

Lesson 28: Double Integrals III

Recall from Last Time, given a function $z = f(x, y)$ and a region, R , in the xy -plane, we have

$$\iint_R f(x, y) dA$$

To solve these integrals, we start by drawing the region. We do this to determine the bounds of our integrals.

So a good question to ask is when to use $dx dy$ or $dy dx$?

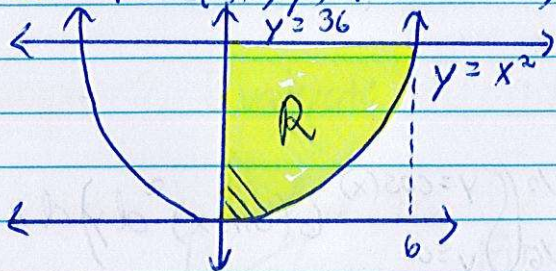
Before we answer that let's practice switching the order of integration.

To switch the order of integration, we need to draw a picture.

Example 1: Switch the order of integration on the following integrals.

$$(a) \int_0^6 \int_{x^2}^{36} f(x,y) dy dx = \int_{x=0}^{x=6} \int_{y=x^2}^{y=36} f(x,y) dy dx$$

So $R = \{(x,y) \mid 0 \leq x \leq 6, x^2 \leq y \leq 36\}$. Let's draw the region.



Looking at the region we can describe R in another way. The y -values are $0 \leq y \leq 36$. As for x , we see that x lies between the y -axis ($x=0$) and

the parabola ($y=x^2$). Let's get $y=x^2$ in terms of $x = \pm \sqrt{y}$.

Since $x \geq 0$, $x = \sqrt{y}$. Hence R can be also described by

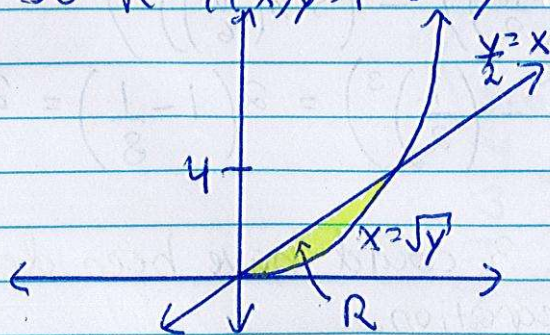
$$R = \{(x,y) \mid 0 \leq y \leq 36, 0 \leq x \leq \sqrt{y}\}$$

Hence we can rewrite the integral to be

$$\int_{y=0}^{y=36} \int_{x=0}^{x=\sqrt{y}} f(x,y) dx dy$$

$$(b) \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y) dx dy = \int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x,y) dx dy$$

So $R = \{(x,y) \mid 0 \leq y \leq 4, y/2 \leq x \leq \sqrt{y}\}$. Let's draw R .



Looking at the region we can describe R in another way. We see $x=0$ is the smallest value to find the largest plug $y=4$ into

$$x = \frac{y}{2} \text{ or } x = \sqrt{y}$$

So $x = \frac{4}{2} = 2$. Hence $0 \leq x \leq 2$. As for y , we see that y

lies between $x = \sqrt{y}$ and $x = \frac{y}{2}$. Let's get both equations

in terms of y .

$$x = \frac{y}{2} \Rightarrow y = 2x$$

$$x = \sqrt{y} \Rightarrow y = x^2$$

Hence R can be also described by

$$R = \{(x, y) \mid 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

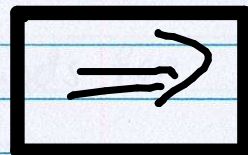
Hence we can rewrite the integral to be

$$\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x, y) dy dx$$

But sometimes the integral we obtain can't be integrated.

Mainly because we don't know it's antiderivative. So when

should you use $dx dy$ or $dy dx$ is vital in these problems.



i.e. This is where switching the order of integration comes in. Given an integral with $dx dy$, we can switch it to $dy dx$ (and vice versa) via the drawing of the region.

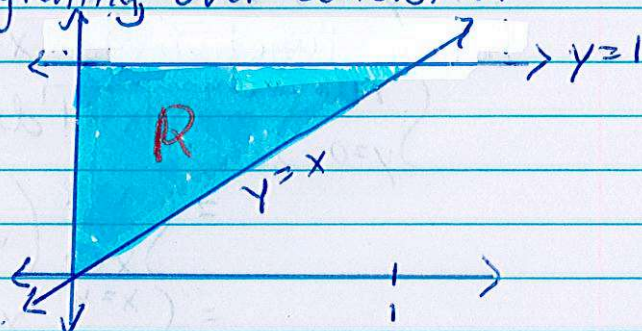
It's easier to see through some examples.

Example 2: Compute

$$(a) \int_0^1 \int_x^1 \sin(y^2) dy dx = \int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2) dy dx$$

So the region we are integrating over consist of $0 \leq x \leq 1, x \leq y \leq 1$

Note: $y=x$ is the bottom function, while $y=1$ is the top.



So the y -values are $0 \leq y \leq 1$. The x -values we see that the largest is $y=x$ (or $x=y$) and smallest is the y -axis (or $x=0$). So $0 \leq x \leq y$. Hence

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2) dy dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} \sin(y^2) dx dy$$

$$= \int_{y=0}^{y=1} \left(\sin(y^2) \cdot x \right) \Big|_{x=0}^{x=y} dy$$

$$= \int_{y=0}^{y=1} y \sin(y^2) dy$$

$$\begin{array}{l} u=y^2 \\ du=2y dy \\ du/2=y dy \end{array} \int \sin(u) \frac{du}{2}$$

$$= -\frac{1}{2} \cos(u)$$

$$= -\frac{1}{2} \cos(y^2) \Big|_{y=0}^{y=1}$$

$$= -\frac{1}{2} \cos(1) + \frac{1}{2} \cos(0)$$

$$= \frac{1}{2} - \frac{1}{2} \cos(1)$$

$$\textcircled{b} \int_0^{16} \int_{\sqrt{y}}^4 \sqrt{x^3+1} dx dy = \int_{y=0}^{y=16} \int_{x=\sqrt{y}}^{x=4} \sqrt{x^3+1} dx dy$$

So the region we are integrating over consist of $0 \leq y \leq 16, \sqrt{y} \leq x \leq 4$

So the x-values are $0 \leq x \leq 4$. The y-values we see the top

Function is $x = \sqrt{y}$ (or $y = x^2$) and the

bottom function is x-axis (or $y=0$). So $0 \leq y \leq x^2$. Hence

$$\begin{aligned} \int_{y=0}^{y=16} \int_{x=\sqrt{y}}^{x=4} \sqrt{x^3+1} dx dy &= \int_{x=0}^{x=4} \int_{y=0}^{y=x^2} \sqrt{x^3+1} dy dx \\ &= \int_{x=0}^{x=4} \left(\sqrt{x^3+1} \cdot y \right) \Big|_{y=0}^{y=x^2} dx \\ &= \int_{x=0}^{x=4} x^2 \sqrt{x^3+1} dx \end{aligned}$$

$$\begin{aligned} u &= x^3+1 \\ du &= 3x^2 dx \\ du/3 &= x^2 dx \end{aligned} \quad \int u^{1/2} \frac{du}{3}$$

$$= \frac{1}{3} \cdot \frac{2}{3} u^{3/2}$$

$$= \frac{2}{9} (x^3+1)^{3/2} \Big|_{x=0}^{x=4}$$

$$= \frac{2}{9} (4^3+1)^{3/2} - \frac{2}{9} (0^3+1)^{3/2}$$

$$= \frac{2}{9} \cdot (65)^{3/2} - \frac{2}{9}$$