Lesson 9: Partial Fractions I

Recall the concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{2}+\frac{1}{3}=\frac{1}{2}\cdot\frac{3}{3}+\frac{1}{3}\cdot\frac{2}{2}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

So we can say that a partial fraction decomposition for 5/6 is

$$\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$$

This concept can be used with functions of x.

Example 1: Combine the following fractions

$$\frac{1}{x-2}+\frac{3}{x-5}=\frac{1}{(x-2)}\cdot\frac{(x-5)}{(x-5)}+\frac{3}{(x-5)}\cdot\frac{(x-2)}{(x-2)}$$

$$=\frac{(x-5)+3(x-2)}{(x-2)(x-5)}$$

$$=\frac{x-5+3x-6}{x^2-2x-5x+10}$$

$$=\frac{4x-11}{x^2-7x+10}$$

Why do we care about partial fraction decomposition? It's because u-sub isn't enough.

Example 2: Evaluate $\displaystyle\int\frac{4x-11}{x^2-7x+10}\,dx$

Let's first try a u-sub.

$$\frac{u=x^2-7x+10}{du=(2x-7)dx}\qquad\int\frac{4x-11}{u}\cdot\frac{du}{2x-7}$$

As you can see there is no way to eliminate the x's.

Now let's try partial fraction decomposition. Using Ex1,

$$\int\frac{4x-11}{x^2-7x+10}\,dx=\int\frac{1}{x-2}\,dx+\int\frac{3}{x-5}\,dx$$

Now we know how to integrate these functions:

- $\int \frac{1}{x-2}\,dx \xrightarrow[du=dx]{u=x-2} \int \frac{1}{u}\,du = \ln|u| = \ln|x-2|$

- $\int \frac{3}{x-5}\,dx \xrightarrow[du=dx]{u=x-5} \int \frac{3}{u}\,du = 3\ln|u| = 3\ln|x-5|$

So $\int \frac{4x-11}{x^2-7x+10}\,dx = \ln|x-2| + 3\ln|x-5| + C$

Method of Decomposing into Partial Fractions

Given a rational function $\frac{N(x)}{D(x)}$

① Factor the denominator as much as possible.

② Write the fraction into decomposition form.

ⓐ Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

We will cover these next class

ⓑ Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

ⓒ Distinct irreducible quadratic terms like x^2+a^2 decompose to

$$\frac{Ax+B}{x^2+a^2}$$

ⓓ Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

③ Combine your decomposition from ② as 1 fraction.

④ Set the original numerator, $N(x)$, equal to the numerator from ③.

⑤ Equate the coefficients of the terms, to yields a system of equations. Then solve the constants. i.e. A, B, C.

⑥ Plug the values found in (5) in (2).

Example 3: Let $f(x) = \dfrac{6x+10}{x^2+5x}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^2+5x completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+5}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+5} = \frac{A}{x} \cdot \frac{(x+5)}{(x+5)} + \frac{B}{x+5} \cdot \frac{x}{x}$$

$$= \frac{A(x+5)+Bx}{x(x+5)} = \frac{Ax+5A+Bx}{x(x+5)} = \frac{(A+B)x+5A}{x(x+5)}$$

④ Set the old numerater = new numerator.

$$6x+10 = (A+B)x + 5A$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=6 & (i) \\ 5A = 10 & (ii) \end{cases}$$

From (ii), we find $A=2$.

Plug $A=2$ into (i)

$$A+B=6$$
$$2+B=6$$
$$B=4$$

⑥ Plug $A=2$ and $B=4$ into ②.

$$\frac{2}{x} + \frac{4}{x+5}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{6x+10}{x^2+5x}dx = \int \frac{2}{x}dx + \int \frac{4}{x+5}dx$$

$$= 2\ln|x| + 4\ln|x+5| + C$$

Example 4: Let $f(x) = \dfrac{40}{x^2-16}$

ⓐ Determine the partial fraction decomposition of $f(x)$.

① Factor x^2-16 completely.

$$x^2-16=(x-4)(x+4)$$

② Write the fraction into decomposition form.

$$\frac{A}{x-4}+\frac{B}{x+4}$$

③ Combine the fractions in ②.

$$\frac{A}{x-4}+\frac{B}{x+4}=\frac{A}{x-4}\cdot\frac{(x+4)}{(x+4)}+\frac{B}{x+4}\cdot\frac{(x-4)}{(x-4)}$$

$$=\frac{A(x+4)+B(x-4)}{(x-4)(x+4)}$$

$$=\frac{Ax+4A+Bx-4B}{(x-4)(x+4)}=\frac{(A+B)x+(4A-4B)}{(x-4)(x+4)}$$

④ Set the old numerator = new numerator

$$0x+40=(A+B)x+(4A-4B)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=0 \\ 4A-4B=40 \end{cases} \Rightarrow \begin{cases} B=-A & (i) \\ A-B=10 & (ii) \end{cases}$$

Plug (i) into (ii).

$$A-B=10$$
$$A-(-A)=10$$
$$A+A=10$$
$$2A=10$$
$$A=5$$

Plug $A=5$ into (i)

$$B=-A$$
$$=-5$$

⑥ Plug $A=5$ and $B=-5$ into ②.

$$\frac{5}{x-4}+\frac{-5}{x+4}$$

ⓑ Using ⓐ, evaluate $\int f(x)\,dx$.

$$\int\frac{40}{x^2-16}\,dx=\int\frac{5}{x-4}\,dx+\int\frac{-5}{x+4}\,dx$$

$$=5\ln|x-4|-5\ln|x+4|+C$$

Example 5: Let $f(x) = \dfrac{x^2+2}{x^3+3x^2+2x}$

(a) Determine the partial fraction decomposition of $f(x)$.

(1) Factor x^3+3x^2+2x completely.

$$x^3+3x^2+2x = x(x^2+3x+2)$$
$$= x(x+1)(x+2)$$

(2) Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

(3) Combine the fractions in (2).

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} = \frac{A}{x} \cdot \frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{B}{x+1} \cdot \frac{x(x+2)}{x(x+2)} + \frac{C}{x+2} \cdot \frac{x(x+1)}{x(x+1)}$$
$$= \frac{A(x^2+3x+2) + B(x^2+2x) + C(x^2+x)}{x(x+1)(x+2)}$$
$$= \frac{(A+B+C)x^2 + (3A+2B+C)x + (2A)}{x(x+1)(x+2)}$$

(4) Set the old numerator = new numerater

$$x^2+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$
$$x^2+0x+2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

(5) Create a system of equations from (4), and solve.

$$\begin{cases} A+B+C=1 & \text{(i)} \\ 3A+2B+C=0 & \text{(ii)} \\ 2A=2 & \text{(iii)} \end{cases}$$

From (iii), we find $A=1$. So we can plug that into (i) and (ii) yielding

$$\begin{cases} 1+B+C=1 \\ 3+2B+C=0 \end{cases} \Rightarrow \begin{cases} B+C=0 \\ 2B+C=-3 \end{cases} \Rightarrow \begin{cases} C=-B & \text{(i')} \\ 2B+C=-3 & \text{(ii')} \end{cases}$$

Plug (i') into (ii').

$$2B+C=-3$$
$$2B-B=-3$$
$$B=-3$$

Plug $B=-3$ into (i')

$$C=-B$$
$$=-(-3)$$
$$=3$$

(6) Plug $A=1$, $B=-3$, $C=3$ into (2).

$$\frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$$

ⓑ Using ⓐ, evaluate $\int f(x)dx$.

$$\int \frac{x^2+2}{x^3+3x^2+2x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-3}{x+1}\,dx + \int \frac{3}{x+2}\,dx$$

$$= \ln|x| - 3\ln|x+1| + 3\ln|x+2| + C$$