Lesson 20: Separation of Variables II

Today's lecture we will be doing more of the same, plus some application problems.

Example 1: Find the general solution of the differential eqn:

(a) $t^2 \frac{dy}{dt} - y = 0$

Rewrite: $t^2 \frac{dy}{dt} = y$

$$\frac{dy}{y} = \frac{dt}{t^2}$$

$$\frac{dy}{y} = t^{-2}\,dt$$

$$\int \frac{dy}{y} = \int t^{-2}\,dt$$

$$\ln|y| = -t^{-1} + c$$

$$\ln|y| = \frac{-1}{t} + c$$

$$|y| = \exp\left[\frac{-1}{t} + c\right]$$

$$\pm y = e^c \exp\left[\frac{-1}{t}\right]$$

$$y = \underbrace{\pm e^c}_{\text{All a constant}} \exp\left[\frac{-1}{t}\right]$$

$$y = C \exp\left[\frac{-1}{t}\right]$$

(b) $-x^3 y + y' = 2x^3$

Rewrite: $y' = 2x^3 + x^3 y$

$$\frac{dy}{dx} = x^3(2 + y)$$

$$\frac{dy}{2+y} = x^3\,dx$$

$$\int \frac{dy}{2+y} = \int x^3\,dx$$

$$\ln|2+y| = \frac{x^4}{4} + C$$

$$|2+y| = \exp\left[\frac{x^4}{4} + C\right]$$

$$\pm(2+y) = e^C \exp\left[\frac{x^4}{4}\right]$$

$$2+y = \underbrace{\pm e^C}_{\text{All a constant}} \exp\left[\frac{x^4}{4}\right]$$

$$2+y = C\exp\left[\frac{x^4}{4}\right]$$

$$y = C\exp\left[\frac{x^4}{4}\right] - 2$$

Example 2: In a particular chemical reaction, a substance is converted into a second substance at a rate proportional to the square of the amount of the first substance present at any time, t. Initially, 50 grams of the first substance was present, and 1 hour later only 14 grams of the first substance remained. What is the amount of the first substance remaining after 7 hours?

Set-Up: $\frac{da}{dt} = a^2 k$; $a(0) = 50$; $a(1) = 14$

Solve: $\frac{da}{a^2} = k\,dt$

$$\int a^{-2}\,da = \int k\,dt$$

$$-a^{-1} = kt + C$$

$$-\frac{1}{a} = kt + C$$

$$\frac{1}{a} = -kt \underbrace{- C}_{\text{All a constant}}$$

$$\frac{1}{a} = -kt + C$$

$$a = \frac{1}{-kt + C}$$

When $a(0) = 50$,

$$50 = a(0) = \frac{1}{C}$$

$$C = 1/50$$

So $a = \frac{1}{1/50 - kt}$

$$= \frac{50}{1 - 50kt}$$

When $a(1) = 14$,

$$14 = a(1) = \frac{50}{1 - 50k}$$

$$14(1 - 50k) = 50$$

$$1 - 50k = \frac{50}{14} = \frac{25}{7}$$

$$-50k = \frac{25}{7} - 1 = \frac{18}{7}$$

$$k = \frac{-1}{50} \cdot \frac{18}{7}$$

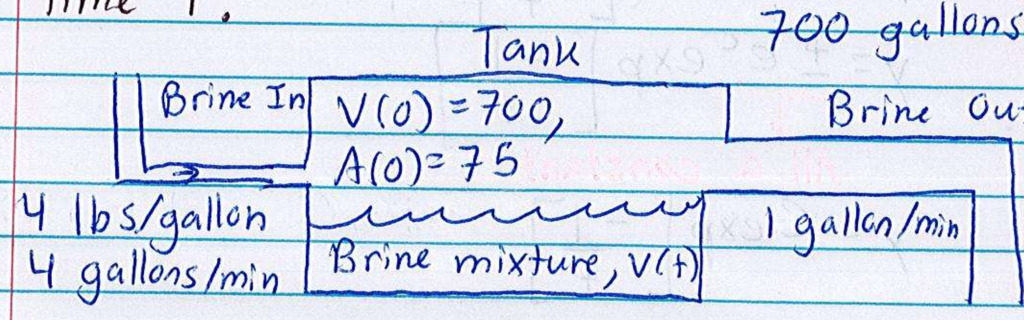
$$= \frac{-18}{350}$$

So $a = \frac{50}{1 - 50\left(\frac{-18}{350}\right)t}$

$$= \frac{350}{7 + 18t}$$

Hence $a(7) \approx 2.6316$ grams

Example 3: A 800 gallon tank initially contains 700 gallons of brine containing 75 pounds of dissolved salt. Brine containing 4 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute and the well-stirred mixture flows out of the tank at the rate of 1 gallon per minute. Set up a differential equation for the amount of salt, $A(t)$, in the tank at time t.



Define: $V(t)$ = amount of brine mixture in tank at time t (in gallons)

$A(t)$ = amount of salt in the tank at time t

t = time in minutes

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate of change of salt}\\ \text{in tank in lbs/min}\end{pmatrix} = \begin{pmatrix}\text{rate in}\\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{rate out}\\ \text{of salt}\end{pmatrix}$$

Rate in: $\left(4 \dfrac{\text{lbs}}{\text{gallons}}\right)\left(4 \dfrac{\text{gallons}}{\text{min}}\right) = 16 \dfrac{\text{lbs}}{\text{min}}$

Rate out: "Well-stirred" means each gallon in the tank has as much salt in it as any other gallon. i.e. Salt is uniformly mixed in the brine mixture

$$= \left(\frac{A(t)}{V(t)} \frac{\text{lbs}}{\text{gallons}}\right)\left(1 \frac{\text{gallons}}{\text{min}}\right) = \frac{A(t)}{V(t)} \frac{\text{lbs}}{\text{min}}$$

$\dfrac{dA}{dt} = 16 - \dfrac{A(t)}{V(t)}$. Now find $V(t)$,

So $\dfrac{dV}{dt} = \begin{pmatrix}\text{rate of change of}\\ \text{brine mix in gallons/min}\end{pmatrix} = \begin{pmatrix}\text{rate in}\\ \text{of brine}\end{pmatrix} - \begin{pmatrix}\text{rate out}\\ \text{of brine}\end{pmatrix}$

$= 4 \dfrac{\text{gallons}}{\text{min}} - 1 \dfrac{\text{gallons}}{\text{min}} = 3 \dfrac{\text{gallons}}{\text{min}}$

Hence $\begin{cases} V'(t) = 3 \\ V(0) = 700 \end{cases}$

So $V(t) = \int V'(t)\,dt = \int 3\,dt = 3t + C$

When $V(0) = 700$,

$700 = V(0) = 3(0) + C$

$700 = C \qquad \Rightarrow V(t) = 3t + 700$

Hence $\dfrac{dA}{dt} = 16 - \dfrac{A}{3t + 700}$