

Lesson 27: Double Integrals I

Let $z = f(x, y)$ be a function of two variables. Similar to taking partial derivatives with respect to x and y , we can take

- $\int_{x=a}^{x=b} f(x, y) dx$ - Integrate with respect to x and Treat y as a constant
- $\int_{y=c}^{y=d} f(x, y) dy$ - Integrate with respect to y and Treat x as a constant

Combining the above integrals, we obtain the following double integrals:

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dx dy = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y) dx \right) dy$$

$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) dy dx = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x, y) dy \right) dx$$

Example 1: Evaluate

$$\begin{aligned} \textcircled{a} \int_0^5 \int_0^3 (x+y) dy dx &= \int_{x=0}^{x=5} \left(\int_{y=0}^{y=3} (x+y) dy \right) dx \\ &= \int_{x=0}^{x=5} \left(\left[xy + \frac{y^2}{2} \right]_{y=0}^{y=3} \right) dx \\ &= \int_{x=0}^{x=5} \left(3x + \frac{9}{2} - \left(0 \cdot x - \frac{0^2}{2} \right) \right) dx \\ &= \int_{x=0}^{x=5} \left(3x + \frac{9}{2} \right) dx \\ &= \left[\frac{3x^2}{2} + \frac{9}{2}x \right]_{x=0}^{x=5} \\ &= \frac{3(5)^2}{2} + \frac{9}{2}(5) - \left(\frac{3(0)^2}{2} + \frac{9}{2}(0) \right) \\ &= 60 \end{aligned}$$

$$\begin{aligned}
 \textcircled{b} \int_1^2 \int_0^1 x^2 y \, dy \, dx &= \int_{x=1}^{x=2} \left(\int_{y=0}^{y=1} x^2 y \, dy \right) dx \\
 &= \int_{x=1}^{x=2} x^2 \cdot \left(\int_{y=0}^{y=1} y \, dy \right) dx \\
 &= \int_{x=1}^{x=2} x^2 \left(\frac{y^2}{2} \right) \Big|_{y=0}^{y=1} dx \\
 &= \int_{x=1}^{x=2} x^2 \left(\frac{1^2}{2} - \frac{0^2}{2} \right) dx \\
 &= \int_{x=1}^{x=2} \frac{1}{2} x^2 \, dx \\
 &= \left. \frac{1}{2} \cdot \frac{x^3}{3} \right|_{x=1}^{x=2} \\
 &= \frac{2^3}{6} - \frac{1^3}{6} \\
 &= \frac{7}{6}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{c} \int_0^{\pi/2} \int_0^1 16y^4 \cos(x) \, dy \, dx &= \int_{x=0}^{x=\pi/2} \left(\int_{y=0}^{y=1} 16y^4 \cos(x) \, dy \right) dx \\
 &= \int_{x=0}^{x=\pi/2} \cos(x) \cdot \left(\int_{y=0}^{y=1} 16y^4 \, dy \right) dx \\
 &= \int_{x=0}^{x=\pi/2} \cos(x) \cdot \left(\frac{16y^5}{5} \right) \Big|_{y=0}^{y=1} dx \\
 &= \int_{x=0}^{x=\pi/2} \cos(x) \left(\frac{16(1)^5}{5} - \frac{16(0)^5}{5} \right) dx \\
 &= \int_{x=0}^{x=\pi/2} \frac{16}{5} \cos(x) \, dx \\
 &= \left. \frac{16}{5} \sin(x) \right|_{x=0}^{x=\pi/2} \\
 &= \frac{16}{5} \left(\sin\left(\frac{\pi}{2}\right) - \sin(0) \right) \\
 &= \frac{16}{5} (1 - 0) \\
 &= 16/5
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \int_{\pi/6}^{\pi/2} \int_{-1}^2 \cos(y) dx dy &= \int_{y=\pi/6}^{y=\pi/2} \left(\int_{x=-1}^{x=2} \cos(y) dx \right) dy \\
 &= \int_{y=\pi/6}^{y=\pi/2} \cos(y) \cdot \left(\int_{x=-1}^{x=2} dx \right) dy \\
 &= \int_{y=\pi/6}^{y=\pi/2} \cos(y) \cdot x \Big|_{x=-1}^{x=2} dy \\
 &= \int_{y=\pi/6}^{y=\pi/2} \cos(y) \cdot (2 - (-1)) dy \\
 &= \int_{y=\pi/6}^{y=\pi/2} 3 \cos(y) dy \\
 &= 3 \sin y \Big|_{y=\pi/6}^{y=\pi/2} \\
 &= 3 \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) \right] \\
 &= 3 \left[1 - \frac{1}{2} \right] \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{e} \int_0^5 \int_0^3 x^2 e^{2y} dx dy &= \int_{y=0}^{y=5} \left(\int_{x=0}^{x=3} x^2 e^{2y} dx dy \right) \\
 &= \int_{y=0}^{y=5} e^{2y} \cdot \left(\int_{x=0}^{x=3} x^2 dx \right) dy \\
 &= \int_{y=0}^{y=5} e^{2y} \left(\frac{x^3}{3} \right) \Big|_{x=0}^{x=3} dy \\
 &= \int_{y=0}^{y=5} e^{2y} \left(\frac{3^3}{3} - \frac{0^3}{3} \right) dy \\
 &= \int_{y=0}^{y=5} 9 e^{2y} dy \\
 &= \frac{9}{2} e^{2y} \Big|_0^5 \\
 &= \frac{9}{2} (e^{10} - e^0) = \frac{9}{2} (e^{10} - 1)
 \end{aligned}$$

Example 2: Evaluate

$$\begin{aligned} \textcircled{a} \int_0^7 \int_0^y 5xy \, dx \, dy &= \int_{y=0}^{y=7} \left(\int_{x=0}^{x=y} 5xy \, dx \right) dy \\ &= \int_{y=0}^{y=7} 5y \cdot \left(\int_{x=0}^{x=y} x \, dx \right) dy \\ &= \int_{y=0}^{y=7} 5y \left(\frac{x^2}{2} \right) \Big|_{x=0}^{x=y} dy \\ &= \int_{y=0}^{y=7} 5y \left(\frac{y^2}{2} - \frac{0^2}{2} \right) dy \\ &= \int_{y=0}^{y=7} 5y \cdot \frac{y^2}{2} dy \\ &= \frac{5}{2} \int_{y=0}^{y=7} y^3 dy \\ &= \frac{5}{2} \cdot \frac{y^4}{4} \Big|_0^7 \\ &= \frac{5}{8} (7^4 - 0^4) = \frac{12005}{8} \end{aligned}$$

$$\begin{aligned} \textcircled{b} \int_3^4 \int_3^x \frac{6x}{y^2} \, dy \, dx &= \int_{x=3}^{x=4} \left(\int_{y=3}^{y=x} 6xy^{-2} \, dy \right) dx \\ &= \int_{x=3}^{x=4} 6x \cdot \left(\int_{y=3}^{y=x} y^{-2} \, dy \right) dx \\ &= \int_{x=3}^{x=4} 6x \cdot \left(-y^{-1} \right) \Big|_{y=3}^{y=x} dx \\ &= \int_{x=3}^{x=4} 6x \left(\frac{-1}{y} \right) \Big|_{y=3}^{y=x} dx \\ &= \int_{x=3}^{x=4} 6x \left(-\frac{1}{x} + \frac{1}{3} \right) dx \end{aligned}$$

Remember we introduced

$$\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y) dx \right) dy \quad \text{and} \quad \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y) dy \right) dx$$

A geometric interpretation of these double integrals is we are finding the volume below $z = f(x,y)$ above the region $R = \{(x,y) \mid a \leq x \leq b; c \leq y \leq d\}$

We can denote the integrals above by just one $\iint_R f(x,y) dA$

where R is the domain of integration.

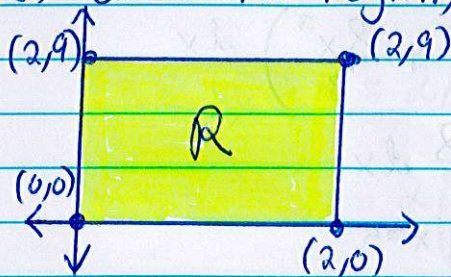
To solve the integrals of the form $\iint_R f(x,y) dA$, we start

by drawing the region. We do this to determine the bounds of our integrals.

Example 3: Evaluate the integral $\iint_R 10x^3y dA$ where R

is the rectangle with vertices $(0,0)$, $(2,0)$, $(0,9)$, and $(2,9)$.

First draw the region, R . We can see that $0 \leq x \leq 2$ and



$0 \leq y \leq 9$. So

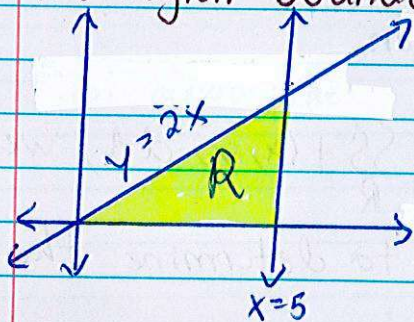
$$R = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 9\}$$

$$\begin{aligned} \text{Hence } \iint_R 10x^3y dA &= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=9} 10x^3 \cdot y dy \right) dx \\ &= \int_{x=0}^{x=2} \left(10x^3 \cdot \left[\frac{y^2}{2} \right]_{y=0}^{y=9} \right) dx \\ &= \int_{x=0}^{x=2} \left(10x^3 \left(\frac{9^2}{2} - \frac{0^2}{2} \right) \right) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{x=0}^{x=2} \left(10x^3 \cdot \frac{81}{2} \right) dx \\
 &= \int_{x=0}^{x=2} 405x^3 dx \\
 &= \frac{405}{4} x^4 \Big|_{x=0}^{x=2} \\
 &= \frac{405}{4} (2^4 - 0^4) = 1620
 \end{aligned}$$

Example 4: Evaluate the integral $\iint_R (x^2 + y^2) dA$ where R is

the region bounded by the lines $y = 2x$, $x = 5$, and the x -axis.



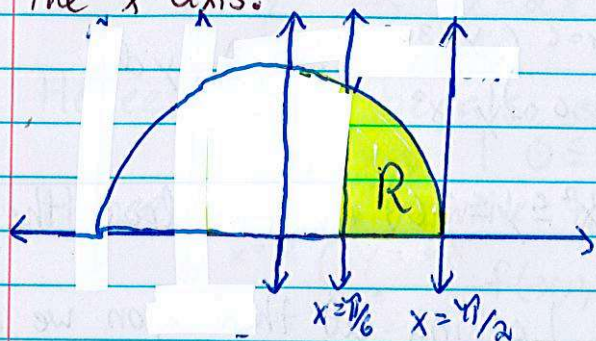
First draw the region, R . We can see that

$$R = \{(x, y) \mid 0 \leq x \leq 5, 0 \leq y \leq 2x\}$$

$$\begin{aligned}
 \text{Hence } \iint_R (x^2 + y^2) dA &= \int_{x=0}^{x=5} \left(\int_{y=0}^{y=2x} (x^2 + y^2) dy \right) dx \\
 &= \int_{x=0}^{x=5} \left(\left[x^2 \cdot y + \frac{y^3}{3} \right]_{y=0}^{y=2x} \right) dx \\
 &= \int_{x=0}^{x=5} \left(x^2(2x) + \frac{(2x)^3}{3} - \left(x^2 \cdot 0 + \frac{0^3}{3} \right) \right) dx \\
 &= \int_{x=0}^{x=5} \left(2x^3 + \frac{8x^3}{3} \right) dx \\
 &= \int_{x=0}^{x=5} \frac{14}{3} x^3 dx \\
 &= \frac{14}{3} \cdot \frac{x^4}{4} \Big|_{x=0}^{x=5} \\
 &= \frac{7}{6} x^4 \Big|_{x=0}^{x=5} \\
 &= \frac{7}{6} (5^4 - 0^4) = \frac{4375}{6}
 \end{aligned}$$

Example 5: Evaluate the integral $\iint_R 6 \sin^2(x) dA$ where R is the

region bounded by the curves $y = \cos(x)$, $x = \pi/6$, $x = \pi/2$ and the x -axis.



First draw the region, R . We can see that

$$R = \left\{ (x, y) \mid \frac{\pi}{6} \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x \right\}$$

$$\begin{aligned} \text{Hence } \iint_R 6 \sin^2(x) dA &= \int_{x=\pi/6}^{x=\pi/2} \int_{y=0}^{y=\cos(x)} 6(\sin x)^2 dy dx \\ &= \int_{x=\pi/6}^{x=\pi/2} \left(6(\sin x)^2 \cdot y \right) \Big|_{y=0}^{y=\cos x} dx \\ &= \int_{x=\pi/6}^{x=\pi/2} \left(6(\sin x)^2 \cdot (\cos x - 0) \right) dx \\ &= \int_{x=\pi/6}^{x=\pi/2} 6(\sin x)^2 \cdot \cos x dx \end{aligned}$$

$$\frac{u = \sin x}{du = \cos x dx} \int 6u^2 du$$

$$= \frac{6u^3}{3} = 2u^3$$

$$= 2(\sin x)^3 \Big|_{x=\pi/6}^{x=\pi/2}$$

$$= 2 \left(\left(\sin\left(\frac{\pi}{2}\right) \right)^3 - \left(\sin\left(\frac{\pi}{6}\right) \right)^3 \right)$$

$$= 2 \left((1)^3 - \left(\frac{1}{2}\right)^3 \right) = 2 \left(1 - \frac{1}{8} \right) = 2 \cdot \frac{7}{8} = \frac{7}{4}$$

Note that Examples 3, 4, and 5 could have been done with $dx dy$ as the order of integration.

So a good question to ask is when to use $dx dy$ or $dy dx$? We will answer that next time.