Lesson 34: Double Integrals II

Recall from Lesson 5, the formula for average value:

For $f(x)$ defined on $[a,b]$, the average value of $f(x)$ on $[a,b]$ is

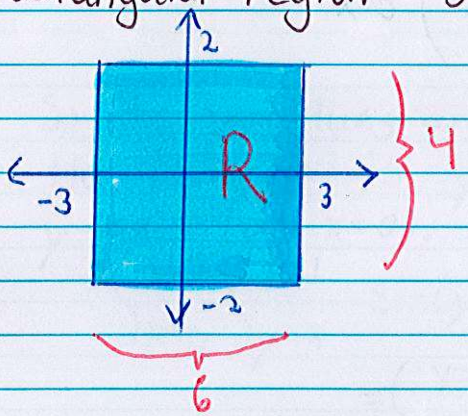
$$f_{AVE}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The multivariable average value formula follows:

For $f(x,y)$ defined on a region, R, the average value of $f(x,y)$ over the region, R, is given by

$$f_{AVE}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$ where A is the area of R.

Example 1: Find the average of $f(x,y) = 12 - x^2 - y^2$ in a rectangular region $-3 \le x \le 3$, $-2 \le y \le 2$.



First draw the region. With the drawing, find the area of R.

Area of R $= 6 \times 4 = 24$

Note we are given the bounds for our integral. So

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\int_{y=-2}^{y=2}(12 - x^2 - y^2)\,dy\,dx$$

Now integrate.

$$f_{AVE}(x,y) = \frac{1}{24}\int_{x=-3}^{x=3}\left(\left(12y - x^2y - \frac{y^3}{3}\right)\right]_{y=-2}^{y=2} dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(12(2) - 2x^2 - \frac{2^3}{3} - \left(12(-2) - x^2(-2) - \frac{(-2)^3}{3}\right)\right)dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(24 - 2x^2 - \frac{8}{3} + 24 - 2x^2 - \frac{8}{3}\right)dx$$

$$= \frac{1}{24}\int_{x=-3}^{x=3}\left(\frac{128}{3} - 4x^2\right)dx$$

$$= \frac{1}{24}\left(\frac{128}{3}x - \frac{4x^3}{3}\right)\Big]_{x=-3}^{x=3}$$

$$= \frac{1}{24}\left(\frac{128}{3}(3) - \frac{4(3)^3}{3} - \left(\frac{128}{3}(-3) - \frac{4(-3)^3}{3}\right)\right) = \frac{28}{3}$$

Last Class, we introduced

$$\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy \quad \text{and} \quad \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$$

A geometric interpertation of these double integrals is we are finding the volume below $z=f(x,y)$ above the region

$$R=\{(x,y) \mid a\le x\le b;\ c\le y\le d\}$$

We can denote the integrals above by just one

$$\iint_R f(x,y)\,dA$$

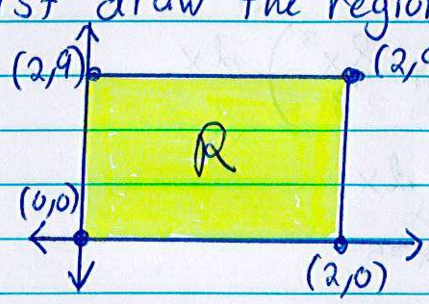
where R is the **domain of integration.**

To solve the integrals of the form $\iint_R f(x,y)\,dA$, we start by drawing the region. We do this to determine the bounds of our integrals.

Example 2: Evaluate the integral $\iint_R 10x^3y\,dA$ where R is the rectangle with vertices $(0,0)$, $(2,0)$, $(0,9)$, and $(2,9)$.

First draw the region, R. We can see that $0\le x\le 2$ and $0\le y\le 9$. So

$$R=\{(x,y) \mid 0\le x\le 2,\ 0\le y\le 9\}$$



Hence

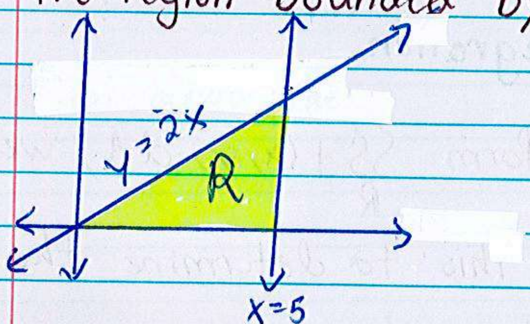
$$\iint_R 10x^3y\,dA = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=9} 10x^3\cdot y\,dy\right)dx$$

$$=\int_{x=0}^{x=2}\left(10x^3\cdot \frac{y^2}{2}\Big]_{y=0}^{y=9}\right)dx$$

$$=\int_{x=0}^{x=2}\left(10x^3\left(\frac{9^2}{2}-\frac{0^2}{2}\right)\right)dx$$

$$= \int_{x=0}^{x=2} \left(10x^3 \cdot \frac{81}{2}\right) dx$$
$$= \int_{x=0}^{x=2} 405x^3\, dx$$
$$= \frac{405}{4} x^4 \Big]_{x=0}^{x=2}$$
$$= \frac{405}{4}(2^4 - 0^4) = 1620$$

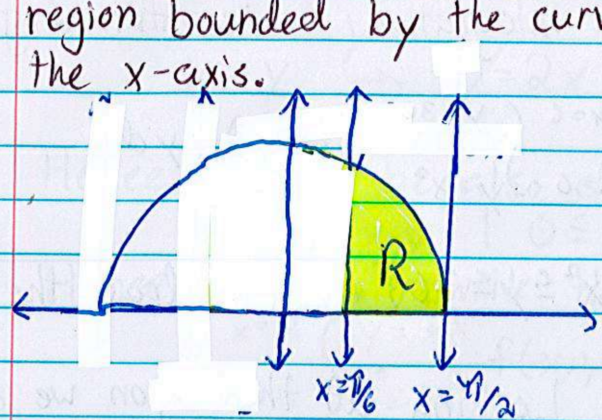
Example **3**: Evaluate the integral $\iint_R (x^2+y^2)\,dA$ where R is the region bounded by the lines $y=2x$, $x=5$, and the x-axis.



First draw the region, R. We can see that
$$R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 2x\}$$

Hence $\iint_R (x^2+y^2)\,dA = \int_{x=0}^{x=5}\left(\int_{y=0}^{y=2x} (x^2+y^2)\,dy\right) dx$
$$= \int_{x=0}^{x=5} \left(\left(x^2 \cdot y + \frac{y^3}{3}\right)\Big]_{y=0}^{y=2x} \right) dx$$
$$= \int_{x=0}^{x=5} \left(x^2(2x) + \frac{(2x)^3}{3} - \left(x^2 \cdot 0 + \frac{0^3}{3}\right)\right) dx$$
$$= \int_{x=0}^{x=5} \left(2x^3 + \frac{8x^3}{3}\right) dx$$
$$= \int_{x=0}^{x=5} \frac{14}{3} x^3\, dx$$
$$= \frac{14}{3} \cdot \frac{x^4}{4} \Big]_{x=0}^{x=5}$$
$$= \frac{7}{6} x^4 \Big]_{x=0}^{x=5}$$
$$= \frac{7}{6}(5^4 - 0^4) = \frac{4375}{6}$$

Example 4: Evaluate the integral $\iint_R 6\sin^2(x)\,dA$ where R is the region bounded by the curves $y=\cos(x)$, $x=\pi/6$, $x=\pi/2$ and the x-axis.



First draw the region, R. We can see that

$$R=\left\{(x,y)\;\middle|\;\frac{\pi}{6}\le x\le\frac{\pi}{2},\ 0\le y\le\cos x\right\}$$

Hence
$$\iint_R 6\sin^2(x)\,dA=\int_{x=\pi/6}^{x=\pi/2}\left(\int_{y=0}^{y=\cos(x)} 6(\sin x)^2\,dy\right)dx$$
$$=\int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2\cdot y\Big]_{y=0}^{y=\cos x}\right)dx$$
$$=\int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2\cdot(\cos x-0)\right)dx$$
$$=\int_{x=\pi/6}^{x=\pi/2} 6(\sin x)^2\cdot\cos x\,dx$$

$$\frac{u=\sin x}{du=\cos x\,dx}\quad\int 6u^2\,du$$
$$=\frac{6u^3}{3}=2u^3$$
$$=2(\sin x)^3\Big]_{x=\pi/6}^{x=\pi/2}$$
$$=2\left(\left(\sin\left(\frac{\pi}{2}\right)\right)^3-\left(\sin\left(\frac{\pi}{6}\right)\right)^3\right)$$
$$=2\left((1)^3-\left(\frac{1}{2}\right)^3\right)=2\left(1-\frac{1}{8}\right)=\frac{2\cdot 7}{8}=\frac{7}{4}$$

Note that Examples **2, 3, and 4** could have been done with dxdy as the order of integration.

So a good question to ask is when to use dxdy or dydx? We will answer that next time.