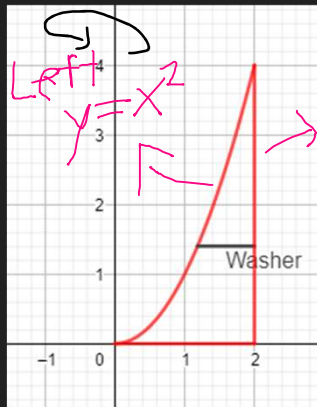
<https://www.geogebra.org/m/znzmqhqq7>

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Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \quad \text{and} \quad y = 0$$

About the y-axis.



Rewrite $y = x^2$ to be $x = \sqrt{y}$

$x = 2$
→ Right

Note no $-$ b/c we are in the first quadrant.

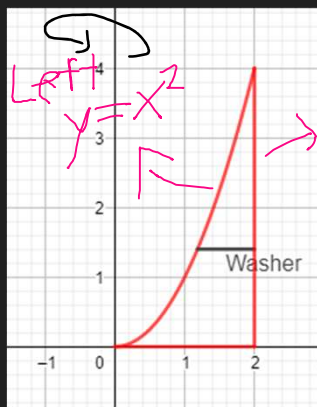
<https://www.geogebra.org/m/znzmqhqq7>

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Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \quad \text{and} \quad y = 0$$

About the y-axis.



$$\text{So } V = \pi \int (2)^2 - (\sqrt{y})^2 dy$$

$$= \pi \int (4 - y) dy$$

From the graph, we see $y = 0$ to $y = 4$.

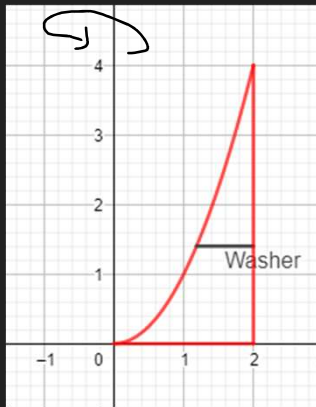
<https://www.geogebra.org/m/znzmqhqq7>

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Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \quad \text{and} \quad y = 0$$

About the y-axis.



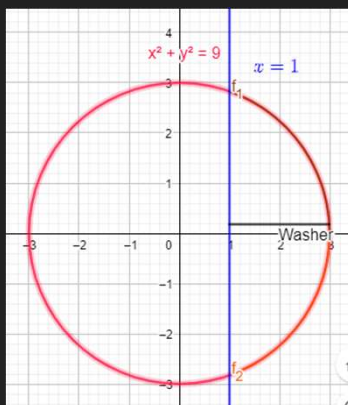
$$\begin{aligned} V &= \pi \int_0^4 (4 - \sqrt{y}) dy \\ &= \pi \left(4y - \frac{\sqrt{y}}{2} \right) \Big|_0^4 \\ &= 8\pi \end{aligned}$$

<https://www.geogebra.org/m/znzmqhqq7>

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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

First draw the region.

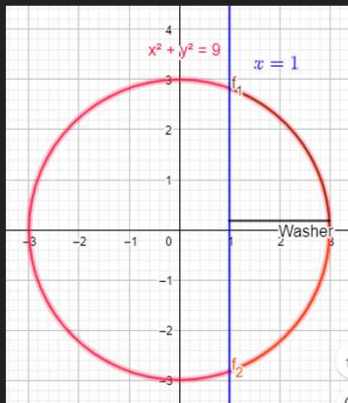


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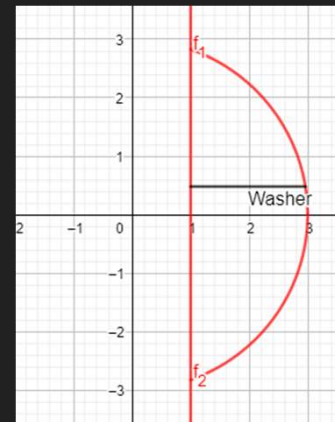
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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y -axis.

First draw the region.



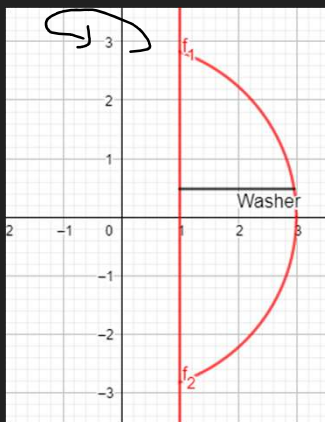
Note we are only looking at the right of $x=1$



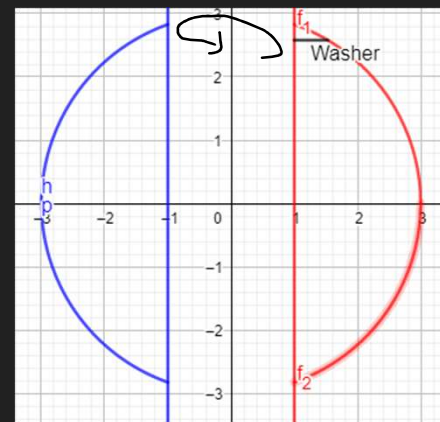
<https://www.geogebra.org/m/c2wzbrbf>

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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y -axis.



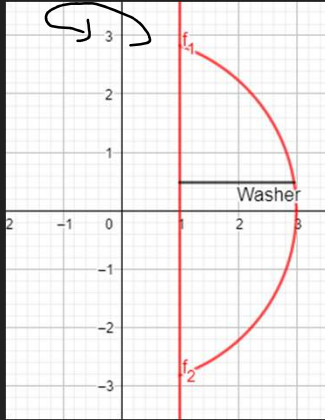
Rotation about y -axis



<https://www.geogebra.org/m/c2wzbrbf>

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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y -axis.



Rewrite $x^2 + y^2 = 9$ to be $x =$

$$x^2 = 9 - y^2$$

$$x = \sqrt{9 - y^2}$$

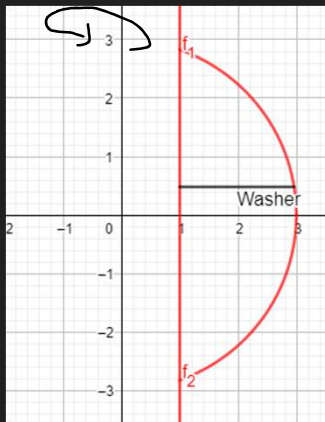
Note no - b/c we are looking at the right of $x = 1$.

<https://www.geogebra.org/m/c2wzbrbf>

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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y -axis.

To find the bounds, plug $x = 1$ into



$$x^2 + y^2 = 9$$

$$1^2 + y^2 = 9$$

$$1 + y^2 = 9$$

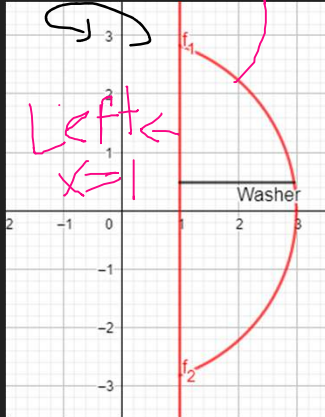
$$y^2 = 8$$

$$y = \pm \sqrt{8}$$

<https://www.geogebra.org/m/c2wzbrbf>

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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y -axis.



$$x = \sqrt{9 - y^2}$$

$$V = \pi \int_{-\sqrt{8}}^{\sqrt{8}} (\sqrt{9 - y^2})^2 - 1^2 dy$$

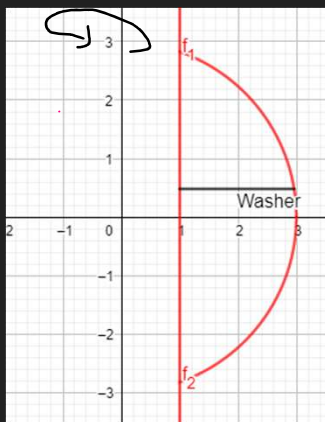
$$= \pi \int_{-\sqrt{8}}^{\sqrt{8}} (9 - y^2 - 1) dy$$

$$= \pi \int_{-\sqrt{8}}^{\sqrt{8}} (8 - y^2) dy$$

<https://www.geogebra.org/m/c2wzbrbf>

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Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y -axis.



$$V = \pi \left(8y - \frac{y^3}{3} \right) \Big|_{-\sqrt{8}}^{\sqrt{8}}$$

$$= \frac{64\sqrt{2}}{3}$$

<https://www.geogebra.org/m/c2wzbrbf>

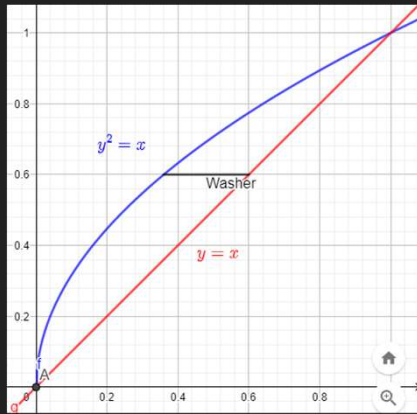
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Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

a) About the y-axis

First draw the region.



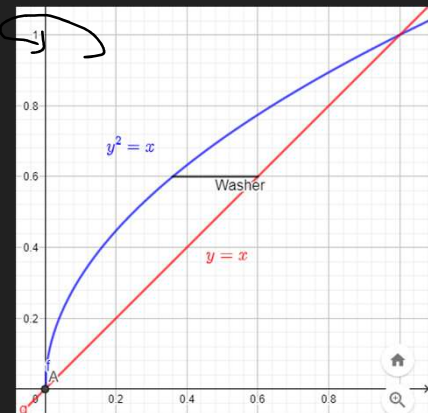
<https://www.geogebra.org/m/qtt49cqx>

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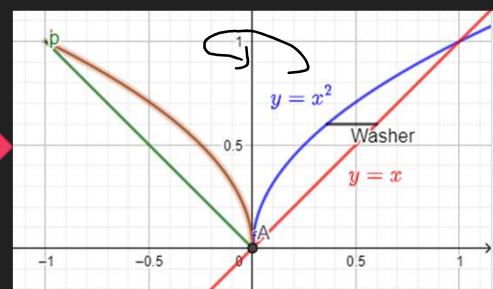
Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

a) About the y-axis



Rotation about y-axis



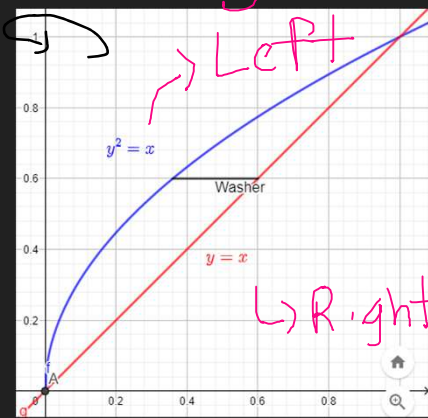
<https://www.geogebra.org/m/qtt49cqx>

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Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

a) About the y-axis



Using the graph, we have

$$V = \pi \int_0^1 (y)^2 - (y^2)^2 dy$$

$$= \pi \int_0^1 (y^2 - y^4) dy$$

$$= \pi \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \Big|_0^1 = \frac{2\pi}{15}$$

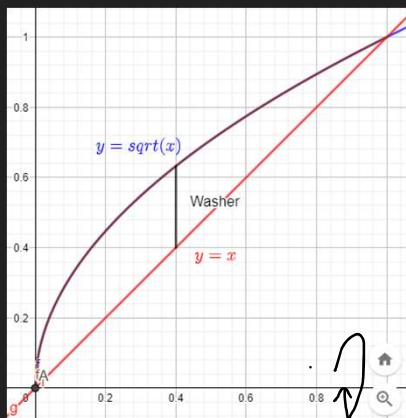
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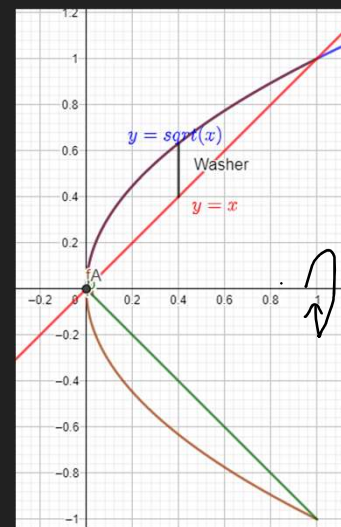
Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

b) About the x-axis



Rotation about x-axis



<https://www.geogebra.org/m/esmxtp7s>

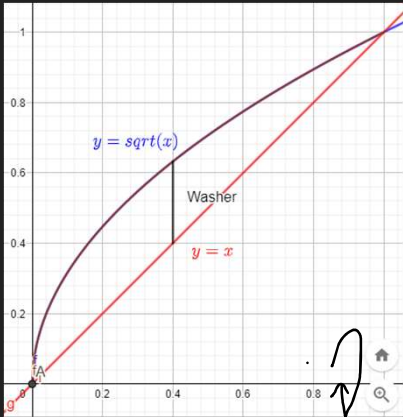
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Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

b) About the x-axis

Note we are using the same graph
Except we rewrite the
lines to be $x =$



Also we are now looking
@ Top & Bottom

<https://www.geogebra.org/m/esmxtp7s>

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Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

b) About the x-axis



$$\begin{aligned} V &= \pi \int_0^1 (\sqrt{x})^2 - (x)^2 dx \\ &= \pi \int_0^1 (x - x^2) dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\pi}{6} \end{aligned}$$

<https://www.geogebra.org/m/esmxtp7s>

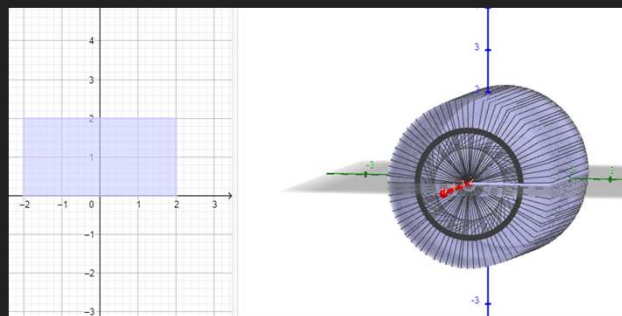
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RECAP: Disk vs. Washer Method

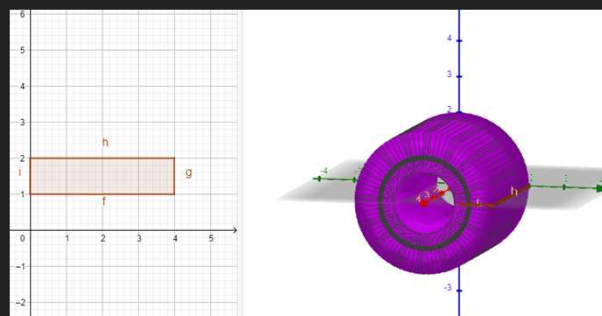
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When do apply Disk Method or Washer Method?

Disk Method



Washer Method



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When do we apply Disk Method or Washer Method?

- When the region “hugs” the axis of rotation
⇒ Disk Method
- When there is a “gap” between the region and axis of rotation
⇒ Washer Method

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GeoGebra Link for Lesson 15

- <https://www.geogebra.org/m/f73zjxfe>
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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