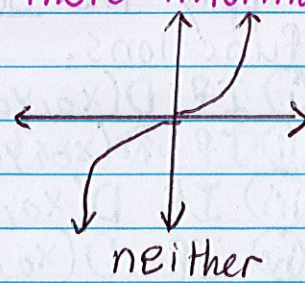
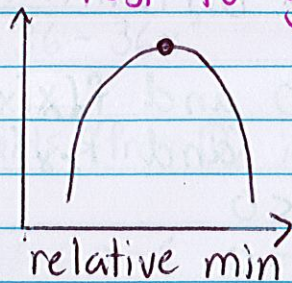
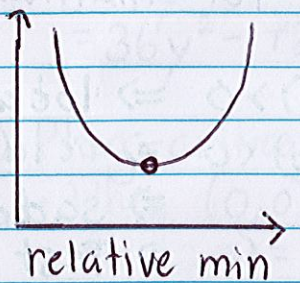


# Lesson 29: Extrema of Functions of Two Variables

Recall from Calculus 1, when  $y = f(x)$

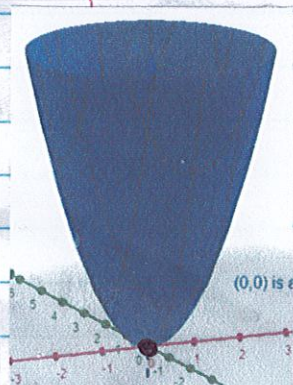
- $x_0$  is a critical number if  $f'(x_0) = 0$
- By the Second Derivative Test,
  - ↳ if  $f''(x_0) > 0 \Rightarrow$  relative min @  $(x_0, f(x_0))$
  - ↳ if  $f''(x_0) < 0 \Rightarrow$  relative max @  $(x_0, f(x_0))$
  - ↳ if  $f''(x_0) = 0 \Rightarrow$  test is inconclusive

Typically we apply the First Derivative Test to gain more information

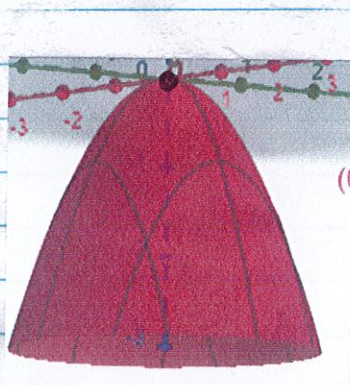


Extrema for two variable function

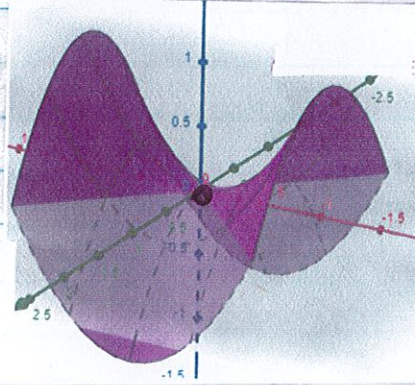
If  $z = f(x, y)$ , the following extrema may occur:



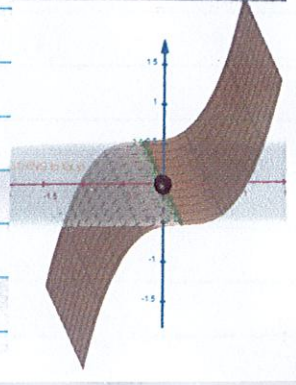
relative min



relative max



Saddle point



neither

We see that in all the cases of extrema, the tangent plane to the graph is parallel to the  $xy$ -plane. We can describe this in terms of first order partials.

i.e.  $\frac{df}{dx}(x_0, y_0) = 0$  or  $\frac{df}{dy}(x_0, y_0) = 0$

To determine what type of extrema is taking place, we use an analogous Second Derivative Test for Multivariable Functions.

## Finding extrema of functions of two variables

Let  $z = f(x, y)$

① Find all the critical points.

i.e. All  $(x_0, y_0)$  such that  $f_x(x_0, y_0) = 0$  and  $f_y(x_0, y_0) = 0$

② Compute  $f_{xx}$ ,  $f_{xy}$ ,  $f_{yy}$  and

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

where  $D$  is known as the discriminant.

③ For every given critical point  $(x_0, y_0)$ , evaluate  $D$  and  $f_{xx}$  at  $(x_0, y_0)$

④ Apply the Second Derivative Test for multivariable functions.

(i) IF  $D(x_0, y_0) > 0$  and  $f_{xx}(x_0, y_0) > 0 \Rightarrow$  relative min

(ii) IF  $D(x_0, y_0) > 0$  and  $f_{xx}(x_0, y_0) < 0 \Rightarrow$  relative max

(iii) IF  $D(x_0, y_0) < 0 \Rightarrow$  saddle point

(iv) IF  $D(x_0, y_0) = 0 \Rightarrow$  Test is inconclusive

Example 1: Find all the relative maxs, relative mins, and saddle points of the following functions:

(a)  $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

① Critical Point(s)

$$\begin{cases} f_x = 6xy - 6x = 0 & \textcircled{\text{I}} \\ f_y = 3x^2 + 3y^2 - 6y = 0 & \textcircled{\text{II}} \end{cases}$$

Solve  $\textcircled{\text{I}}$

$$6xy - 6x = 0$$

$$6x(y-1) = 0$$

$$x=0 \quad y=1$$

Plug  $x=0$  and  $y=1$  into  $\textcircled{\text{II}}$

• When  $x=0$ ,

$$3(0)^2 + 3y^2 - 6y = 0$$

$$3y^2 - 6y = 0$$

$$3y(y-2) = 0$$

$$y=0, y=2$$

Critical Pts:  $(0,0), (0,2)$

• When  $y=1$ ,

$$3x^2 + 3(1)^2 - 6(1) = 0$$

$$3x^2 + 3 - 6 = 0$$

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$3(x-1)(x+1) = 0$$

$$x = \pm 1$$

Critical Pts:  $(1,1), (-1,1)$

② Compute  $f_{xx}$ ,  $f_{xy}$ ,  $f_{yy}$ , and  $D$ .

$$f_x = 6xy - 6x$$

$$f_{xx} = 6y - 6 \quad f_{xy} = 6x$$

$$f_y = 3x^2 + 3y^2 - 6y$$

$$f_{yy} = 6y - 6$$

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= (6y-6)(6y-6) - (6x)^2 \\ &= (6y-6)^2 - (6x)^2 \\ &= 36y^2 - 72y + 36 - 36x^2 \end{aligned}$$

③ Plug critical points (from ①) into  $f_{xx}$  and  $D$ .

• When  $(0,0)$ ,

$$f_{xx}(0,0) = 6(0) - 6 = -6$$

$$D(0,0) = 36(0)^2 - 72(0) + 36 - 36(0)^2 = 36$$

• When  $(0,2)$ ,

$$f_{xx}(0,2) = 6(2) - 6 = 6$$

$$D(0,2) = 36(2)^2 - 72(2) + 36 - 36(0)^2 = 36$$

• When  $(1,1)$ ,

$$f_{xx}(1,1) = 6(1) - 6 = 0$$

$$D(1,1) = 36(1)^2 - 72(1) + 36 - 36(1)^2 = -36$$

• When  $(-1,1)$ ,

$$f_{xx}(-1,1) = 6(1) - 6 = 0$$

$$D(-1,1) = 36(1)^2 - 72(1) + 36 - 36(-1)^2 = -36$$

④ Second Derivative Test.

Using ③, we make the following conclusions:

- $(0,0)$ :  $D(0,0) = 36 > 0$   $f_{xx}(0,0) = -6 < 0 \Rightarrow$  relative max
- $(0,2)$ :  $D(0,2) = 36 > 0$   $f_{xx}(0,2) = 6 > 0 \Rightarrow$  relative min
- $(1,1)$ :  $D(1,1) = -36 < 0 \Rightarrow$  saddle pt
- $(-1,1)$ :  $D(-1,1) = -36 < 0 \Rightarrow$  saddle pt

$$\textcircled{b} f(x,y) = \frac{2}{3}y^3 + x^2 - 4yx - 10y + 6$$

① Critical Point(s)

$$f_x = 2x - 4y = 0 \quad \textcircled{I}$$

$$f_y = 2y^2 - 4x - 10 = 0 \quad \textcircled{II}$$

Solve ① for  $x$ .

$$2x - 4y = 0$$

$$2x = 4y$$

$$x = 2y$$

Plug  $y = 5, -1$  into  $x = 2y$

$$\bullet y = 5$$

$$x = 2(5) = 10$$

$$\bullet y = -1$$

$$x = 2(-1) = -2$$

Plug  $x = 2y$  into ②

$$2y^2 - 4(2y) - 10 = 0$$

$$2y^2 - 8y - 10 = 0$$

$$2(y^2 - 4y - 5) = 0$$

$$2(y - 5)(y + 1) = 0$$

$$y = 5, -1$$

Hence the critical points are  $(10, 5), (-2, -1)$

② Compute  $f_{xx}, f_{xy}, f_{yy}$  and  $D$ .

$$f_x = 2x - 4y$$

$$f_{xx} = 2$$

$$f_{xy} = -4$$

$$f_y = 2y^2 - 4x - 10$$

$$f_{yy} = 4y$$

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= 2(4y) - (-4)^2 \\ &= 8y - 16 \end{aligned}$$

③ Plug critical points (from ①) into  $f_{xx}$  and  $D$ .

$$\bullet (10, 5): f_{xx}(10, 5) = 2$$

$$D(10, 5) = 8(5) - 16 = 34$$

$$\bullet (-2, -1): f_{xx}(-2, -1) = 2$$

$$D(-2, -1) = 8(-1) - 16 = -24$$

#### ④ Second Derivative Test

Using ③, we make the following conclusions:

- $(10, 5)$ :  $D(10, 5) = 34 > 0$   $f_{xx}(10, 5) = 2 > 0 \Rightarrow$  relative min
- $(-2, -1)$ :  $D(-2, -1) = -24 < 0 \Rightarrow$  saddle point

$$\textcircled{c} f(x, y) = \frac{3}{2}x^4 - yx^2 + 20x^2 + \frac{1}{2}y^2 - 3$$

#### ① Critical Point(s)

$$f_x = 6x^3 - 2xy + 40x \quad \textcircled{I}$$

$$f_y = -x^2 + y = 0 \quad \textcircled{II}$$

Solve ② for  $y$ .

$$-x^2 + y = 0$$

$$y = x^2$$

Plug  $y = x^2$  into ①.

$$6x^3 - 2x(x^2) + 40x = 0$$

$$6x^3 - 2x^3 + 40x = 0$$

$$4x^3 + 40x = 0$$

$$4x(x^2 + 10) = 0$$

$$4x = 0 \quad x^2 + 10 = 0$$

$$x = 0 \quad \underbrace{\hspace{2cm}}_{\text{no solution}}$$

$$\text{Plug } x=0 \text{ into } y=x^2$$

$$y = 0^2 = 0$$

Hence  $(0, 0)$  is the only critical point.

#### ② Compute $f_{xx}$ , $f_{xy}$ , $f_{yy}$ , and $D$ .

$$f_x = 6x^3 - 2xy + 40x$$

$$f_{xx} = 18x^2 - 2y + 40 \quad f_{xy} = -2x$$

$$f_y = -x^2 + y$$

$$f_{yy} = 1$$

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

$$= (18x^2 - 2y + 40)(1) - (-2x)^2$$

$$= 18x^2 - 2y + 40 - 4x^2$$

$$= 14x^2 - 2y + 40$$

③ Plug critical point (from ①) into  $f_{xx}$  and  $D$ .

$$(0,0): f_{xx}(0,0) = 18(0)^2 - 2(0) + 40 = 40$$

$$D(0,0) = 14(0)^2 - 2(0) + 40 = 40$$

④ Second Derivative Test

Using ③, we make the following conclusion:

$$(0,0): D(0,0) = 40 > 0 \quad f_{xx}(0,0) = 40 > 0 \Rightarrow \text{relative min}$$