Lesson 14: Differential Equations

Definition: A differential equation is an equation that relates one or more functions and their derivatives.

e.g. ⓐ $y' = ky$ ⓑ $y' = C$ ⓒ $y'' = y$

Definition: The general solution to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Answers are of the form $y = \underline{\qquad} + C$

Definition: The particular solution is similar to the general solution but it does take initial condition.

i.e. Find the general solution. Then using the initial condition to find C. Plug C back into the general solution and done.

Note: Solving the Initial Value Problem (IVP) is the same to finding the particular solution.

Ⓐ Growth & Decay

Example 1: Consider the differential equation $\frac{dy}{dx} = ky$ where the proportionally constant $k>0$. Find the general solution.

Idea: Try to get terms w/ y on one-sided and x on the other.

$$\frac{dy}{dx} = ky$$

$$dx\,\frac{dy}{dx} = ky\,dx$$

$$dy = ky\,dx$$

$$\frac{1}{y}\,dy = \frac{1}{y}(ky)$$

$$\frac{1}{y}\,dy = k\,dx$$

Now integrate

$$\int \frac{1}{y}\,dy = \int k\,dx$$

$$\ln|y| = kx + c$$

$$e^{\ln|y|} = e^{kx+c}$$

$$|y| = e^{kx}e^{c}$$

$$\pm y = e^{kx}e^{c}$$

$$y = \underbrace{\pm e^{c}}e^{kx}$$

All of this is a constant... So call it all C.

$$y = Ce^{kx}$$

In the future, proportionality $\Rightarrow y' = ky \Rightarrow y = Ce^{kt}$

Example 2: Suppose that $y' = ky$, $y(0) = 5$, and $y'(0) = 10$. What is y as a function of t?

$$y' = ky \Rightarrow y = Ce^{kt}$$

When $y(0) = 5$,

$$5 = Ce^{k(0)} = C \quad \Rightarrow \quad y = 5e^{kt}$$

Find y': $y' = 5ke^{kt}$

When $y'(0) = 10$,

$$10 = 5ke^{k(0)} = 5k$$

$$k = 2 \quad\quad \Rightarrow y = 5e^{2t}$$

Also recall that half-life constant is denoted as

$$k = \frac{\ln(1/2)}{\text{half-life}} = \frac{-\ln 2}{\text{half-life}}$$

Example 3: A radioactive element decays with a half-life of 8 years. If a mass of the element weighs 6 pounds at $t=0$, find the amount of the element after 11.9 yrs.

Recall $y' = ky \Rightarrow y = Ce^{kt}$ and $k = \frac{-\ln(2)}{\text{half-life}} = \frac{-\ln 2}{8}$

So putting them together,

$$y = C\exp\left[-\frac{\ln 2}{8}t\right]$$

We also know $y=6$ when $t=0$

$$6 = C\exp\left[-\frac{\ln 2}{8}\cdot 0\right]$$
$$6 = Ce^0$$
$$6 = C$$

So $y = 6\exp\left[-\frac{\ln 2}{8}t\right]$

$$y(11.9) = 6\exp\left[-\frac{\ln 2}{8}\cdot(11.9)\right] \approx 2.1398 \text{ years}$$

Example 4: Let y denote the mass of a radioactive substance at time t. Suppose this substance obeys the equation $y' = -18y$. Assume that, initially, the mass of the substance is $y(0) = M > 0$. At what time does half of the mass remain?

Recall $y' = ky \Rightarrow y = Ce^{kt}$

$y' = -18y \Rightarrow y = Ce^{-18t}$

When $y(0) = M$,

$$M = y(0) = Ce^{-18\cdot 0}$$
$$M = C$$

So $y = Me^{-18t}$

Note we want t when $y(t) = \frac{1}{2}M$

$$Me^{-18t} = y(t) = \tfrac{1}{2}M$$
$$e^{-18t} = \tfrac{1}{2}$$
$$-18t = \ln(\tfrac{1}{2})$$
$$t = \frac{\ln(\frac{1}{2})}{-18} \approx 0.0385$$

⑬ Separation of Variables Intro

The technique used in Example 1 is called Separation of Variables.

Example 5: Solve the IVP:

$$\frac{dy}{dx} = 5x \quad \text{when } y=10,\ x=0$$

Solve like we did in example 1.

$$dy = 5x\,dx$$

$$\int dy = \int 5x\,dx$$

$$y = \frac{5x^2}{2} + C$$

Plug $y=10$, $x=0$ to find C.

$$10 = \frac{5}{2}(0)^2 + C$$

$$10 = C$$

So $y = \frac{5x^2}{2} + 10$

Example 6: Find the general solution of the differential equation $\frac{dy}{dx} = \frac{-x}{y}$

Solve like we did in Example 1.

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$y^2 = -x^2 + \underbrace{2C}$$

This is a constant... So let it be C.

$$y^2 = -x^2 + C$$

$$y = \pm\sqrt{C - x^2}$$