

## Reminders

- TODAY QUIZ 6 on
  - Volume of Revolutions
    - Disks (Lesson 14)
    - Washers (Lesson 15)
- 1-WEEK REMINDER
  - Exam 2 on WEDNESDAY March 1 @ 6:30pm – 7:30pm
  - Location: LILY 1105

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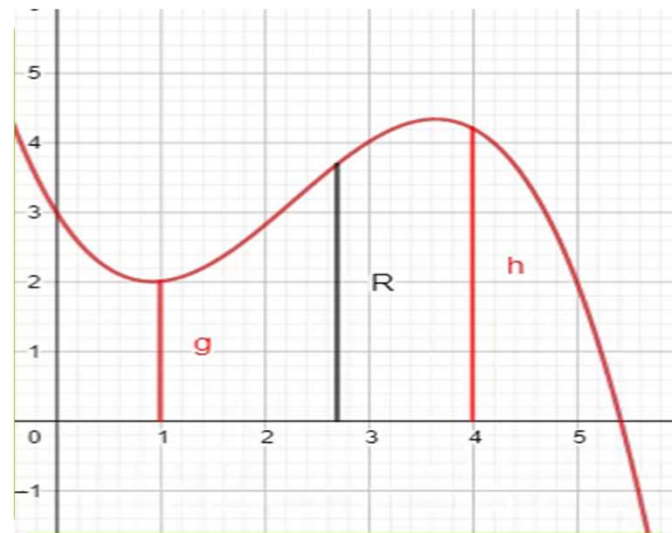
## MA 16020: Lesson 17 Volume By Revolution Shell Method

By Alexandra Cuadra

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## So far...

- We have learned how to find the volume of a solid of revolution by integrating
- In the same way, we calculate the area under a curve
- Running a line segment of varying length across the region, and adding them up

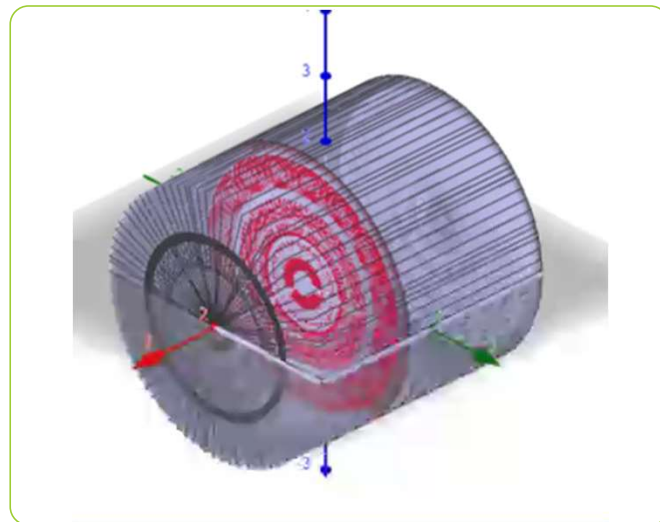


<https://www.geogebra.org/m/tgceabb2#material/tnnhu7gz>

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## In other words,

- We learned to find the volume of a solid of revolution by
- Running some area across a shape and add them up.
- Like in the case of the cylinder shown on the right.



<https://www.geogebra.org/m/tgceabb2#material/qcwutumt>

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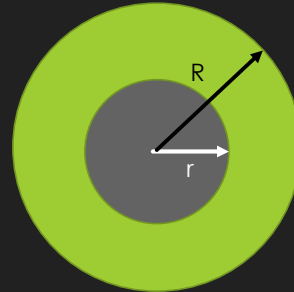
## But what were those “shapes”? ANSWER: CROSS-SECTIONS

Sometimes it was a disk



Whose area is  $\pi R^2$

Sometimes it was a washer



Whose area is  $\pi(R^2 - r^2)$

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In Today's Lecture, we will be covering the case, when neither method (Disk nor Washer) is easy.

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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

<https://www.geogebra.org/m/jqfyndpu>

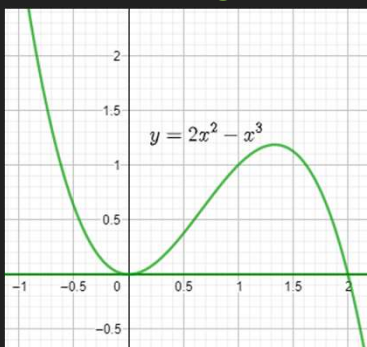
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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

Draw the region.



<https://www.geogebra.org/m/jqfyndpu>

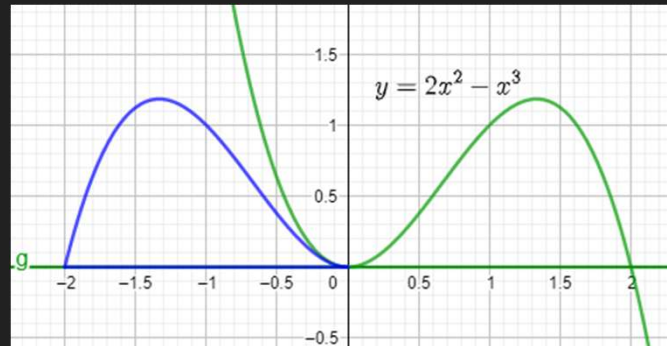
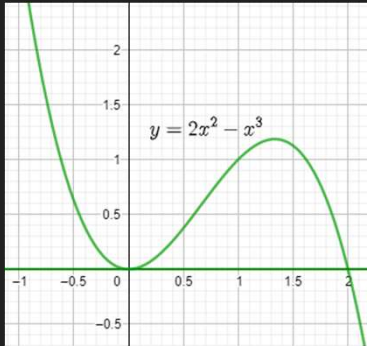
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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

Rotation about y-axis



<https://www.geogebra.org/m/jafyndpu>

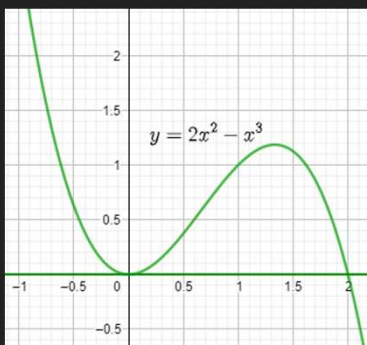
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Example 1: Find the volume obtained by revolving the region bounded by the curves

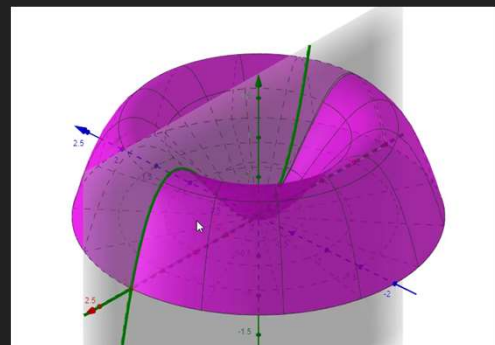
$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

WASHER ???



Furthermore, 3-D



<https://www.geogebra.org/m/jafyndpu>

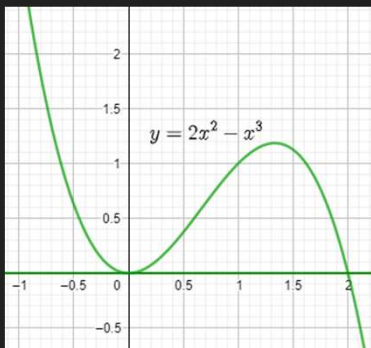
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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the  $y$ -axis.

Technically, yes. It is a Washer Problem.  
But there are two issues:



1. Given we are revolving around  $y$ -axis, we want to solve our equations for  $x$ .

i.e. Solve  $y = 2x^2 - x^3$  for  $x$ .

But that is easier said than done.

2. For washer problems, we need two equations for each radius.

Here we have both radius depend on the same function.

<https://www.geogebra.org/m/jafyndpu>

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So how can I do this kind of problem without giving myself a headache?

**ANSWER: SHELL METHOD**

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## GEOMETRY: Finding The Volume of A Hollow Cylinder

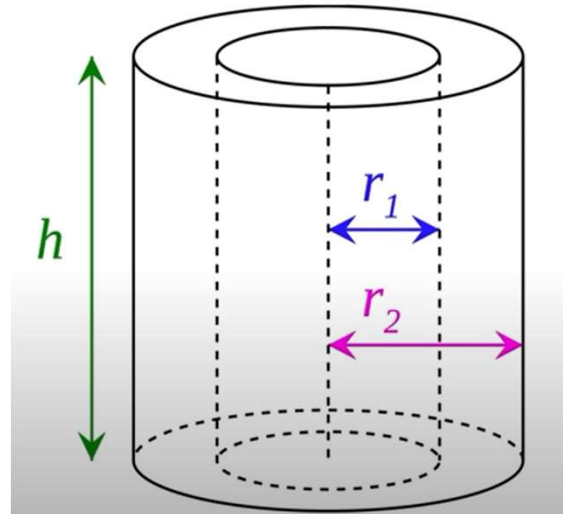
- To find the volume of this hollow cylinder, we used the same idea when washers were first introduced.

$$V_{total} = V_{outer} - V_{inner}$$

- Remember the volume of a cylinder is  $\pi r^2 h$ . So

$$V_{outer} = \pi(r_2)^2 h \quad \text{and} \quad V_{inner} = \pi(r_1)^2 h$$

- Hence 
$$\begin{aligned} V_{total} &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi h(r_2^2 - r_1^2) \\ &= \pi h(r_2 - r_1)(r_2 + r_1) \end{aligned}$$



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## GEOMETRY: Finding The Volume of A Hollow Cylinder

So let's be clever

- Let's take the sum  $r_2 + r_1$  and express it as an average.

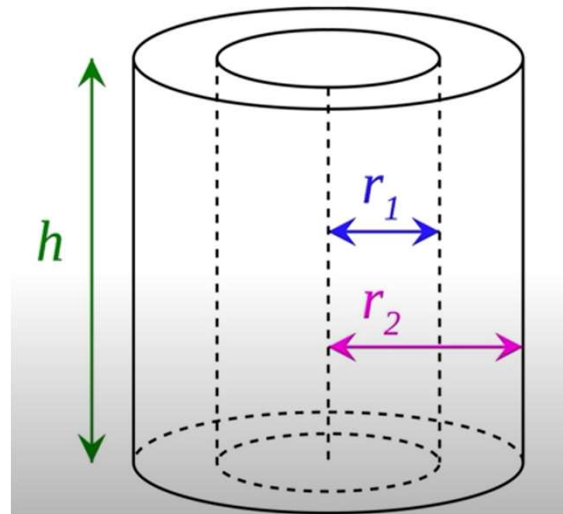
$$\text{i.e. } (r_1 + r_2)/2$$

- To do that multiple the equation below by  $2/2$ .

$$\begin{aligned} V_{total} &= \pi h(r_2 - r_1)(r_2 + r_1) \\ &= 2\pi h(r_2 - r_1) \left( \frac{r_2 + r_1}{2} \right) \end{aligned}$$

- Since we have the average radius in our equation, we can now call  $r = \frac{r_1 + r_2}{2}$ .

$$V_{total} = 2\pi h(r_2 - r_1)r$$



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## GEOMETRY: Finding The Volume of A Hollow Cylinder

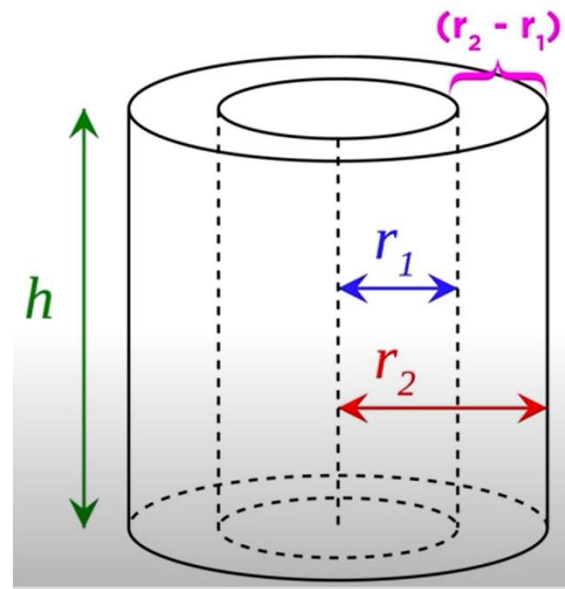
- Note that the difference of the radii gives us the thickness of the cylinder.

- Let  $\Delta r$  be that difference

$$\Delta r = r_2 - r_1$$

- Hence we can say that the volume of the hollow cylinder is

$$V_{total} = 2\pi r h \cdot \Delta r$$



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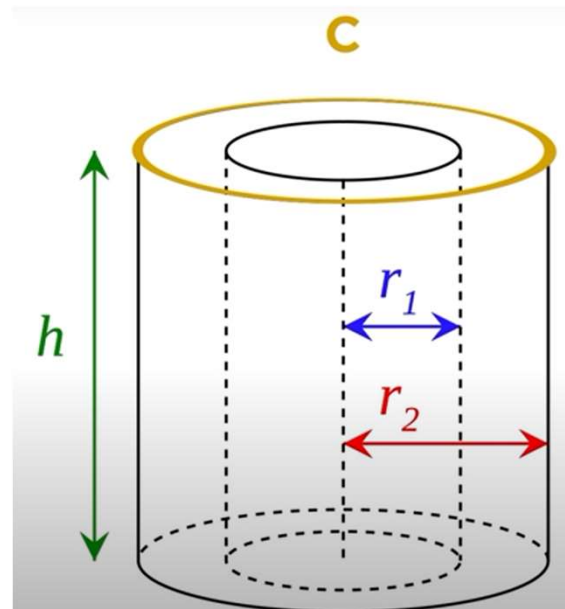
## GEOMETRY: Finding The Volume of A Hollow Cylinder

- One way to remember this

$$V_{total} = 2\pi r h \cdot \Delta r$$

is to see that  $2\pi r$  is the same as the circumference,  $C$ , (as shown in the image) of the cylinder.

- So this is just the circumference  $\times$  height  $\times$  thickness.



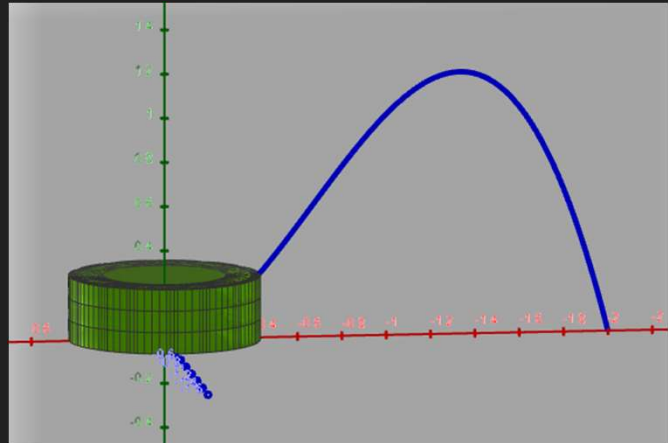
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## So how does this help us answer Example 1?

- The reason is USEFUL is that
  - For this problem, we no longer have to solve for  $x$  in terms of  $y$ .
- If we picture one possible shell, it will have a
  - *height* =  $f(x)$
  - *circumference* =  $2\pi x$
- As this shell spans the volume, we then have

$$V = \int_a^b 2\pi x \cdot f(x) dx$$



<https://www.geogebra.org/m/jafyndpu>

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## Even more... Shell Method Formulas

Since we are just cutting out parallel to the axis, we choose  $dx$  or  $dy$  in the following way:

- Rotating around  $y$ -axis

⇒ “  $dx$  ” problem

$$V = 2\pi \int_a^b x \cdot (\text{Top} - \text{Bottom}) dx$$

- Rotating around  $x$ -axis

⇒ “  $dy$  ” problem

$$V = 2\pi \int_c^d y \cdot (\text{Right} - \text{Left}) dy$$

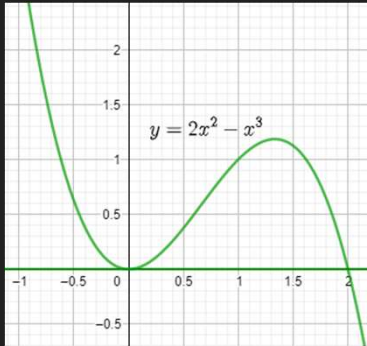
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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

$\Rightarrow dx$  problem



Find the bounds by setting the eqns equal

$$2x^2 - x^3 = 0$$

$$x^2(2-x) = 0$$

$$x = 0, 2$$

<https://www.geogebra.org/m/jafyndpu>

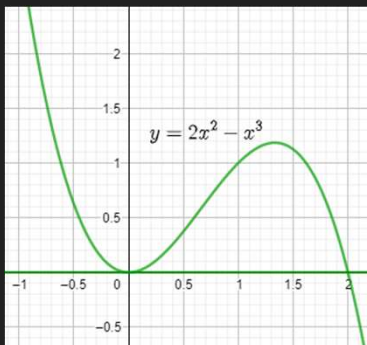
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Example 1: Find the volume obtained by revolving the region bounded by the curves

$$y = 2x^2 - x^3 \quad \text{and} \quad y = 0$$

About the y-axis.

$\Rightarrow dx$  problem



$$V = 2\pi \int_0^2 x(2x^2 - x^3) dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4) dx$$

$$= 2\pi \left( \frac{2x^4}{4} - \frac{x^5}{5} \right) \Big|_0^2 = \frac{16\pi}{5}$$

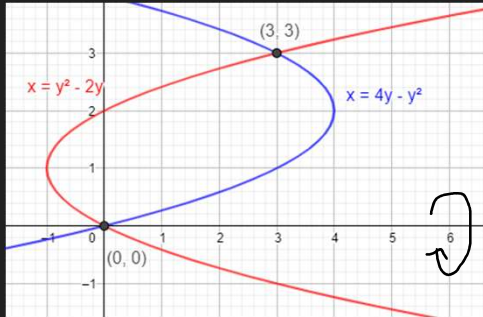
<https://www.geogebra.org/m/jafyndpu>

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Example 2: Find the volume obtained by revolving the region bounded by the curves

About the x-axis.  $x = y^2 - 2y$  and  $x = 4y - y^2$

Draw the region.



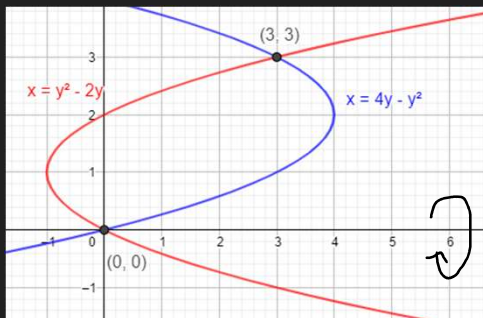
<https://www.geogebra.org/m/f3wrypfh#material/qrar4br5>

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Example 2: Find the volume obtained by revolving the region bounded by the curves

About the x-axis.  $x = y^2 - 2y$  and  $x = 4y - y^2$

Draw the region.



<https://www.geogebra.org/m/f3wrypfh#material/qrar4br5>

Bounds

$$y^2 - 2y = 4y - y^2$$

$$2y^2 - 6y = 0$$

$$2y(y - 3) = 0$$

$$y = 0, 3$$

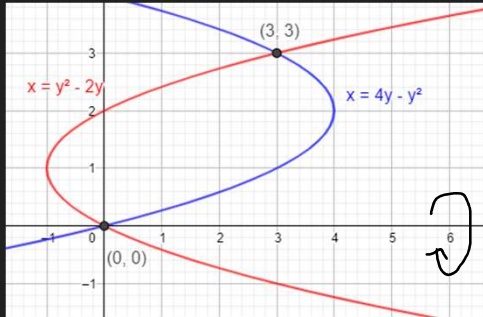
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Example 2: Find the volume obtained by revolving the region bounded by the curves

$$x = y^2 - 2y \quad \text{and} \quad x = 4y - y^2$$

About the x-axis.

Draw the region.



<https://www.geogebra.org/m/f3wrypfh#material/qrar4br5>

$$\begin{aligned} V &= 2\pi \int_0^3 y(y^2 - 2y - (4y - y^2)) dy \\ &= 2\pi \int_0^3 y(2y^2 - 6y) dy \\ &= 2\pi \int_0^3 (2y^3 - 6y^2) dy \\ &= 2\pi \left( \frac{2y^4}{4} - \frac{6y^3}{3} \right) \Big|_0^3 \end{aligned}$$

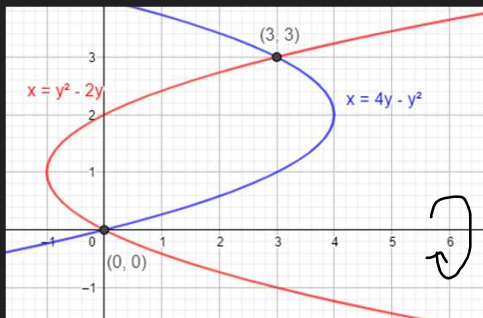
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Example 2: Find the volume obtained by revolving the region bounded by the curves

$$x = y^2 - 2y \quad \text{and} \quad x = 4y - y^2$$

About the x-axis.

Draw the region.



<https://www.geogebra.org/m/f3wrypfh#material/qrar4br5>

$$\begin{aligned} V &= 2\pi \left( \frac{2y^4}{4} - \frac{6y^3}{3} \right) \Big|_0^3 \\ &= \pi (y^4 - 4y^3) \Big|_0^3 \\ &= 27\pi \end{aligned}$$

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Example 3: Consider the region bounded by:

$$y = 4x, \quad y = 0, \quad \text{and} \quad x = 10$$

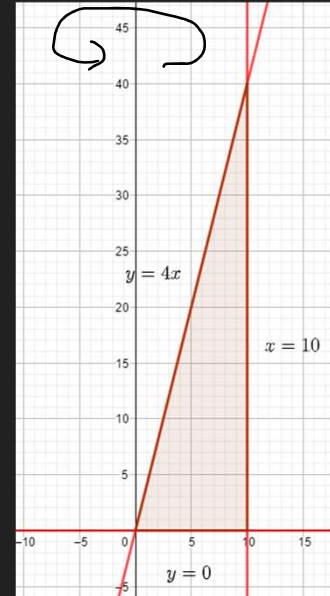
Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

ONLY SET-UP

A) Disk/Washer Method

Draw the region.

y-axis + Disk/Washer  
= dy problem



<https://www.geogebra.org/m/f3wrypfh#material/cb7ffa8n>

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Example 3: Consider the region bounded by:

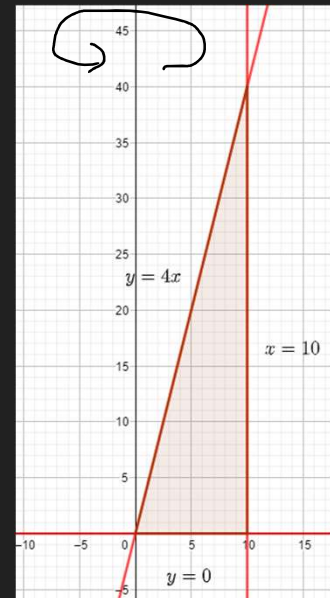
$$y = 4x, \quad y = 0, \quad \text{and} \quad x = 10$$

Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

A) Disk/Washer Method

Right  $\Rightarrow x = 10$   
Left  $\Rightarrow y = 4x \Rightarrow x = \frac{y}{4}$

$$V = \pi \int_0^{40} (10)^2 - \left(\frac{y}{4}\right)^2 dy$$



<https://www.geogebra.org/m/f3wrypfh#material/cb7ffa8n>

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Example 3: Consider the region bounded by:

$$y = 4x, \quad y = 0, \quad \text{and} \quad x = 10$$

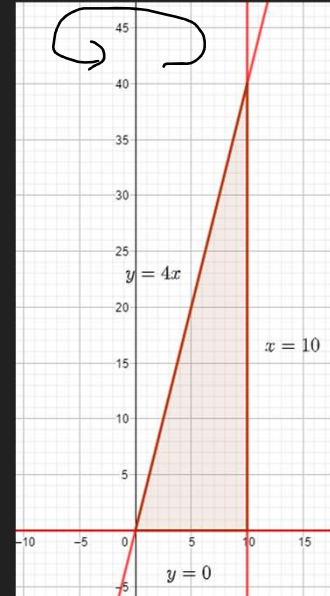
Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

ONLY SET-UP

B) Shell Method

Draw the region.

y-axis + Shell  
= dx problem



<https://www.geogebra.org/m/f3wrypfh#material/cb7ffa8n>

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Example 3: Consider the region bounded by:

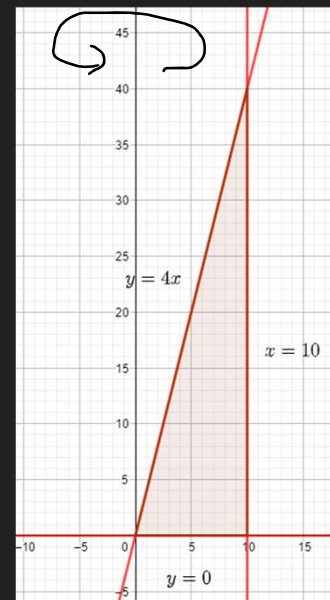
$$y = 4x, \quad y = 0, \quad \text{and} \quad x = 10$$

Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

B) Shell Method

Top  $\Rightarrow y = 4x$   
Bottom  $\Rightarrow y = 0$

$$V = 2\pi \int_0^{10} x(4x - 0) dx$$



<https://www.geogebra.org/m/f3wrypfh#material/cb7ffa8n>

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Example 3: Consider the region bounded by:

$$y = 4x, \quad y = 0, \quad \text{and} \quad x = 10$$

Set up the integral that represents the volume of solid obtained by the rotating the region about the y-axis using

Interesting Question: Which integral is easier to compute?

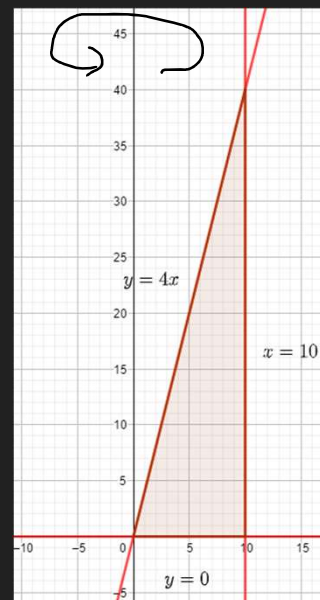
A) Disk/Washer Method

$$V = \pi \int_0^{40} (10)^2 - \left(\frac{y}{4}\right)^2 dy$$

B) Shell Method

$$V = 2\pi \int_0^{10} x(4x) dx$$

<https://www.geogebra.org/m/f3wrypfh#material/cb7ffa8n>



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## When do we apply Disk Method or Washer Method or Shell Method?

- When the region “hugs” the axis of rotation  
⇒ Disk Method
- When there is a “gap” between the region and axis of rotation  
⇒ Washer Method
- But if you find solving for  $x$  or  $y$ , in either method, is hard  
⇒ Shell Method

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## Formulas from Lessons 14 and 15 and 17

For rotation around x-axis:

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2 dx$$

- Washer Method:

$$V = \pi \int_a^b (R^2 - r^2) dx$$

- Shell Method:

$$V = 2\pi \int_c^d y \cdot (\text{Right} - \text{Left}) dy$$

For rotation around y-axis:

- Disk Method:

$$V = \pi \int_c^d [g(y)]^2 dy$$

- Washer Method:

$$V = \pi \int_c^d (R^2 - r^2) dy$$

- Shell Method:

$$V = 2\pi \int_a^b x \cdot (\text{Top} - \text{Bottom}) dx$$

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## Formulas from Lesson 16 Rotation around any non-Axis Formulas

For rotation around the line  $y = \#$ :

- Disk Method:

$$V = \pi \int_a^b [f(x) - \#]^2 dx$$

- Washer Method:

$$V = \pi \int_a^b [(R - \#)^2 - (r - \#)^2] dx$$

For rotation around the line  $x = \#$ :

- Disk Method:

$$V = \pi \int_c^d [g(y) - \#]^2 dy$$

- Washer Method:

$$V = \pi \int_c^d [(R - \#)^2 - (r - \#)^2] dy$$

**Note:** That these formulas work for the case of x-axis ( $y = 0$ ) and y-axis ( $x = 0$ ).

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## GeoGebra Link for Lesson 17

- <https://www.geogebra.org/m/f3wrypfh>
- Note click on the play buttons on the left-most screen and the animation will play/pause.