Lesson 7: Integration by Parts I

Recall the Product Rule,

$$(uv)' = u'v + uv'$$

what if we integrate both sides with respect to x.

$$\int (uv)'\,dx = \int (u'v + uv')\,dx$$

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

Remember $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$. So

$$uv = \int v \frac{du}{dx}\,dx + \int u \frac{dv}{dx}\,dx$$

$$uv = \int v\,du + \int u\,dv$$

$$uv - \int v\,du = \int u\,dv$$

Interesting enough is that for many cases an integral will have the form

$$\int u\,dv$$

This technique of turning one integral into another is called Integration by Parts.

It's formula is: $\int u\,dv = uv - \int v\,du$

To use this technique

- Choose u to be the one to differentiate
- Choose dv to be integrated

Remark: Sometimes picking u and dv can be tricking. There is an acronym that makes picking u easier. Think of it as a sort of order of operation for choosing u.

L - Logarithmic
A - Algebraic
T - Trigonometric
E - Exponential

Example 1: Evaluate

ⓐ $\int x \ln x \, dx$ $\quad \dfrac{u = \ln x}{du = \frac{1}{x}dx} \quad \dfrac{dv = x\,dx}{v = \frac{x^2}{2}} \quad uv - \int v\,du$

$$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^3}{2}\ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

ⓑ $\int x \cos x \, dx$ $\quad \dfrac{u = x}{du = dx} \quad \dfrac{dv = \cos x\,dx}{v = \sin x} \quad uv - \int v\,du$

$$= x\sin x - \int \sin x\,dx = x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

ⓒ $\int x e^{2x} \, dx$ $\quad \dfrac{u = x}{du = dx} \quad \dfrac{dv = e^{2x}dx}{v = \frac{1}{2}e^{2x}} \quad uv - \int v\,du$

$$= \frac{x}{2}e^{2x} - \underbrace{\int \frac{1}{2}e^{2x}\,dx}_{\text{found by u-sub}} = \frac{x}{2}e^{2x} - \frac{1}{2} \cdot \frac{1}{2}e^{2x} + C$$

$$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

Example 2: Evaluate

ⓐ $\int_0^{\pi/2} 8x\sin x\,dx \quad \frac{u=8x}{du=8dx} \quad \frac{dv=\sin x\,dx}{v=-\cos x}$

$= -8x\cos x\Big]_0^{\pi/2} - \int_0^{\pi/2} -8\cos x\,dx$

$= -8x\cos x\Big]_0^{\pi/2} + \int_0^{\pi/2} 8\cos x\,dx$

$= -8x\cos x\Big]_0^{\pi/2} + 8\sin x\Big]_0^{\pi/2}$

$= -\frac{8\pi}{2}\cos\left(\frac{\pi}{2}\right) - \left(-8(0)\cos(0)\right) + 8\sin\left(\frac{\pi}{2}\right) - 8\sin(0)$

$= 0 - 0 + 8 - 0 = 8$

ⓑ $\int_1^e x\ln\sqrt[3]{x}\,dx = \int_1^e x\ln(x^{1/3})\,dx = \int_1^e \frac{1}{3}x\ln x\,dx$

$\frac{u=\ln x}{du=\frac{1}{x}dx} \quad \frac{dv=\frac{1}{3}x\,dx}{v=\frac{1}{3}\cdot\frac{x^2}{2}} \quad \frac{x^2}{6}\ln x\Big]_1^e - \int_1^e \frac{x^2}{6}\cdot\frac{1}{x}\,dx$

$= \frac{x^2}{6}\ln x\Big]_1^e - \int_1^e \frac{x}{6}\,dx = \frac{x^2}{6}\ln x\Big]_1^e - \frac{x^2}{12}\Big]_1^e$

$= \frac{e^2}{6}\ln e - \frac{1}{6}\ln(1) - \left(\frac{e^2}{12} - \frac{1}{12}\right)$

$= \frac{e^2}{6} - 0 - \frac{e^2}{12} + \frac{1}{12}$

ⓒ $\int_1^e \frac{\ln x}{x^4}\,dx = \int_1^e x^{-4}\ln x\,dx \quad \frac{u=\ln x}{du=\frac{1}{x}dx} \quad \frac{dv=x^{-4}dx}{v=\frac{x^{-3}}{-3}=\frac{-1}{3x^3}}$

$= \frac{-1}{3x^3}\ln x\Big]_1^e - \int_1^e \frac{-1}{3x^3}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{3x^3}\Big]_1^e + \int_1^e \frac{1}{3x^4}\,dx$

$= -\frac{\ln x}{3x^3}\Big]_1^e + \frac{1}{3}\int_1^e x^{-4}\,dx = -\frac{\ln x}{3x^3}\Big]_1^e + \frac{1}{3}\,\frac{x^{-3}}{-3}\Big]_1^e$

$= -\frac{\ln e}{3e^3} - \left(\frac{-\ln(1)}{3\cdot 1}\right) - \frac{1}{9}\left((e)^{-3} - 1^{-3}\right)$

$= \frac{-1}{3e^3} - 0 - \frac{1}{9}\left(\frac{1}{e^3} - 1\right) = \frac{-1}{3e^3} - \frac{1}{9e^3} + \frac{1}{9}$