Lesson 27: Partial Derivatives

A partial derivative is a derivative where we hold some variable constant.

Let's think about a function of one variable.

ex. $f(x) = x^2 \Rightarrow f'(x) = 2x$

But what about a function of two variables?

$$f(x,y) = x^2 + y^3$$

We find its partial derivative with respect to x by treating y as a constant.

$$f_x(x,y) = 2x + 0 = 2x$$

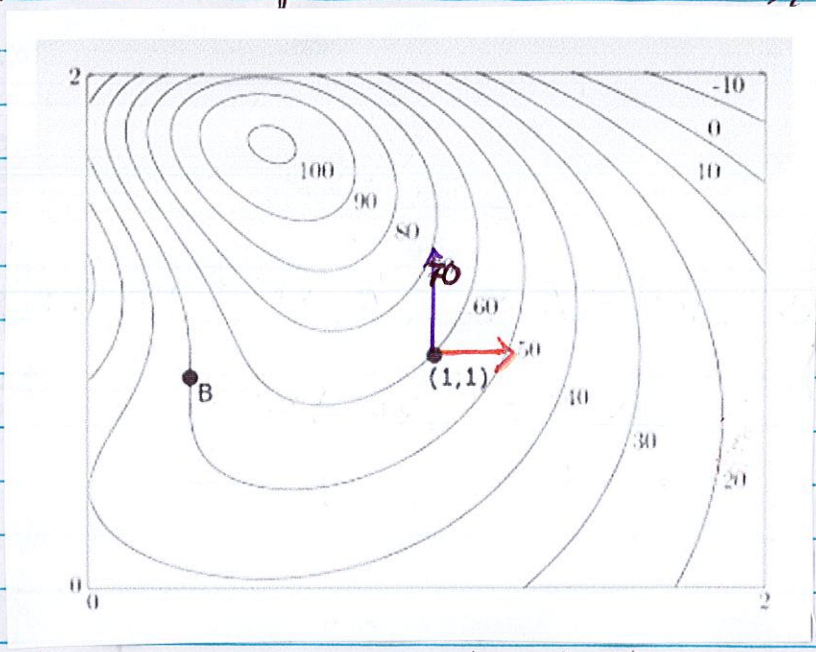
To find the partial derivative with respect to y, we treat x as a constant.

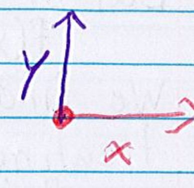
$$f_y(x,y) = 0 + 3y^2 = 3y^2$$

Definition: • The (first) partial derivative f_x describes the rate of change of f as x changes, where y remains constant. i.e. Find the derivative with respect to x, where we treat y as a constant.

• The (first) partial derivative f_y describes the rate of change of f as y changes, where x remains constant. i.e. Find the derivative with respect to y, where we treat x as a constant.

Example 1: The accompanying figure shows a contour plot for an unspecified function $f(x,y)$.





ⓐ What can you say about the signs of $f_x(1,1)$?
(i) Positive $\boxed{\text{(ii) Negative}}$ (iii) Zero

We always ask the question what would happen if the variable that we are differentiating with respect to was increased?
i.e. What if we took a super small step from the point we're at to the right if that's the direction of positive or increasing x?

From the graph, we'd be going from a line with value 60 towards a line with value 50 and that would be going down. Hence that means when x is increasing, f is decreasing. i.e.

$$f_x(1,1) = \frac{\Delta f}{\Delta x}(1,1) = \frac{-}{+} = - \Rightarrow \text{negative}$$

ⓑ What can you say about the signs of $f_y(1,1)$?
$\boxed{\text{(i) Positive}}$ (ii) Negative   (iii) Zero

From the graph, we'd be going from a line with value 60 towards a line with value 70 and that would be going up. Hence that means when y is increasing, f is also increasing. i.e.

$$f_y(1,1) = \frac{\Delta f}{\Delta y}(1,1) = \frac{+}{+} = + \Rightarrow \text{positive}$$

Example 2: Compute the first order partial derivatives.

ⓐ $f(x,y) = x^3 + 3xy$

First order partials $\Rightarrow$ We need to find f_x and f_y.

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f(x,y) = x^3 + (3y)x$$
$$f_x(x,y) = 3x^2 + 3y$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f(x,y) = x^3 + (3x)y$$
$$f_y(x,y) = 0 + 3x = 3x$$

Chain Rule Problem

ⓑ $f(x,y) = \ln(x+2y)$

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f_x(x,y) = \frac{1}{x+2y} \cdot \frac{d}{dx}(x+2y) = \frac{1}{x+2y} \cdot (1+0) = \frac{1}{x+2y}$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f_y(x,y) = \frac{1}{x+2y} \cdot \frac{d}{dy}(x+2y) = \frac{1}{x+2y} \cdot (0+2) = \frac{2}{x+2y}$$

ⓒ $f(x,y) = \dfrac{9xy}{\sqrt{y-1}}$

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f(x,y) = \frac{9y}{\sqrt{y-1}}(x)$$

$$f_x(x,y) = \frac{9y}{\sqrt{y-1}} \frac{d}{dx}(x) = \frac{9y}{\sqrt{y-1}}$$

Next find f_y, i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f(x,y) = 9x\left(\frac{y}{\sqrt{y-1}}\right)$$

$$f_y(x,y) = 9x \frac{d}{dy}\left(\frac{y}{\sqrt{y-1}}\right) = 9x\left(\frac{1 \cdot \sqrt{y-1} - y(\frac{1}{2}(y-1)^{-1/2})}{(\sqrt{y-1})^2}\right)$$

$$= 9x\left(\frac{\sqrt{y-1} - \frac{1}{2}y(y-1)^{-1/2}}{y-1}\right)$$