Lesson 15: Separation of Variables

Recall from Last Time,

If a differential equation can be written in the form $g(y)\,dy = h(x)\,dx$ from $\dfrac{dy}{dx} = \dfrac{h(x)}{g(y)}$

Then
$$\int g(y)\,dy = \int h(x)\,dx$$
$$G(y) + C_1 = H(x) + C_2$$
$$G(y) = H(x) + C_2 - C_1$$
$$G(y) = H(x) + C$$

Example 1: Solve the IVP:
$$\frac{dV}{dt} = 2\sqrt{V} \qquad V(0) = 0$$

Rewrite:
$$\frac{dV}{\sqrt{V}} = 2\,dt$$
$$V^{-1/2}\,dV = 2\,dt$$
$$\int V^{-1/2}\,dV = \int 2\,dt$$
$$2V^{1/2} = 2t + C$$
$$V^{1/2} = t + \underbrace{\frac{C}{2}}_{\text{All of this is a constant.}}$$
$$V^{1/2} = t + C$$
$$V = (t + C)^2$$

When $V(0) = 0$,
$$0 = V(0) = (0 + c)^2$$
$$0 = c^2$$
$$0 = c$$

Hence $V = t^2$

Example 2: Find the general solution for the following differential equations:

ⓐ $\frac{dy}{dt} = y\sin(t)$

Rewrite: $\frac{dy}{y} = \sin(t)\,dt$

$$\int \frac{dy}{y} = \int \sin(t)\,dt$$

$$\ln|y| = -\cos(t) + C$$

$$\exp[\ln|y|] = \exp[-\cos(t) + C]$$

$$|y| = \exp[-\cos(t)] \cdot e^C$$

$$\pm y = e^C \cdot \exp[-\cos(t)]$$

$$y = \underbrace{\pm e^C}_{\text{All a constant}} \cdot \exp[-\cos(t)]$$

$$y = C\exp[-\cos(t)]$$

ⓑ $\frac{dy}{dt} = 7e^{-4t-y}$

Rewrite: $\frac{dy}{dt} = 7e^{-4t}e^{-y}$

$$e^y\,dy = 7e^{-4t}\,dt$$

$$\int e^y\,dy = \int 7e^{-4t}\,dt$$

Done with a u-sub.

$$e^y = 7 \cdot \frac{-1}{4}e^{-4t} + C$$

$$e^y = \frac{-7}{4}e^{-4t} + C$$

$$\ln(e^y) = \ln\left(\frac{-7}{4}e^{-4t} + C\right)$$

$$y = \ln\left(\frac{-7}{4}e^{-4t} + C\right)$$

ⓒ $\frac{dy}{dx} = 3x^2(5+y)$

Rewrite: $\frac{dy}{5+y} = 3x^2\,dx$

$$\int \frac{dy}{5+y} = \int 3x^2\,dx$$

$$\ln|5+y| = \frac{3x^3}{3} + C$$

$$\ln|5+y| = x^3 + C$$

$$\exp[\ln|5+y|] = \exp[x^3 + C]$$

$$|5+y| = \exp[x^3]\cdot e^C$$

$$\pm(5+y) = e^C \cdot \exp[x^3]$$

$$5+y = \underbrace{\pm e^C}_{\text{All a constant}} \cdot \exp[x^3]$$

$$5+y = C\exp[x^3]$$

$$y = C\exp[x^3] - 5$$

ⓓ $\frac{dy}{dx} = \frac{5x+1}{4y^2}$

Rewrite: $4y^2\,dy = (5x+1)\,dx$

$$\int 4y^2\,dy = \int (5x+1)\,dx$$

$$\frac{4y^3}{3} = \frac{5x^2}{2} + x + C$$

$$y^3 = \frac{3}{4}\left(\frac{5}{2}x^2 + x + C\right)$$

$$y^3 = \frac{15}{8}x^2 + \frac{3}{4}x + \underbrace{\frac{3}{4}C}_{\text{All a constant}}$$

$$y^3 = \frac{15}{8}x^2 + \frac{3}{4}x + C$$

$$y = \left(\frac{15}{8}x^2 + \frac{3}{4}x + C\right)^{1/3}$$

Example 3: Find the general solution of the differential eqn:

(a) $t^2 \frac{dy}{dt} - y = 0$

Rewrite: $t^2 \frac{dy}{dt} = y$

$$\frac{dy}{y} = \frac{dt}{t^2}$$

$$\frac{dy}{y} = t^{-2} dt$$

$$\int \frac{dy}{y} = \int t^{-2} dt$$

$$\ln|y| = -t^{-1} + C$$

$$\ln|y| = \frac{-1}{t} + C$$

$$|y| = \exp\left[\frac{-1}{t} + C\right]$$

$$\pm y = e^C \exp\left[\frac{-1}{t}\right]$$

$$y = \underbrace{\pm e^C}_{\text{All a constant}} \exp\left[\frac{-1}{t}\right]$$

$$y = C \exp\left[\frac{-1}{t}\right]$$

(b) $-x^3 y + y' = 2x^3$

Rewrite: $y' = 2x^3 + x^3 y$

$$\frac{dy}{dx} = x^3(2+y)$$

$$\frac{dy}{2+y} = x^3 dx$$

$$\int \frac{dy}{2+y} = \int x^3 dx$$

$$\ln|2+y| = \frac{x^4}{4} + C$$

$$|2+y| = \exp\left[\frac{x^4}{4} + C\right]$$

$$\pm(2+y) = e^C \exp\left[\frac{x^4}{4}\right]$$

$$2+y = \underbrace{\pm e^C}_{\text{All a constant}} \exp\left[\frac{x^4}{4}\right]$$

$$2+y = C\exp\left[\frac{x^4}{4}\right]$$

$$y = C\exp\left[\frac{x^4}{4}\right] - 2$$

Example 4: In a particular chemical reaction, a substance is converted into a second substance at a rate proportional to the square of the amount of the first substance present at any time, t. Initially, 50 grams of the first substance was present, and 1 hour later only 14 grams of the first substance remained. What is the amount of the first substance remaining after 7 hours?

Set-Up: $\frac{da}{dt} = a^2 k$; $a(0) = 50$; $a(1) = 14$

Solve: $\frac{da}{a^2} = k\,dt$

$$\int a^{-2}\,da = \int k\,dt$$

$$-a^{-1} = kt + C$$

$$-\frac{1}{a} = kt + C$$

$$\frac{1}{a} = -kt \underbrace{- C}_{\text{All a constant}}$$

$$\frac{1}{a} = -kt + C$$

$$a = \frac{1}{-kt + C}$$

When $a(0) = 50$,

$$50 = a(0) = \frac{1}{C}$$

$$C = 1/50$$

$$\text{So } a = \frac{1}{1/50 - kt} = \frac{50}{1 - 50kt}$$

When $a(1) = 14$,

$$14 = a(1) = \frac{50}{1 - 50k}$$

$$14(1 - 50k) = 50$$

$$1 - 50k = \frac{50}{14} = \frac{25}{7}$$

$$-50k = \frac{25}{7} - 1 = \frac{18}{7}$$

$$k = \frac{-1}{50} \cdot \frac{18}{7} = \frac{-18}{350}$$

$$\text{So } a = \frac{50}{1 - 50\left(\frac{-18}{350}\right)t} = \frac{350}{7 + 18t}$$

Hence $a(7) \approx 2.6316$ grams

Example 5: A wet towel hung on a clothesline to dry outside loses moisture at a rate proportional to its moisture content. After 1 hour, the towel has lost 32% of its original moisture content. After how long will the towel have lost 74% of its moisture content?

Let $M(t)$ = % moisture in t hrs, and

proportional $\Rightarrow M' = kM \Rightarrow M = Ce^{kt}$

and $M(0) = 1$ (b/c 100%) $M(1) = 1 - 0.32 = 0.68$

When $M(0) = 1$,
$1 = Ce^0 = C$
So $M = 1 \cdot e^{kt} = e^{kt}$

When $M(1) = 0.68$
$0.68 = e^k$
$\ln(0.68) = k$
So $M = \exp[t \ln(0.68)]$

Solve $M(t) = 0.26$ for t. b/c 74% lost $\Rightarrow$ 26% moisture

$$0.26 = \exp[t \ln(0.68)]$$
$$\ln(0.26) = t \ln(0.68)$$
$$t = \frac{\ln(0.26)}{\ln(0.68)} \approx 3.493$$