MA 16020 LESSON 5: INTEGRATION BY SUBSTITUTION

(Handout)

Example 1: It is estimated that t hours after 8:00 am, the population of a certain bacterial sample will be changing at a rate of:

$$N'(t) = \frac{3t}{\sqrt{t+4}} \quad \text{bacteria per hour.}$$

Find the increase in the bacteria population from 11:00 am to 1:00 pm.

11:00 am $\Rightarrow t=3$
1:00 pm $\Rightarrow t=5$ $\Rightarrow \int_3^5 \frac{3t}{\sqrt{t+4}}\,dt$

So $u=t+4 \Leftrightarrow u-4=t$, $du=dt$

$$\int \frac{3(u-4)}{u^{1/2}}\,du = \int \left(3u^{1/2} - 12u^{-1/2}\right)du$$

$$= 3\cdot\frac{2}{3}u^{3/2} - 12\cdot\frac{2}{1}u^{1/2} = \left(2(t+4)^{3/2} - 24(t+4)^{1/2}\right)\Big]_3^5$$

$$= \left(2(5+4)^{3/2} - 24(5+4)^{1/2}\right) - \left(2(3+4)^{3/2} - 24(3+4)^{1/2}\right)$$

≈ 8.458

Example 2: It is estimated that t – weeks into a semester, the average amount of sleep a college math student gets per day $S(t)$ at a rate of

$$-\frac{6t}{e^{t^2}} \quad \text{hours per day.}$$

When the semester begins, math students sleep on average of 8.1 hours per day. What is $S(t)$, 10 week(s) into the semester?

$$S(t) = \int -\frac{6t}{e^{t^2}}\,dt = \int -6t e^{-t^2}\,dt \quad \frac{u = -t^2}{du = -2t\,dt} \quad \int -6\cancel{t} e^{u} \cdot \frac{du}{-2\cancel{t}}$$

$$\frac{du}{-2t} = dt$$

$$= \int 3e^{u}\,du = 3e^{u} + C = 3e^{-t^2} + C$$

Now find C with $S(0) = 8.1$

$$8.1 = S(0) = 3e^{-0^2} + C$$

$$8.1 = 3 + C$$

$$5.1 = C$$

So $S(t) = 3e^{-t^2} + 5.1$

$$S(10) = 3e^{-100} + 5.1 = 5.1$$

<u>Example 3:</u> A certain plant grows at the rate $H'(t) = \frac{1}{\sqrt[3]{8t+3}}$ inches per day, t days after it was planted. How many inches will the height of the plant change on the third day? Round answer to 3 decimal places.

$$\int_2^3 \frac{dt}{\sqrt[3]{8t+3}} \quad \frac{u = 8t+3}{du = 8dt} \quad \int \frac{1}{u^{1/3}} \cdot \frac{du}{8} = \frac{1}{8} \int u^{-1/3}\, du$$

$$\frac{du}{8} = dt$$

$$= \frac{1}{8} \cdot \frac{3}{2} u^{2/3} = \frac{3}{16} (8t+3)^{2/3} \Big]_2^3$$

$$= \frac{3}{16} (8(3)+3)^{2/3} - \frac{3}{16} (8(2)+3)^{2/3}$$

$$= \frac{3}{16} (27)^{2/3} - \frac{3}{16} (19)^{2/3} \approx 0.352$$

Definition: For $f(x)$ defined on $[a, b]$, the average value of $f(x)$ on $[a, b]$ is:

$$f_{AVE}(x) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Example 4: Find the average value of $f(x) = 6x^2 + 2$ over $[1, 3]$.

$$f_{AVE}(x) = \frac{1}{3-1}\int_1^3 (6x^2+2)\,dx = \frac{1}{2}\left(\frac{6x^3}{3} + 2x\right)\Big]_1^3$$

$$= \frac{1}{2}\left(2x^3 + 2x\right)\Big]_1^3 = \left(x^3 + x\right)\Big]_1^3$$

$$= (3^3+3) - (1^3+1)$$

$$= 28$$

Example 5: Find the average value of $f(x) = xe^{x^2}$ over $[0, 2]$.

$$f_{AVE}(x) = \frac{1}{2-0}\int_0^2 xe^{x^2}\,dx \;\overline{\underline{\;u = x^2\;}}\; \frac{1}{2}\int xe^u \cdot \frac{du}{2\cancel{x}}$$

$$du = 2x\,dx$$

$$\frac{du}{2x} = dx$$

$$= \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u = \frac{1}{4}e^{x^2}\Big]_0^2$$

$$= \frac{1}{4}e^{2^2} - \frac{1}{4}e^{0^2}$$

$$= \frac{1}{4}e^4 - \frac{1}{4}$$

Example 6: After t months on the job, a postal clerk can sort

$$Q(t) = 700 - 400e^{-0.5t}$$

Letters per hour. What is the average rate at which the clerk sorts mail during the first 3 months on the job? Round your answer to two decimal places.

$$Q_{AVE}(t) = \frac{1}{3-0}\int_0^3 (700 - 400e^{-0.5t})\,dt$$

$$= \int_0^3 \frac{700}{3}\,dt - \underbrace{\int_0^3 \frac{400}{3} e^{-0.5t}\,dt}$$

$u = -0.5t$

$du = -0.5t$

$-2du = t$

$$= \frac{700}{3}t\Big]_0^3 - \int \frac{400}{3} e^u \cdot (-2)\,du$$

$$= \frac{700}{3}t\Big]_0^3 + \frac{800}{3} e^u$$

$$= \frac{700t}{3}\Big]_0^3 + \frac{800}{3} e^{-0.5t}\Big]_0^3$$

$$= \frac{700}{3}(3-0) + \frac{800}{3}\left(e^{-0.05(3)} - e^{-0.05(0)}\right)$$

$$= 700 + \frac{800}{3} e^{-0.15} - \frac{800}{3}$$

$$\approx 492.83$$