Lesson 0: Review

Exponential Rules

① $x^a x^b = x^{a+b}$

② $\frac{x^a}{x^b} = x^{a-b}$

③ $(x^a)^b = x^{a \cdot b}$

④ $x^1 = x$

⑤ $x^0 = 1$

⑥ $x^{-1} = \frac{1}{x}$

Logarithmic Rules

① $\ln 1 = 0$

② $\ln(e^x) = x$

③ $e^{\ln x} = x$

④ $\ln(xy) = \ln x + \ln y$

⑤ $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

⑥ $\ln(x^m) = m \ln x$

Rational Powers

① $\sqrt{x} = x^{1/2}$

② $\sqrt[3]{x} = x^{1/3}$

③ $\sqrt[q]{x^p} = x^{p/q}$

Trigonometry

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	$0 = \frac{0}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2} = 1$
cos	$1 = \frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{0}{2} = 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$

Lesson 0: Review of Differentiation

Constant Rule: $\frac{d}{dx}(c) = 0$ where c is a constant

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ where n is any real #

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$ where C is a constant

Sum/Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Example 1: Find the derivative for the following:

ⓐ $f(x) = x^3$

$f'(x) = 3x^2$

ⓑ $f(x) = 1/x$

$f(x) = x^{-1}$

$f'(x) = -x^{-2} = -\frac{1}{x^2}$

ⓒ $f(x) = \sqrt{x^3}$

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$

ⓓ $f(x) = \frac{3}{x^4} - 2x^2 + 6x - 7$

$f(x) = 3x^{-4} - 2x^2 + 6x - 7$

$f'(x) = 3(-4)x^{-5} - 2(2)x + 6$

$= -12x^{-5} - 4x + 6$

$= -\frac{12}{x^5} - 4x + 6$

ⓔ $f(x) = \dfrac{x^{2.5} - 2x^{-3}}{x}$

$f(x) = \dfrac{x^{2.5}}{x} - \dfrac{2x^{-3}}{x} = x^{1.5} - 2x^{-4}$

$f'(x) = 1.5x^{0.5} - 2(-4)x^{-5}$

$= 1.5x^{0.5} + 8x^{-5}$

Position / Velocity / Acceleration

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

Other Differentiation Rules

$\dfrac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

$\dfrac{d}{dx}(\sec x) = \tan x \sec x$ $\qquad$ $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

$\dfrac{d}{dx}(e^x) = e^x$ $\qquad$ $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Product Rule: $\dfrac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x)$

Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{u(x)}{v(x)}\right] = \dfrac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$

Chain Rule: Let $h(x) = f(g(x))$. Then

$h'(x) = f'(g(x)) \cdot g'(x)$

Example 2: Find the derivative of the following

ⓐ $y = (3x+1)^2$

$f(x) = x^2 \qquad g(x) = 3x+1$

$f'(x) = 2x \qquad g'(x) = 3$

$y' = f'(g(x))\, g'(x)$

$= f'(3x+1) \cdot 3$

$= 2(3x+1) \cdot 3 = 6(3x+1) = 18x+6$

ⓑ $y = 2\cos^3 x$

$y = 2(\cos x)^3$

$f(x) = 2x^3 \qquad g(x) = \cos x$

$f'(x) = 6x^2 \qquad g'(x) = -\sin x$

$y' = f'(g(x)) \cdot g'(x)$

$= f'(\cos x) \cdot (-\sin x)$

$= 6(\cos x)^2 \cdot (-\sin x) = -6\cos^2 x \sin x$

ⓒ $y = \dfrac{15}{\sqrt[3]{x^2+1}}$

$y = 15(x^2+1)^{-1/3}$

$f(x) = 15x^{-1/3} \qquad g(x) = x^2+1$

$f'(x) = 15(-1/3)x^{-4/3} \qquad g'(x) = 2x$

$= -5x^{-4/3}$

$y' = f'(g(x))\, g'(x)$

$= f'(x^2+1) \cdot 2x$

$= -5(x^2+1)^{-4/3} \cdot 2x = \dfrac{-10x}{(x^2+1)^{4/3}}$

ⓓ $y = \sin(x^3)$

$f(x) = \sin x \qquad g(x) = x^3$

$f'(x) = \cos x \qquad g'(x) = 3x^2$

$y' = f'(g(x)) \cdot g'(x)$

$= f'(x^3) \cdot (3x^2)$

$= \cos(x^3) \cdot 3x^2 = 3x^2 \cos(x^3)$

Lesson 0: Review of Integration

Indefinite Integration: $\int f(x)\,dx = F(x) + C$ where C is a constant

Basic Integration Rules

- $\int 0\,dx = C$
- $\int k\,dx = kx + C$
- $\int kf(x)\,dx = k\int f(x)\,dx$
- $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1}$ ← Power Rule
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$

Recall you can check your answer by taking the derivative of it and seeing if it matches the original function.

Example 1: Evaluate the following

ⓐ $\int (6\sec^2 x - 5e^x)\,dx = 6\int \sec^2 x\,dx - 5\int e^x\,dx$

$= 6\tan x - 5e^x + C$

ⓑ $\int (x^2 + 2\sqrt{x})\,dx = \int (x^2 + 2x^{1/2})\,dx$

$= \frac{x^3}{3} + \frac{2 \cdot 2}{3}x^{3/2} + C$

$= \frac{x^3}{3} + \frac{4}{3}x^{3/2} + C$

ⓒ $\int \left(\frac{3}{x} + \sqrt[3]{x^2}\right) dx = 3\int \frac{1}{x}\,dx + \int x^{2/3}\,dx$

$= 3\ln|x| + \frac{3}{5}x^{5/3} + C$

Differential Equations

Example 2: Solve the differential equation $y' = 3x$.

Recall $y' = \frac{dy}{dx}$. So

$\int y'\,dx = \int 3x\,dx$

$\int \frac{dy}{\cancel{dx}}\,\cancel{dx} = \int 3x\,dx$

$\int dy = \int 3x\,dx$

$y = \frac{3}{2}x^2 + C$ ⇒ This is called the general solution.

What if we are given an initial condition (such as $y(0) = 2$)?

Example 3: Solve the initial value problem (IVP) $y' = 3x$ with $y(0) = 2$.

From Ex 2, $y = \frac{3}{2}x^2 + C$.

Using $y(0) = 2$, we can find C.

$2 = \frac{3}{2}(0)^2 + C \Rightarrow C = 2$

Hence $y = \frac{3}{2}x^2 + 2 \Rightarrow$ This is can a particular solution.

Definite Integrals

A definite integral looks like $\int_a^b f(x)\,dx$. Remember $+C$ isn't necessary for definite integrals.

Fundamental Theorem of Calculus (FTC)

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

where F is the antiderivative of f.

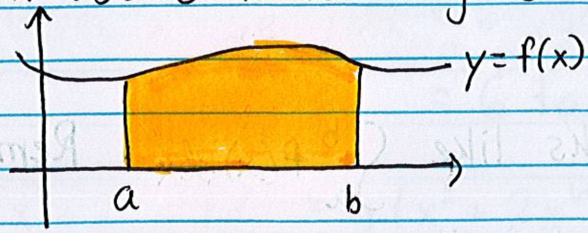
Example 4: Evaluate the following

ⓐ
$$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4}$$
$$= \tan\left(\frac{\pi}{4}\right) - \tan(0)$$
$$= 1 - 0$$
$$= 1$$

ⓑ
$$\int_1^4 \frac{x^2+x}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{x^2}{x^{1/2}} + \frac{x}{x^{1/2}}\right) dx$$
$$= \int_1^4 \left(x^{3/2} + x^{1/2}\right) dx$$
$$= \left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right)\Big]_1^4$$
$$= \left(\frac{2}{5}(4)^{5/2} + \frac{2}{3}(4)^{3/2}\right) - \left(\frac{2}{5}(1)^{5/2} + \frac{2}{3}(1)^{3/2}\right)$$
$$= \frac{2}{5}\cdot 2^5 + \frac{2}{3}\cdot 2^3 - \frac{2}{5} - \frac{2}{3}$$
$$= \frac{256}{15}$$

Area under a Curve

We can use definite integrals to find area under a curve

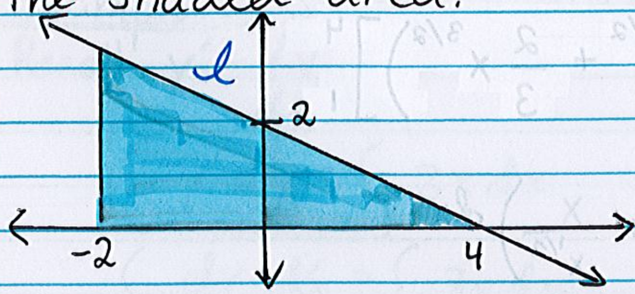


Bounded by
$y=0$, $y=f(x)$
$x=a$, $x=b$

Example 5: Find the area of the region bounded by
$y=2x+1$; $y=0$; $x=1$; $x=3$

$$\int_1^3 (2x+1)\,dx = \left(\frac{2x^2}{2}+x\right)\Big]_1^3$$
$$= (x^2+x)\Big]_1^3$$
$$= (3^2+3)-(1^2+1) = 10$$

Example 6: Write the definite integral that represents the shaded area.



We can see the bounds of the integral will be -2 to 4. So,
$$\int_{-2}^{4} \boxed{}\,dx$$

Now we need to determine the equation of ℓ. Note from the graph we have 2 points on ℓ, which are $(0,2)$ and $(4,0)$.
So the slope of ℓ is
$$m=\frac{0-2}{4-0}=\frac{-2}{4}=\frac{-1}{2}$$
Note we are also given the y-intercept of ℓ, $(0,2)$. So
$$\ell = \frac{-1}{2}x+2$$
Hence the definite integral is
$$\int_{-2}^{4}\left(\frac{-1}{2}x+2\right)dx$$