Lesson 1: Integration By Substitution I

Warm-Up: Determine the Inner/Outer Functions for the following:

ⓐ $h(x) = \sin^3 x$

Note $h(x) = (\sin x)^3$

So outer function $\Rightarrow f(x) = x^3$

inner function $\Rightarrow g(x) = \sin x$

Check $h(x) = f(g(x))$ is true.

$$f(g(x)) = f(\sin x)$$
$$= (\sin x)^3 = h(x)$$

ⓑ $h(x) = \sqrt{3x+2}$

So outer function $\Rightarrow f(x) = \sqrt{x}$

inner function $\Rightarrow g(x) = 3x+2$

Check $h(x) = f(g(x))$ is true

$$f(g(x)) = f(3x+2)$$
$$= \sqrt{3x+2} = h(x)$$

ⓒ $h(x) = \tan(3x^2)$

So outer function $\Rightarrow f(x) = \tan x$

inner function $\Rightarrow g(x) = 3x^2$

Check $h(x) = f(g(x))$ is true

$$f(g(x)) = f(3x^2)$$
$$= \tan(3x^2) = h(x)$$

Integration By Substitution is kinda of the Integration Version of the Chain Rule. Also referred to as "Change of Variables."

Example 1: Find $\int 2x\cos(x^2)\,dx$.

Idea to solve Example 1 is to undo the chain rule.
First determine if you have a function within a function.
In this example,

$$\cos(x^2)$$

where $\cos(x)$ is the outer function, and
x^2 is the inner function.

What's the derivative of the inner function?

$$2x$$

Do you see $2x$ in the integrand?
YES!!!

Let's recall chain rule: $y' = f'(g(x))\cdot g'(x)$ for $y = f(g(x))$
So from our example: $y' = \cos(x^2)\cdot 2x$
i.e. $f(x) =$ $\qquad$ $g(x) = x^2$
$f'(x) = \cos x$ $\qquad$ $g'(x) = 2x$

So what's $f(x)$?

$$f(x) = \int f'(x)\,dx = \int \cos x\,dx = \sin x$$

Hence $y = f(g(x)) = \sin(x^2)$.
i.e. $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

How to fast track this method?
Do a change of variable using the inner function.

Solution to Example 1:

$$\int \underbrace{\cos(x^2)}_{u}\cdot\underbrace{2x\,dx}_{du} \xrightarrow[du=2x\,dx]{u=x^2} \int \cos(u)\,du$$

$$= \sin(u) + C$$

But if we start w/ a function of x. We want to end w/ a function of x. So sub back $u = x^2$.

$$\int \cos(x^2) \cdot 2x\,dx = \sin(x^2) + C$$

Note we may not see what du equals in our integrand. So you may have to do some equation manipulation.

Example 2: Compute the following integrals:

ⓐ $\int \sqrt{4x+1}\,dx \quad \dfrac{u = 4x+1}{du = 4dx} \quad \int \sqrt{u} \cdot \dfrac{du}{4} = \dfrac{1}{4}\int u^{1/2}\,du$

$\dfrac{du}{4} = dx$

$$= \frac{1}{4} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{6}(4x+1)^{3/2} + C$$

ⓑ $\int 4e^{2x}\,dx \quad \dfrac{u = 2x}{du = 2dx} \quad \int 4e^{u} \cdot \dfrac{du}{2} = 2\int e^{u}\,du$

$\dfrac{du}{2} = dx$

$$= 2e^{u} + C = 2e^{2x} + C$$

ⓒ $\int 3x^5 e^{x^6} \quad \dfrac{u = x^6}{du = 6x^5 dx} \quad \int 3x^5 e^{u} \cdot \dfrac{du}{6x^5} = \int \dfrac{1}{2} e^{u}\,du$

$\dfrac{du}{6x^5} = dx$

$$= \frac{1}{2}e^{u} + C = \frac{1}{2}e^{x^6} + C$$

ⓓ $\int \dfrac{3\sin x}{\cos^8 x}\,dx = \int \dfrac{3\sin x}{(\cos x)^8}\,dx$

$\dfrac{u = \cos x}{du = -\sin x\,dx} \quad \int \dfrac{3\sin x}{u^8} \cdot \dfrac{du}{-\sin x} = -3\int \dfrac{du}{u^8}$

$\dfrac{du}{-\sin x} = dx$

$$= -3\int u^{-8}\,du = -3\frac{u^{-7}}{-7} + C = \frac{3}{7}(\cos x)^{-7} + C$$

ⓔ $\int 5e^{\tan(14x)} \cdot \sec^2(14x)\,dx$ $\quad u = \tan(14x)$

$du = \sec^2(14x) \cdot 14\,dx$

$\frac{du}{\sec^2(14x) \cdot 14} = dx$

$$= \int 5e^u \cancel{\sec^2(14x)} \cdot \frac{du}{\cancel{\sec^2(14x)} \cdot 14} = \int \frac{5}{14} e^u\,du$$

$$= \frac{5}{14} e^u + C = \frac{5}{14} e^{\tan(14x)} + C$$

Example 3: Find the function $f(x)$ whose tangent line has the slope $\frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$ for any $x > 0$ and whose graph passes through the point $(9, 5/3)$.

$$f'(x) = \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}$$

So $f(x) = \int f'(x)\,dx = \int \frac{(1+\sqrt{x})^{1/2}}{4\sqrt{x}}\,dx$ $\quad u = 1+\sqrt{x} = 1 + x^{1/2}$

$du = \frac{1}{2} x^{-1/2}\,dx = \frac{dx}{2\sqrt{x}}$

$2\sqrt{x}\,du = dx$

$$= \int \frac{u^{1/2}}{4\sqrt{x}} \cdot 2\sqrt{x}\,du = \int \frac{1}{2} u^{1/2}\,du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3}(1+\sqrt{x})^{3/2} + C$$

Now we need to find C. (w/ $(9, 5/3)$)

$$\frac{5}{3} = f(9) = \frac{1}{3}(1+\sqrt{9})^{3/2} + C$$

$$\frac{5}{3} = \frac{8}{3} + C$$

$$-1 = \frac{-3}{3} = C$$

So $f(x) = \frac{1}{3}(1+\sqrt{x})^{3/2} - 1$