Lesson 25: La Grange Multipliers
Constrained Min/Max

Previously, we optimized a function on a region that contained its boundary via the Second Derivative Test for Multi-variable Functions. However, as we saw in the examples finding potential optimal points on the boundary was LONG and MESSY process.

Now we are going to take a look at another way of optimizing a function, $f(x,y)$, subject to the given constraint $g(x,y)=k$.

Method of La Grange Multipliers

① Solve the following system of equations.

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$$

② Evaluate f at every point (x,y) found in ①.

↳ The largest is the maximum value of f.

↳ The smallest is the minimum value of f.

Note the equations from ① are known as LaGrange Equations and the constant, λ, is called the La Grange Multiplier.

Example 1: Find the minimum value of $f(x,y) = x^2+y^2$ subject to the constraint $7y = 6-2x$

Here $g(x,y) = 7y+2x = 6$. Using La Grange Multipliers,

$$f_x = 2x \qquad \lambda g_x = \lambda 2$$
$$f_y = 2y \qquad \lambda g_y = \lambda 7$$

So our system is

$$\begin{cases} 2x = \lambda 2 \\ 2y = \lambda 7 \\ 7y + 2x = 6 \end{cases} \Rightarrow \begin{cases} x = \lambda & ① \\ y = (7/2)\lambda & ② \\ 7y + 2x = 6 & ③ \end{cases}$$

Plug ① and ② into ③.

$$7\left(\frac{7}{2}\lambda\right) + 2\lambda = 6$$
$$\frac{49}{2}\lambda + 2\lambda = 6$$
$$49\lambda + 4\lambda = 12$$
$$53\lambda = 12$$
$$\lambda = \frac{12}{53}$$

Plug $\lambda = 12/53$ into ① and ②.

① $x = \lambda = \dfrac{12}{53}$

② $y = \dfrac{7}{2}\lambda = \dfrac{7}{2} \cdot \dfrac{12}{53} = \dfrac{42}{53}$

So $f\left(\frac{12}{53}, \frac{42}{53}\right) = \left(\frac{12}{53}\right)^2 + \left(\frac{42}{53}\right)^2 \approx 0.6792$

Since there is only one value, 0.6792 is the min.

Example 2: Find the maximum value of $f(x,y) = 31x^{3/2}y$ subject to the constraint $x + y = 46$

Here $g(x,y) = x + y = 46$. Using La Grange Multipliers,

$f_x = 31\left(\frac{3}{2}\right)x^{1/2}y = \frac{93}{2}x^{1/2}y \qquad \lambda g_x = \lambda(1)$

$f_y = 31x^{3/2} \qquad \lambda g_y = \lambda(1)$

So our system is

$$\begin{cases} \frac{93}{2}x^{1/2}y = \lambda & ① \\ 31x^{3/2} = \lambda & ② \\ x + y = 46 & ③ \end{cases}$$

Set ① and ② equal.

$$\frac{93}{2}x^{1/2}y = \lambda = 31x^{3/2}$$
$$93x^{1/2}y = 62x^{3/2}$$
$$93x^{1/2}y - 62x^{3/2} = 0$$
$$31x^{1/2}(3y - 2x) = 0$$
$$x = 0 \qquad y = \frac{2x}{3}$$

Plug $x = 0$ into ③.

$$0 + y = 46$$
$$y = 46$$

$(0, 46)$

Plug $y = 2/3\,x$ into ③

$$x + \left(\frac{2}{3}x\right) = 46$$

$$\frac{5x}{3} = 46$$

$$x = \frac{138}{5}$$

$$\left(\frac{138}{5}, \frac{92}{5}\right)$$

Testing for Max

$$f(x,y) = 31x^{3/2}y$$

$$f(0,46) = 0$$

$$f\left(\frac{138}{5}, \frac{92}{5}\right) \approx 82707 \Rightarrow \text{max}$$

Example 3: Find the maximum value of the function $f(x,y) = 3x - 11y^2$ subject to the constraint $x^2 + 11y^2 = 81$.

Here $g(x,y) = x^2 + 11y^2 = 81$. Using La Grange Multipliers,

$$f_x = 3 \qquad \lambda g_x = \lambda(2x)$$

$$f_y = -22y \qquad \lambda g_y = \lambda(22y)$$

So our system is

$$\begin{cases} 3 = 2\lambda x \\ -22y = \lambda 22y \\ x^2 + 11y^2 = 81 \end{cases} \Rightarrow \begin{cases} 3 = 2\lambda x & ① \\ \lambda 22y + 22y = 0 & ② \\ x^2 + 11y^2 = 81 & ③ \end{cases}$$

Solve ②.

$$\lambda 22y + 22y = 0$$

$$22y(\lambda + 1) = 0$$

$$y = 0, \ \lambda = -1$$

Plug $y = 0$ into ③

$$x^2 + 11(0)^2 = 81$$

$$x^2 = 81$$

$$x = \pm 9$$

$$(9,0), (-9,0)$$

Plug $\lambda = -1$ into ②

$$3 = 2(-1)x$$

$$3 = -2x$$

$$x = -3/2$$

Plug $x = -3/2$ into ③.

$$(-3/2)^2 + 11y^2 = 81$$

$$9/4 + 11y^2 = 81$$

$$11y^2 = 315/4$$

$$y^2 = 315/44$$

$$y = \pm\sqrt{\frac{315}{44}}$$

So points to test are:

- $(9,0)$
- $(-9,0)$
- $\left(-\frac{3}{2}, \sqrt{\frac{315}{44}}\right)$
- $\left(-\frac{3}{2}, -\sqrt{\frac{315}{44}}\right)$

Testing for Max

$f(x,y) = 3x - 11y^2$

$f(9,0) = 27 \Rightarrow$ max

$f(-9,0) = -27$

$f\left(\frac{-3}{2}, \sqrt{\frac{315}{44}}\right) = -\frac{333}{4}$

$f\left(\frac{-3}{2}, -\sqrt{\frac{315}{44}}\right) = -\frac{333}{4}$

Example 4: Find the minimum and maximum of the function $f(x,y) = e^{-xy}$ subject to the constraint $5x^2 + 19y^2 = 17$.

Here $g(x,y) = 5x^2 + 19y^2 = 17$. Using LaGrange Multipliers,

$f_x = -ye^{-xy}$ $\quad \lambda g_x = \lambda(10x)$

$f_y = -xe^{-xy}$ $\quad \lambda g_y = \lambda(38y)$

So our system is

$$\begin{cases} -ye^{-xy} = 10\lambda x & (1) \\ -xe^{-xy} = 38\lambda y & (2) \\ 5x^2 + 19y^2 = 17 & (3) \end{cases}$$

Multiply e^{-xy} to both (1) and (2).

$$\begin{cases} -y = 10\lambda x e^{-xy} & (1) \\ -x = 38\lambda y e^{-xy} & (2) \\ 5x^2 + 19y^2 = 17 & (3) \end{cases}$$

Next, multiply (1) by $38y$ and (2) by $10x$.

$$\begin{cases} -38y^2 = 380\lambda xye^{-xy} & (1) \\ -10x^2 = 380\lambda xye^{-xy} & (2) \\ 5x^2 + 19y^2 = 17 & (3) \end{cases}$$

Note the RHS of (1) and (2) match. So set (1) = (2).

$-38y^2 = -10x^2$

$19y^2 = 5x^2$

$y = \pm\sqrt{\frac{5}{19}}\, x$

Plug $5x^2 = 19y^2$ into ③

$$5x^2 + 5x^2 = 17$$
$$10x^2 = 17$$
$$x^2 = \pm\sqrt{\frac{17}{10}}$$

When $x = \sqrt{\frac{17}{10}}$, $y = \pm\sqrt{\frac{5}{19}} \cdot \sqrt{\frac{17}{10}} = \pm\sqrt{\frac{17}{38}}$

which gives us $\left(\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}}\right), \left(\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}}\right)$

When $x = -\sqrt{\frac{17}{10}}$, $y = \pm\sqrt{\frac{5}{19}}\left(-\sqrt{\frac{17}{10}}\right) = \mp\sqrt{\frac{17}{38}}$

Which gives us $\left(-\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}}\right), \left(-\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}}\right)$

<u>Test for min/max</u>

$f\left(\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}}\right) = 0.41 \Rightarrow$ min

$f\left(\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}}\right) = 2.39 \Rightarrow$ max

$f\left(-\sqrt{\frac{17}{10}}, \sqrt{\frac{17}{38}}\right) = 2.39 \Rightarrow$ max

$f\left(-\sqrt{\frac{17}{10}}, -\sqrt{\frac{17}{38}}\right) = 0.41 \Rightarrow$ min