

Lesson 20: Maclaurin Series

A special case of a Power Series is called a Maclaurin Series.

Definition: A Maclaurin Series has the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \frac{f^{(n)}(0)}{n!}, \text{ and } |x| < R$$

In other words,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Let's see how to use this.

Example 1: Find the Maclaurin series representation of $f(x) = \sin x$
Start by getting the derivatives and evaluate at $x=0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$f(x) = \sin x$	$f(0) = 0$
1	$f'(x) = \cos x$	$f'(0) = 1$
2	$f''(x) = -\sin x$	$f''(0) = 0$
3	$f'''(x) = -\cos x$	$f'''(0) = -1$
4	$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
5	$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$
6	$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$
7	$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -1$

$$\begin{aligned} \text{So } & f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots \\ &= 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \frac{0}{6!} x^6 + \frac{-1}{7!} x^7 + \dots \\ &= \frac{1}{1!} x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x) \end{aligned}$$

The Maclaurin series for other functions can be found in a similar fashion. We will be kind enough to give them to you now and on the exam.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example 2: Find a power series expansion for

(a) $f(x) = \ln(1+4x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

Replace x by $4x$ in the formula.

$$\ln(1+4x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (4x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4^n x^n}{n}$$

(b) $f(x) = e^{-3x^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Replace x by $-3x^2$ in the formula

$$e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{2n}}{n!}$$

(c) $f(x) = 4x \cos(2x)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Replace x by $2x$ in the formula

$$\begin{aligned} \cos(2x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} \end{aligned}$$

Multiply by $4x$

$$4x \cos(2x) = 4x \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1} x^{2n+1}}{(2n)!}$$

Estimating Function Values Using Power Series

Power series can be used to estimate a function's value at a given value of x .

Example 4: Estimate the value of $\ln(1.62)$ using the first four terms of the Maclaurin series for $f(x) = \ln(1+x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

First four terms

Note that this is for $\ln(x+1)$ and we want $\ln(1.62)$. So
 $x+1 = 1.62 \rightarrow x = 0.62$

So at $x = 0.62$,

$$(0.62) - \frac{(0.62)^2}{2} + \frac{(0.62)^3}{3} - \frac{(0.62)^4}{4} \approx 0.47030$$

Example 5: Use the first three nonzero terms of the Maclaurin series for $f(x) = \sin x$ to estimate $\sin(0.49)$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

First three terms

So at $x = 0.49$,

$$\frac{(0.49)}{1!} - \frac{(0.49)^3}{3!} + \frac{(0.49)^5}{5!} \approx 0.470627$$

Using Maclaurin Series to Estimate Definite Integrals

Steps to Solve Integrals Using Maclaurin

- ① Convert the function into a series.
- ② Integrate the series (remember x is the variable)
- ③ Write out the number of terms to be used.
- ④ Substitute in the bounds.

Example 6: Use the first 4 terms of the Maclaurin series to estimate the integral

$$\int_0^{0.55} \cos(\sqrt{x}) dx$$

① Write $\cos(x^{1/2})$ as a Maclaurin Series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Replace x with $x^{1/2}$,

$$\cos(x^{1/2}) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{1/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

② Integrate

$$\begin{aligned} \int \cos(x^{1/2}) dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{x^{n+1}}{n+1} \end{aligned}$$

③ Write out 4 first terms,

$$n=0: \frac{(-1)^0}{(2 \cdot 0)!} \cdot \frac{x^{0+1}}{0+1} = x$$

$$n=1: \frac{(-1)^1}{(2 \cdot 1)!} \cdot \frac{x^{1+1}}{1+1} = -\frac{x^2}{4}$$

$$n=2: \frac{(-1)^2}{(2 \cdot 2)!} \cdot \frac{x^{2+1}}{2+1} = \frac{x^3}{72}$$

$$n=3: \frac{(-1)^3}{(2 \cdot 3)!} \cdot \frac{x^{3+1}}{3+1} = -\frac{x^4}{2880}$$

④ Evaluate ③ from 0 to 0.55

$$\left(\frac{0.55 - (0.55)^2}{4} + \frac{(0.55)^3}{72} - \frac{(0.55)^4}{2880} \right) - 0 \approx 0.47665$$