Lesson 22: Partial Derivatives

A partial derivative is a derivative where we hold some variables constant.

Let's think about a function of one variable.

ex. $f(x) = x^2 \Rightarrow f'(x) = 2x$

But what about a function of two variables?

$$f(x,y) = x^2 + y^3$$

We find its partial derivative with respect to x by treating y as a constant.

$$f_x = 2x + 0 = 2x$$

To find the partial derivative with respect to y, we treat x as a constant.

$$f_y = 0 + 3y^2 = 3y^2$$

Definition: • The (first) partial derivative f_x describes the rate of change of f as x changes, where y remains constant. i.e. Find the derivative with respect to x, where we treat y as a constant

• The (first) partial derivative f_y describes the rate of change of f as y changes, where x remains constant. i.e. Find the derivative with respect to y, where we treat x as a constant.

Example 1: Compute the first order partial derivatives

(a) $f(x,y) = x^3 + 3xy$

First order partials $\Rightarrow$ We need to find f_x and f_y.

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f = x^3 + (3y)x$$
$$f_x = 3x^2 + 3y$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f = x^3 + (3x)y$$
$$f_y = 0 + 3x \quad = 3x$$

Chain Rule Problem

ⓑ $f(x,y) = \ln(x+2y)$

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f_x = \frac{1}{x+2y} \cdot \frac{d}{dx}(x+2y) = \frac{1}{x+2y} \cdot (1+0) = \frac{1}{x+2y}$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f_y = \frac{1}{x+2y} \cdot \frac{d}{dy}(x+2y) = \frac{1}{x+2y} \cdot (0+2) = \frac{2}{x+2y}$$

ⓒ $f(x,y) = \dfrac{9xy}{\sqrt{y-1}}$

First find f_x. i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f(x,y) = \frac{9y}{\sqrt{y-1}}(x)$$
$$f_x = \frac{9y}{\sqrt{y-1}} \cdot \frac{d}{dx}(x) = \frac{9y}{\sqrt{y-1}}$$

Next find f_y. i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f(x,y) = 9x\left(\frac{y}{\sqrt{y-1}}\right)$$

Apply Quotient Rule

$$f_y = 9x \cdot \frac{d}{dy}\left(\frac{y}{\sqrt{y-1}}\right) = 9x\left(\frac{1 \cdot \sqrt{y-1} - y \cdot \frac{1}{2}(y-1)^{-1/2}}{(\sqrt{y-1})^2}\right)$$
$$= 9x\left(\frac{\sqrt{y-1} - \frac{1}{2}y(y-1)^{-1/2}}{y-1}\right)$$

Example 2: Evaluate the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ at the given point $P_0(x_0, y_0)$.

$$f(x,y) = x^3y^2 + 6x^2 \quad ; \quad P_0(1,-1)$$

First find f_x, i.e. Find the derivative w/ respect to x and treat y as a constant.

$$f_x = 3x^2y^2 + 12x^2$$

Plug $(1,-1)$ into f_x.

$$f_x(1,-1) = 3(1)^2(-1)^2 + 12(1)^2 = 15$$

Next find f_y, i.e. Find the derivative w/ respect to y and treat x as a constant.

$$f_y = x^3 \cdot 2y = 2x^3y$$

Plug $(1,-1)$ into f_y.

$$f_y(1,-1) = 2(1)^3(-1) = -2$$

Lesson 22: Higher Order Partial Derivatives

Recall from last class, the partial derivatives of $z=f(x,y)$ are

$$f_x = \frac{df}{dx} \qquad f_y = \frac{df}{dy}$$

Higher Order Partial Derivatives are similar to Higher Order Derivatives (Lesson 13) from Applied Calculus I. The main difference is that we can mix partials (i.e. first do x then y or vice versa).

Notation/Definition: Taking the partial derivatives of the partials f_x and f_y, we have the following second-order partials.

① $\frac{d}{dx}(f_x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2} = f_{xx}$

② $\frac{d}{dy}(f_x) = \frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d^2f}{dy\,dx} = f_{xy}$

③ $\frac{d}{dx}(f_y) = \frac{d}{dx}\left(\frac{df}{dy}\right) = \frac{d^2f}{dx\,dy} = f_{yx}$

④ $\frac{d}{dy}(f_y) = \frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d^2f}{dy^2} = f_{yy}$

Fact: While not true in general, but for the purpose of this class, we have

$$f_{xy} = f_{yx}$$

So ② and ③, in the definition above, can be combined

Example 1: Compute the second order partial derivatives

ⓐ $f(x,y) = 5x^2y + 2xy^3 + 3y^2$

First find f_x and f_y.

$$f_x = 10xy + 2y^3 \qquad f_y = 5x^2 + 6xy^2 + 6y$$

Next find f_{xx}, f_{xy}, and f_{yy} via the definition.

$$f_{xx} = \frac{d}{dx}(f_x) = \frac{d}{dx}(10xy + 2y^3) = 10y + 0 = 10y$$

$$f_{xy} = \frac{d}{dy}(f_x) = \frac{d}{dy}(10xy + 2y^3) = 10x + 6y^2$$

$$f_{yy} = \frac{d}{dy}(f_y) = \frac{d}{dy}(5x^2 + 6xy^2 + 6y) = 0 + 12xy + 6 = 12xy + 6$$

ⓑ $f(x,y) = x^2 y e^{7x}$

First find f_x and f_y.

$$f(x,y) = y(x^2 e^{7x})$$

$$f_x = y(2xe^{7x} + x^2 e^{7x} \cdot 7) = ye^{7x}(2x + 7x^2)$$

Product Rule

$$f(x,y) = (x^2 e^{7x})y$$

$$f_y = x^2 e^{7x}$$

Next find f_{xx}, f_{xy}, and f_{yy} via the definition.

$$f_{xx} = \frac{d}{dx}(f_x) = \frac{d}{dx}\left(ye^{7x}(2x+7x^2)\right) = y\frac{d}{dx}\left(e^{7x}(2x+7x^2)\right)$$

$$= y\left(7e^{7x}(2x+7x^2) + e^{7x}(2+14x)\right)$$

$$= ye^{7x}[7(2x+7x^2) + (2+14x)] = ye^{7x}[14x + 49x^2 + 2 + 14x]$$

$$= ye^{7x}(49x^2 + 28x + 2)$$

$$f_{xy} = \frac{d}{dy}(f_y) = \frac{d}{dy}(ye^{7x}(2x+7x^2)) = e^{7x}(2x+7x^2)\frac{d}{dy}(y)$$

$$= e^{7x}(2x+7x^2)$$

$$f_{yy} = \frac{d}{dy}(x^2 e^{7x}) = 0$$ b/c there is no y terms

ⓒ $f(x,y) = \ln(6x^2 + 7y)$

Chain Rule → First find f_x and f_y.

$$f_x = \frac{1}{6x^2+7y} \cdot \frac{d}{dx}(6x^2+7y) = \frac{12x}{6x^2+7y}$$

$$f_y = \frac{1}{6x^2+7y} \cdot \frac{d}{dy}(6x^2+7y) = \frac{7}{6x^2+7y}$$

Next find f_{xx}, f_{xy}, and f_{yy} via the definition.

$$f_{xx} = \frac{d}{dx}(f_x) = \frac{d}{dx}\left(\frac{12x}{6x^2+7y}\right) = \frac{12(6x^2+7y) - 12x(12x+0)}{(6x^2+7y)^2}$$

$$= \frac{72x^2+84y-144x^2}{(6x^2+7y)^2}$$

$$f_x = 12x(6x^2+7y)^{-1}$$

$$f_{xy} = \frac{d}{dy}(f_x) = 12x\cdot(-1)(6x^2+7y)^{-2}\cdot(0+7) = \frac{-84x}{(6x^2+7y)^2}$$

$$f_y = 7(6x^2+7y)^{-1}$$

$$f_{yy} = \frac{d}{dy}(f_y) = 7\cdot(-1)(6x^2+7y)^{-2}\cdot(0+7) = \frac{-49}{(6x^2+7y)^2}$$