Lesson 19: Power Series

Recall from Last Time

A geometric series has the form

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{where } |r|<1$$

Today will be focussing on when $a=1$ and $r=x$.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{where } |x|<1$$

series of this form are called Power Series.

One more definition

If $|x|<R$ guarantees the power series $\sum_{n=0}^{\infty} ax^n$ converges, then the value R is called the radius of convergence.

Example 1: Express $f(x)$ as a power series and it's radius of convergence.

ⓐ $f(x) = \frac{1}{1+x}$

Let's rewrite $f(x)$ to look our Power Series Formula.

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

So replace x with $-x$ in our formula. i.e.

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} \quad \text{where } |-x|<1$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} \quad \text{where } |x|<1$$

Moreover, we see the radius of convergence is

$$R=1 \quad b/c \quad |x|<1$$

(b) $f(x) = \dfrac{1}{3-x}$

Let's rewrite $f(x)$ to look like our Power Series

$$\frac{1}{3-x} = \frac{1}{3(1-x/3)} = \frac{1}{3} \cdot \frac{1}{1-x/3}$$

So replace x with $x/3$ in our formula. i.e.

$$\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{1-x/3} \quad \text{where } \left|\frac{x}{3}\right| < 1$$

$$\frac{1}{3}\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{1}{3} \cdot \frac{1}{1-x/3} \quad \text{where } \frac{|x|}{3} < 1$$

$$\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} = \frac{1}{3-x} \quad \text{where } |x| < 3$$

Moreover, we see the radius of convergence is $R = 3$ b/c $|x| < 3$

(c) $f(x) = \dfrac{2x}{5+x^2}$

Let's rewrite $f(x)$ to look like our Power Series.

$$\frac{2x}{5+x^2} = \frac{2x}{5(1+x^2/5)} = \frac{2x}{5} \cdot \frac{1}{1-(-x^2/5)}$$

So replace x with $-x^2/5$ in our formula. i.e.

$$\sum_{n=0}^{\infty} \left(\frac{-x^2}{5}\right)^n = \frac{1}{1-(-x^2/5)} \quad \text{where } \left|\frac{-x^2}{5}\right| < 1$$

$$\frac{2x}{5}\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{5^n} = \frac{2x}{5} \cdot \frac{1}{1+x^2/5} \quad \text{where } \frac{|x^2|}{5} < 1$$

$$\frac{2x}{5}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{5^n} = \frac{2}{5+x^2} \quad \text{where } |x^2| < 5$$

$$\sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+1}}{5^{n+1}} = \frac{2}{5+x^2} \quad \text{where } -5 < x^2 < 5$$

$$0 < x^2 < 5$$

$$-\sqrt{5} < x < \sqrt{5}$$

Moreover, we see the radius of converge is $R = \sqrt{5}$ b/c $|x| < \sqrt{5}$

Application of Power Series

- Some integrals cannot necessarily be evaluated by the techniques used earlier in the course. BUT it is possible to evaluate them as a power series.

Estimating Integrals Using Power Series

① Rewrite the integrand as a power series
② Integrate the power series

If dealing with an indefinite integral, stop here. If not, proceed with the following steps.

③ Write out the terms that are required for evaluation
④ Evaluate by substituting the bounds into the determined terms.

Example 2: Evaluate as a power series: $\int \frac{4}{3+x^2}\,dx$

① Rewrite $\frac{4}{3+x^2}$ as a power series.

$$\frac{4}{3+x^2} = \frac{4}{3(1+x^2/3)} = \frac{4}{3}\cdot\frac{1}{1-(-x^2/3)}$$

$$\text{So } \sum_{n=0}^{\infty}\left(\frac{-x^2}{3}\right)^n = \frac{1}{1-(-x^2/3)}$$

$$\frac{4}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{3^n} = \frac{4}{3}\cdot\frac{1}{1+x^2/3}$$

$$\sum_{n=0}^{\infty}\frac{4(-1)^n x^{2n}}{3^{n+1}} = \frac{4}{3+x^2}$$

② Integrate

$$\int\frac{4}{3+x^2}\,dx = \int\sum_{n=0}^{\infty}\frac{4(-1)^n x^{2n}}{3^{n+1}}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{4(-1)^n}{3^{n+1}}\int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{4(-1)^n}{3^{n+1}} \cdot \frac{x^{2n+1}}{2n+1}$$

Example 3: Evaluate using the first three terms of the series. Round to four decimals.

$$\int_0^{0.15} \frac{4}{3+x^2}\,dx$$

Note Steps ① and ② in Example 4. So

$$\int_0^{0.15} \frac{4}{3+x^2}\,dx = \sum_{n=0}^{\infty} \frac{4(-1)^n}{3^{n+1}} \cdot \frac{x^{2n+1}}{2n+1}$$

③ Write out the first 3 terms

$$\frac{4(-1)^0}{3^{0+1}} \cdot \frac{x^{2(0)+1}}{2(0)+1} + \frac{4(-1)^1}{3^{1+1}} \cdot \frac{x^{2(1)+1}}{2(1)+1} + \frac{4(-1)^2}{3^{2+1}} \cdot \frac{x^{2(2)+1}}{2(2)+1}$$

$$= \frac{4x}{3} - \frac{4x^3}{27} + \frac{4x^5}{135}$$

④ Evaluate ③ from 0 to 0.15

$$\left(\frac{4(0.15)}{3} - \frac{4(0.15)^3}{27} + \frac{4(0.15)^5}{135}\right) - (0) \approx 0.1995$$