

Lesson 18: Geometric Series & Convergence

Definition: Given an infinite sequence a_0, a_1, a_2, \dots of #s if we add all of the numbers in the sequence together we have an infinite series.

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \dots$$

Definition: If we look at just the first n terms in the series, this is called the n -th partial sum.

$$S_n = \sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n$$

$$S_n = \sum_{k=0}^{n-1} a_k = a_0 + a_1 + \dots + a_{n-1}$$

Note that in either case, we are adding up the first n terms of the series. The only difference is the indexing.

The series is said to be convergent if $\lim_{n \rightarrow \infty} S_n$ exists and

is equal to a finite real number. If $\lim_{n \rightarrow \infty} S_n$ is infinite

or does not exist, then the series is said to be divergent.

Example 1: Find the fourth partial sum of the series of

$$\sum_{n=1}^{\infty} n^2$$

Remember we want to find the first four terms and sum them.

$$S_4 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

Example 2: Use summation notation to write the series in compact form.

$$\textcircled{a} e + \frac{e^2}{2} + \frac{e^3}{6} + \frac{e^4}{24} + \frac{e^5}{120} + \dots$$

First, look at the numerators, we see $e, e^2, e^3, e^4, e^5, \dots$
If we start the series at $n=1$, we can see a_n will

have e^n in the numerator.

Now the denominator is tricky. So let's find a pattern

2 \rightarrow 6 multiply by 3

6 \rightarrow 24 multiply by 4

24 \rightarrow 120 multiply by 5

\vdots

\vdots
multiply by n

Overall, we can say the series is

$$\sum_{n=1}^{\infty} \frac{e^n}{2 \cdot 3 \cdot 4 \cdot \dots \cdot n}$$

Note we can rewrite the bottom using factorial

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$

So
$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

⑥
$$-3 + \frac{9}{4} - \frac{27}{9} + \frac{81}{16} - \dots$$

First notice that each term alternates from $-$ and $+$. So a_n will have $(-1)^n$ term, assuming we start the sum with $n=1$.

Secondly, the numerators are just powers of 3. So a_n 's numerator will have 3^n .

Lastly, the denominators are perfect squares. So a_n 's denominator will have n^2 .

Put all those points together and we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n^2}$$

Example 3: Use summation notation to write the series in compact form.

$0.\overline{2}$

$$\text{Note } 0.\overline{2} = 0.2222\dots = \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \dots$$

$$= \frac{2}{10} \left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \right)$$

$$= \frac{2}{10} \left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots \right)$$

$$= \frac{2}{10} \sum_{n=0}^{\infty} \left(\frac{1}{10} \right)^n$$

Definition: If $0 < |r| < 1$, then $\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r}$ is a geometric sum.

Example 4: Compute

$$\textcircled{a} \sum_{n=1}^{\infty} \left(\frac{9}{22} \right)^n$$

First we see that n starts at 1 not 0. So let's rewrite the sum.

$$\sum_{n=1}^{\infty} \left(\frac{9}{22} \right)^n = \frac{9}{22} + \left(\frac{9}{22} \right)^2 + \left(\frac{9}{22} \right)^3 + \dots$$

$$= \frac{9}{22} \left(1 + \frac{9}{22} + \left(\frac{9}{22} \right)^2 + \dots \right)$$

$$= \frac{9}{22} \sum_{n=0}^{\infty} \left(\frac{9}{22} \right)^n$$

Check if $r = \frac{9}{22}$ is between $0 < |r| < 1$, which it is. So

apply the formula.

$$\frac{9}{22} \sum_{n=0}^{\infty} \left(\frac{9}{22} \right)^n = \frac{9/22}{1 - 9/22} = \frac{9/22}{13/22} = \frac{9}{13}$$

$$\textcircled{b} \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$$

We see $n=0$. So we immediately jump to checking if $r = \frac{3}{2}$ is indeed $0 < |r| < 1$. Which it doesn't \Rightarrow diverges

$$\textcircled{c} \sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}}$$

First we see that n starts at 1 not 0, so let's rewrite the sum.

$$\sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} = \frac{3^1}{4^3} + \frac{3^2}{4^4} + \frac{3^3}{4^5} + \dots$$

$$= \frac{3}{4^3} \left(1 + \frac{3}{4} + \frac{3^2}{4^2} + \dots \right)$$

$$= \frac{3}{64} \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots \right)$$

$$= \frac{3}{64} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

Check if $r = \frac{3}{4}$ is between $0 < |r| < 1$, which it is. So

apply the formula.

$$\frac{3}{64} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3/64}{1-3/4} = \frac{3/64}{1/4} = \frac{3}{16}$$

$$\textcircled{d} \sum_{n=0}^{\infty} \left(\frac{3}{7^n} + \frac{4}{5^n}\right)$$

First, let's rewrite the sum.

$$\sum_{n=0}^{\infty} \left(\frac{3}{7^n} + \frac{4}{5^n}\right) = \sum_{n=0}^{\infty} 3\left(\frac{1}{7}\right)^n + \sum_{n=0}^{\infty} 4\left(\frac{1}{5}\right)^n$$

Now check if both $()^n$ terms satisfy $0 < |r| < 1$. So $r_1 = \frac{1}{7}$ is between 0 and 1, and so $r_2 = \frac{1}{5}$. Hence

we can use the formula on each.

$$\sum_{n=0}^{\infty} 3 \left(\frac{1}{7}\right)^n = \frac{3}{1-1/7} = \frac{3}{6/7} = \frac{3}{1} \cdot \frac{7}{6} = \frac{7}{2}$$

$$\sum_{n=0}^{\infty} 4 \left(\frac{1}{5}\right)^n = \frac{4}{1-1/5} = \frac{4}{4/5} = \frac{4}{1} \cdot \frac{5}{4} = 5$$

Hence $\sum_{n=0}^{\infty} \left(\frac{3}{7^n} + \frac{4}{5^n}\right) = \frac{7}{2} + 5 = \frac{17}{2}$

(e) $\sum_{n=0}^{\infty} 10 e^{-0.6n}$

First, let's rewrite the sum.

$$\sum_{n=0}^{\infty} 10 e^{-0.6n} = \sum_{n=0}^{\infty} 10 (e^{-0.6})^n$$

Check if $r = e^{-0.6}$ is between $0 < |r| < 1$, which it is. So apply the formula,

$$\sum_{n=0}^{\infty} 10 (e^{-0.6})^n = \frac{10}{1 - e^{-0.6}}$$

(f) $\sum_{n=0}^{\infty} \frac{6(-1)^n}{3^{2n}}$

First, let's rewrite the sum.

$$\sum_{n=0}^{\infty} \frac{6(-1)^n}{3^{2n}} = \sum_{n=0}^{\infty} \frac{6(-1)^n}{(3^2)^n} = \sum_{n=0}^{\infty} 6 \left(\frac{-1}{3^2}\right)^n = \sum_{n=0}^{\infty} 6 \left(\frac{-1}{9}\right)^n$$

Check if $r = -\frac{1}{9}$ is between $0 < |r| < 1$, which it is. So apply

the formula.

$$\sum_{n=0}^{\infty} 6 \left(\frac{-1}{9}\right)^n = \frac{6}{1 - (-1/9)} = \frac{6}{10/9} = \frac{6}{10} \cdot \frac{9}{1} = \frac{27}{5}$$