MATH 16020: Applied Calculus 2 Practice Exam 2 Wed., Mar. 1, 2023

Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided.

Name: Solutions

1. Determine if the following integral is proper or improper.

$$\int_0^{\pi/2} \frac{\sin x}{1-\cos x}\,dx$$

(A) It is improper because of a discontinuity at $x = \pi/6$
(B) It is improper because of a discontinuity at $x = \pi/4$
(C) It is improper because of a discontinuity at $x = \pi/3$
(D) It is improper because of a discontinuity at $x = 0$
(E) It is improper because of a discontinuity at $x = \pi/2$
(F) It is proper since it is defined on the interval $[0, \pi/2]$.

$$1-\cos x = 0$$
$$1 = \cos x$$
$$x = 0, \pi, 2\pi$$

2. Determine if the following integral is proper or improper.

$$\int_0^{\pi/2} \tan(x)\,dx$$

(A) It is improper because of a discontinuity at $x = \pi/6$
(B) It is improper because of a discontinuity at $x = \pi/4$
(C) It is improper because of a discontinuity at $x = \pi/3$
(D) It is improper because of a discontinuity at $x = 0$
(E) It is improper because of a discontinuity at $x = \pi/2$
(F) It is proper since it is defined on the interval $[0, \pi/2]$.

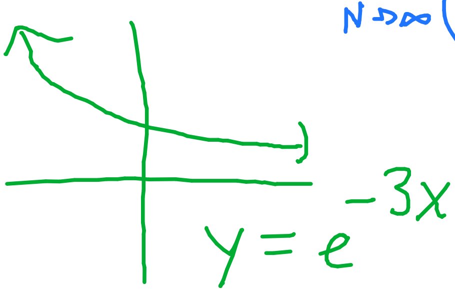
$$\tan x = \frac{\sin x}{\cos x}$$
$$\cos x = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

3. Evaluate the following integral;

$$\int_0^{\infty} e^{-3x}dx$$

$$\int_0^\infty e^{-3x}dx = \lim_{N\to\infty}\int_0^N e^{-3x}dx = \lim_{N\to\infty}\left(\frac{e^{-3x}}{-3}\right)\Bigg]_0^N$$

$$= \lim_{N\to\infty}\left(\frac{e^{-3N}}{-3}+\frac{1}{3}\right) = 0 + 1/3$$



$$\int_0^{\infty} e^{-3x}dx \quad = \quad \boxed{1/3}$$

4. Evaluate the following integral;

$$\int_1^\infty \frac{5}{\sqrt{x}}dx$$

$$\int_1^\infty \frac{5}{\sqrt{x}}dx = \lim_{N\to\infty}\int_1^N 5x^{-1/2}dx = \lim_{N\to\infty}\left(5\cdot 2x^{1/2}\right)\Big]_1^N$$

$$= \lim_{N\to\infty}\left(10(N)^{1/2} - 10\right) = \infty$$

$$\int_1^\infty \frac{5}{\sqrt{x}}dx \quad = \quad \boxed{\infty}$$

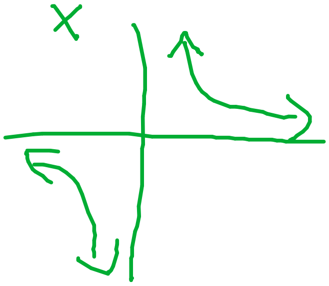
5. Evaluate the following integral;

$$\int_1^\infty \frac{3}{x^2}dx$$

$$\int_1^\infty \frac{3}{x^2}dx = \lim_{N\to\infty}\int_1^N 3x^{-2}dx = \lim_{N\to\infty}\left(\frac{3x^{-1}}{-1}\right]_1^N$$

$$= \lim_{N\to\infty}\left(-\frac{3}{x}\right)\Big]_1^N = \lim_{N\to\infty}\left(-\frac{3}{x} + 3\right)$$

$y = \frac{1}{x}$



$$\int_1^\infty \frac{3}{x^2}dx \quad = \quad \boxed{3}$$

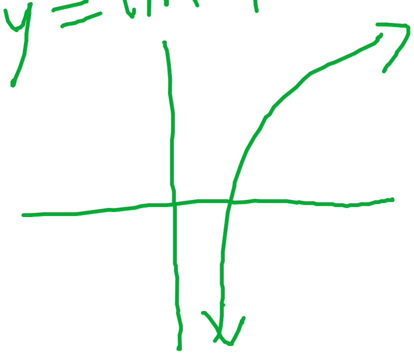
6. Evaluate the following integral;

$$\int_1^\infty \frac{10}{x}dx$$

$$\int_1^\infty \frac{10}{x}dx = \lim_{N\to\infty}\int_1^N \frac{10}{x}\,dx = \lim_{N\to\infty}\left(10\ln|x|\right)\Big]_1^N$$

$$= \lim_{N\to\infty}\left(10\ln|N| - 0\right)$$

$y = \ln|x|$



$$\int_1^\infty \frac{10}{x}dx \quad = \quad \boxed{\infty}$$

7. Set up the integral that computes the **AREA** shown to the right with respect to x.

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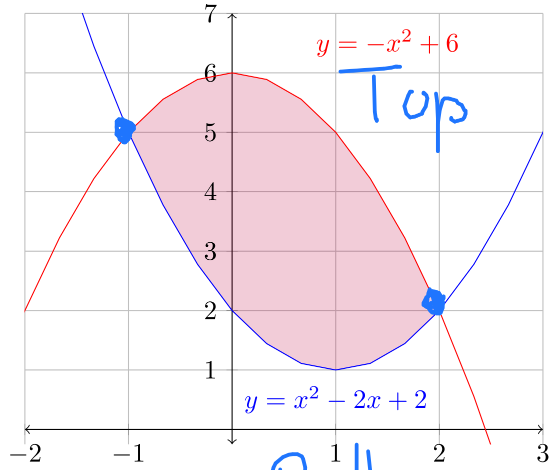

Top

Bottom

$$\boxed{\int_{-1}^{2} (-x^2+6) - (x^2-2x+2)\, dx}$$

Area = ______________________

8. Set up the integral that computes the **AREA** shown to the right with respect to y. → dy

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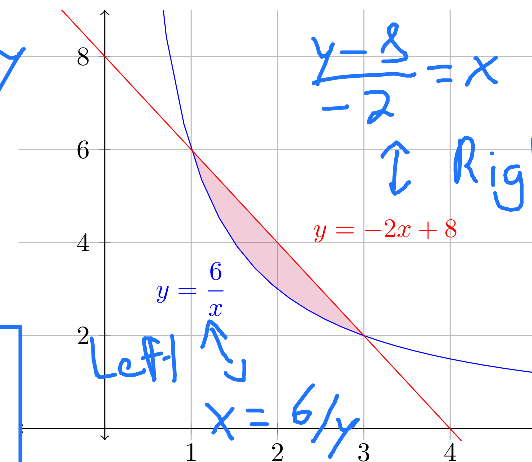


$\frac{y-8}{-2} = x$

Right

Left

$x = 6/y$

Area $= \boxed{\int_2^6 \left(\frac{y-8}{-2}\right) - \frac{6}{y}\, dy}$

9. Set up the integral that computes the **AREA** with respect to x of the region bounded by → dx problem

$$y = \frac{2}{x} \quad \text{and} \quad y = -x + 3$$

Bounds:

$\frac{2}{x} = -x + 3$

$2 = -x^2 + 3x$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1, 2$

Test Pt: $x = 1.5$

$y = \frac{2}{x} \Rightarrow y = \frac{2}{1.5} = \frac{4}{3} \approx 1.33 \to$ Bottom

$y = -x + 3 \Rightarrow y = -1.5 + 3 = 1.5 \to$ Top

Area $= \boxed{\int_1^2 \left(-x + 3 - \frac{2}{x}\right) dx}$

10. Calculate the **AREA** of the region bounded by the following curves.

$$x = 100 - y^2 \quad \text{and} \quad x = 2y^2 - 8$$

Bounds:

$100 - y^2 = 2y^2 - 8$

$108 = 3y^2$

$36 = y^2$

$y = \pm 6$

Test Pt: $y = 0$

$x = 100 - y^2 \rightarrow x = 0 \rightarrow$ Right

$x = 2y^2 - 8 \rightarrow x = -8 \rightarrow$ Left

$A = \int_{-6}^{6} (100 - y^2) - (2y^2 - 8)\, dy$

$= \int_{-6}^{6} (108 - 3y^2)\, dy$

$= (108y - y^3)\Big]_{-6}^{6}$

$= 864$

Area = $\boxed{864}$

11. Calculate the **AREA** of the region bounded by the following curves.

$$y = x^3 \quad \text{and} \quad y = x^2$$

Bounds:

$x^3 = x^2$

$x^3 - x^2 = 0$

$x^2(x - 1) = 0$

$x = 0, 1$

Test Pt: $x = \frac{1}{2}$

$y = x^3 \rightarrow y = \frac{1}{8} \rightarrow$ Bottom

$y = x^2 \rightarrow y = \frac{1}{4} \rightarrow$ Top

$A = \int_0^1 (x^2 - x^3)\, dx$

$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big]_0^1$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

Area = $\boxed{1/12}$

12. After t hours studying, one student is working $Q_1(t) = 25 + 9t - t^2$ problems per hour, and a second student is working on $Q_2(t) = 5 - t + t^2$ problems per hour. How many more problems will the first student have done than the second student after 10 hours?

$$\int_0^{10} Q_1(t) - Q_2(t)\, dt$$

$$= \int_0^{10} (25+9t-t^2)-(5-t+t^2)\, dt$$

$$= \int_0^{10} (20+10t-2t^2)\, dt$$

$$= \left(20t + 5t^2 - \frac{2}{3}t^3\right)\Big]_0^{10}$$

$$= \frac{100}{3}$$

Answer: $\boxed{100/3}$

13. Set up the integral that computes the **VOLUME** of the region bounded by

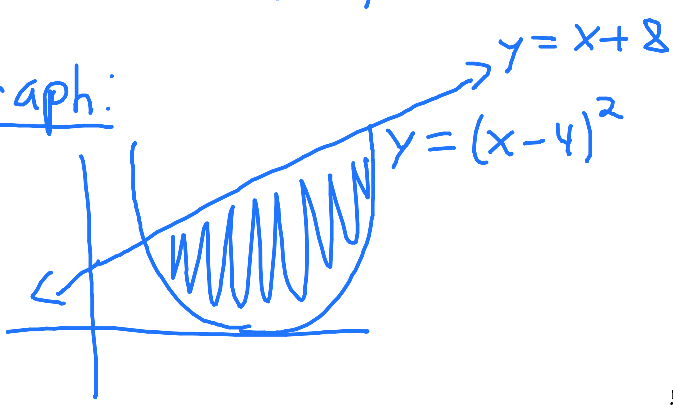
$$y = x + 8, \quad \text{and} \quad y = (x-4)^2$$

about the x-axis

Bounds:

$$x+8 = (x-4)^2$$
$$x+8 = x^2-8x+16$$
$$0 = x^2-9x+8$$
$$0 = (x-8)(x-1)$$
$$x = 1, 8$$

Graph:



Volume $= \boxed{\pi \int_1^8 \left[(x+8)^2 - (x-4)^4\right] dx}$

14. Let R be the region shown below. Set up the integral that computes the **VOLUME** as R is rotated around the x-axis.

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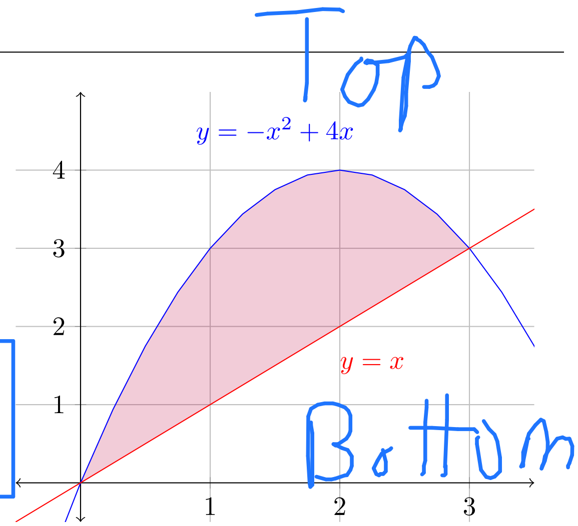


$$\boxed{\pi \int_0^3 \left[(-x^2+4x)^2 - (x)^2\right] dx}$$

Volume $=$ ____________________

15. Set up the integral that computes the **VOLUME** of the region bounded by

$$y = \sqrt{16 - x}, \quad y = 0 \quad \text{and} \quad x = 0$$

about the y-axis $\Rightarrow$ dy problem

Disk

$y = \sqrt{16-x}$
$y^2 = 16 - x$
$x = 16 - y^2$

Bounds: Given $y = 0$
Plug $x = 0$ into $y = \sqrt{16-x}$
$y = \sqrt{16-x}$
$y = \sqrt{16}$
$y = 4$

Volume $=$ $\boxed{\pi \int_0^4 (16-y^2)^2 \, dy}$

16. Find the **VOLUME** of the region bounded by

$$y = 7x, \quad y = 0 \quad x = 1 \quad \text{and} \quad x = 3$$

around the x-axis

Disk

$$V = \pi \int_1^3 (7x)^2\,dx$$
$$= \pi \int_1^3 49x^2\,dx$$
$$= \pi \left(\frac{49x^3}{3}\right)\Big]_1^3$$
$$= \frac{49\pi}{3}(3^3 - 1)$$
$$= \boxed{\frac{1274}{3}\pi}$$

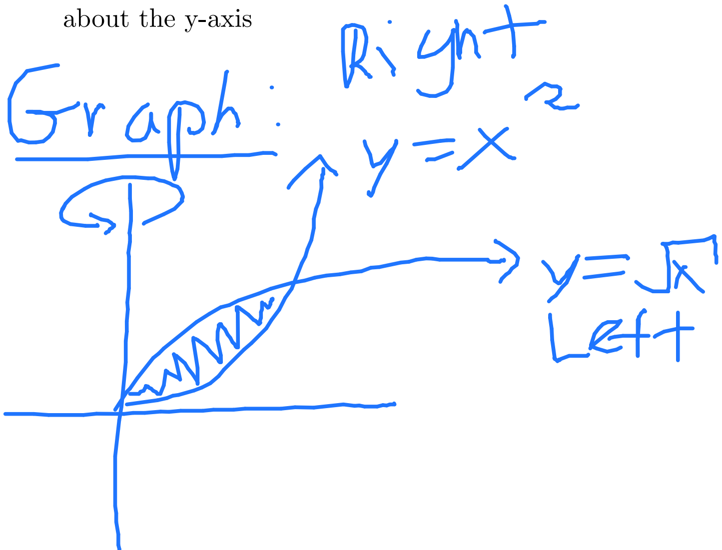
Volume = $\boxed{\frac{1274}{3}\pi}$

17. Set up the integral that computes the **VOLUME** of the region bounded by

$$y = x^2, \quad \text{and} \quad y = \sqrt{x}$$

about the y-axis

Graph:



Bounds:

$$\sqrt{y} = y^2$$
$$y = y^4$$
$$0 = y^4 - y$$
$$0 = y(y^3 - 1)$$
$$y = 0, 1$$

But y-axis $\Rightarrow dy$

Right $\to y = x^2 \to x = \sqrt{y}$

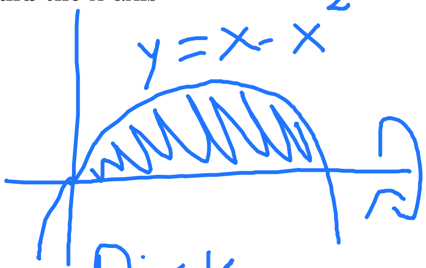
Left $\to y = \sqrt{x} \to x = y^2$

Volume = $\boxed{\pi \int_0^1 \left[(\sqrt{y})^2 - (y^2)^2\right] dy}$

18. Find the **VOLUME** of the region bounded by

$$y = x - x^2, \quad \text{and} \quad y = 0$$

around the x-axis



Disk

Bounds:

$x - x^2 = 0$

$x(1-x) = 0$

$x = 0, 1$

$V = \pi \int_0^1 (x - x^2)^2 \, dx$

$= \pi \int_0^1 (x^2 - 2x^3 + x^4) \, dx$

$= \pi \left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right) \Big]_0^1$

$= \frac{\pi}{30}$

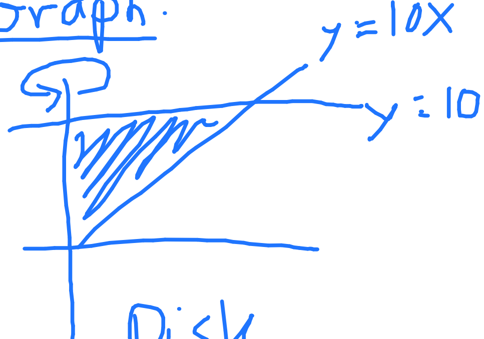
Volume $=$ $\boxed{\pi/30}$

19. Find the **VOLUME** of the region bounded by

$$y = 10x, \quad x = 0, \quad y = 10$$

around the y-axis

Graph:



Disk

But y-axis $\Rightarrow$ dy problem

$y = 10x$

$\frac{y}{10} = x$

$V = \pi \int_0^{10} \left(\frac{y}{10} \right)^2 dy$

$= \pi \int_0^{10} \frac{y^2}{100} \, dy$

$= \frac{\pi}{100} \left(\frac{y^3}{3} \right) \Big]_0^{10}$

$= \frac{10\pi}{3}$

Volume $=$ $\boxed{10\pi/3}$

20. Find the **VOLUME** of the region bounded by

$$x + 3y = 9, \quad x = 0, \quad y = 0$$

around the y-axis

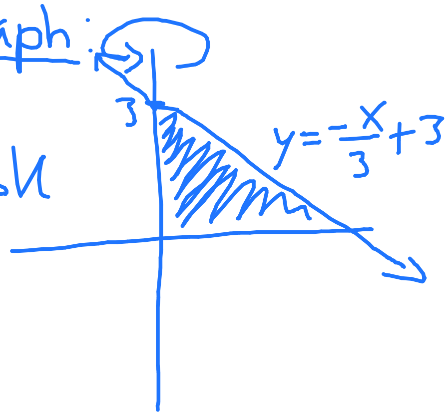
$X+3y=9$

$3y=-X+9$

$y=\frac{-X}{3}+3$

Graph:

Disk



$V=\pi\int_0^3 (9-3y)^2\,dy$

$=\pi\int_0^3 (81-54y+9y^2)\,dy$

$=\pi\left(81y-27y^2+3y^3\right)\Big]_0^3$

$=81\pi$

But y-axis $\Rightarrow$ dy

So $x+3y=9$

$x=9-3y$

Volume $=$ $\boxed{81\pi}$

21. Let R be the region shown to the right. Set up the integral that computes the **VOLUME** as R is rotated around the line $x = 4$. $\Rightarrow$ dy problem

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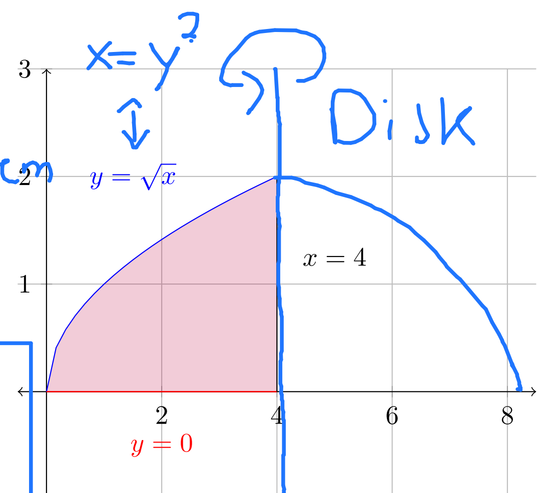


Volume $=$ $\boxed{\pi\int_0^2 (y^2-4)^2\,dy}$

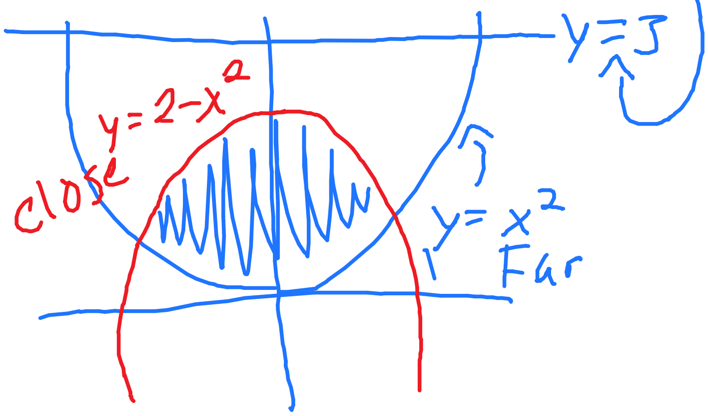
22. Set up the integral needed to find the volume of the solid obtained when the region bounded by

$$y = 2 - x^2 \quad \text{and} \quad y = x^2$$

is rotated about the line $y = 3$.

$y = 3 \Rightarrow dx$ problem

Graph   Washer



Bounds: $2 - x^2 = x^2$

$2 = 2x^2$

$1 = x^2$

$x = \pm 1$

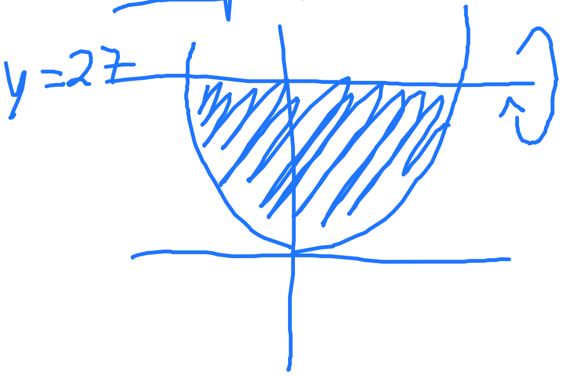
Volume = $\boxed{\pi \int_{-1}^{1} (2-x^2-3)^2 - (x^2-3)^2 \, dx}$

23. Find the **VOLUME** of the region bounded by

$$y = 3x^2, \quad x = 0, \quad y = 27$$

around the line $y = 27$

Graph   Disk



$y = 27 \Rightarrow dx$ problem

Bounds: Given $x = 0$

$27 = 3x^2$

$9 = x^2 \to x = 3$

$V = \pi \int_0^3 (3x^2 - 27)^2 \, dx$

$= \pi \int_0^3 (9x^4 - 162x^2 + 729) \, dx$

$= \pi \left(\frac{9x^5}{5} - 54x^3 + 729x \right]_0^3$

$= 11664.4\pi$

Volume = $\boxed{\frac{8322}{5}\pi}$

24. Find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Rewrite: $y\,dy = 3x^2\,dx$

$$\int y\,dy = \int 3x^2\,dx$$

$$\frac{y^2}{2} = x^3 + c$$

$$y^2 = 2x^3 + c$$

$$y = \pm\sqrt{2x^3+c}$$

$y = \boxed{\pm\sqrt{2x^3+c}}$

25. Find the general solution to the differential equation:

$$\frac{dy}{dx} = 5y$$

Rewrite $dy = 5y\,dx$

$$\frac{dy}{y} = 5\,dx$$

$$\int \frac{dy}{y} = \int 5\,dx$$

$$\ln|y| = 5x + c$$

$$|y| = e^{5x+c}$$

$$\pm y = e^c e^{5x}$$

$$y = \underbrace{\pm e^c}_{=c} e^{5x}$$

$$y = ce^{5x}$$

or memorize

$$\frac{dy}{dx} = ky \Rightarrow y = Ce^{kx}$$

$y = \boxed{Ce^{5x}}$

26. Find the general solution to the differential equation:

$$\frac{dy}{dx} = \frac{-x}{y}$$

Rewrite: $y\,dy = -x\,dx$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$y^2 = -x^2 + C$$

$$y = \pm\sqrt{C - x^2}$$

y = $\boxed{\pm\sqrt{C-x^2}}$

27. Let y denote the mass of a radioactive substance at time t. Suppose this substance obeys the equation

$$y' = -18y$$

Assume that initially, the mass of the substance is $y(0) = 20$ grams. At what time t in hours does half the original mass remain? Round your answer to 3 decimal places.

$$y' = -18y \Rightarrow y = Ce^{-18t}$$

$$y(0) = 20 \Rightarrow \cancel{28}\,20 = Ce^{-18(0)}$$

$$20 = C \qquad \Rightarrow y = 28e^{-18t}$$

We want solve $\frac{1}{2}(\cancel{28}\,20) = y(t)$ for t.

$$10 = 20e^{-18t}$$

$$\frac{1}{2} = e^{-18t}$$

$$\ln(\tfrac{1}{2}) = -18t$$

$$\frac{\ln(\frac{1}{2})}{-18} = t$$

t = $\boxed{0.039}$