

Lesson 2: Integration By Substitution I

Last time, when $\int h(x) dx = \int f(u(x)) \cdot u'(x) dx$

(1) Choose $u(x)$

(2) Then differentiate to get $du = u'(x) dx$

To produce a new integral

$$\int h(x) dx = \int f(u) du$$

Then integrate and plug back $u(x)$ to get our answer.

Example 1 Compute the following integrals:

(a) $\int 5 \sin(3t) \cos^8(3t) dt = \int 5 \sin(3t) \cdot (\cos(3t))^8 dt$

$$\begin{aligned} u &= \cos(3t) \\ du &= -\sin(3t) \cdot 3 dt \\ \frac{du}{-3\sin(3t)} &= dt \end{aligned} \quad \int 5 \cancel{\sin(3t)} \cdot u^8 \cdot \frac{du}{-3\cancel{\sin(3t)}}$$

$$\begin{aligned} &= -\frac{5}{3} \int u^8 du = -\frac{5}{3} \cdot \frac{u^9}{9} + C \\ &= -\frac{5}{27} \cos^9(3t) + C \end{aligned}$$

(b) $\int 8e^{7x} \csc^2(e^{7x}) dx$

$$\begin{aligned} u &= e^{7x} \\ du &= 7e^{7x} dx \\ \frac{du}{7e^{7x}} &= dx \end{aligned} \quad \int 8 \cancel{e^{7x}} \csc^2(u) \cdot \frac{du}{7\cancel{e^{7x}}}$$

$$\begin{aligned} &= \frac{8}{7} \int \csc^2(u) du = -\frac{8}{7} \cot(u) + C \\ &= -\frac{8}{7} \cot(e^{7x}) + C \end{aligned}$$

Sometimes we need to do extra work with u , as shown in the next example(s).

Example 2: Compute the following integrals:

① $\int x\sqrt{x-5} dx$ $\frac{u=x-5}{du=dx}$ $\int x \cdot \sqrt{u} du$

Issue: Our integral has x and u !

Can I rewrite x into some function of u ? Yes

$$u = x - 5 \Leftrightarrow u + 5 = x$$

$$= \int (u+5)u^{1/2} du = \int (u^{3/2} + 5u^{1/2}) du$$

$$= \frac{2}{5} u^{5/2} + 5 \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{5} (x-5)^{5/2} + \frac{10}{3} (x-5)^{3/2} + C$$

② $\int \frac{2x}{(x+7)^{3/2}} dx$ $\frac{u=x+7}{du=dx} \Leftrightarrow x=u-7$ $\int \frac{2(u-7)}{u^{3/2}} du$

$$= \int (2u^{-1/2} - 14u^{-3/2}) du$$

$$= 2 \cdot \frac{2}{1} u^{1/2} - 14 \cdot \frac{2}{-1} u^{-1/2} + C$$

$$= 4(x+7)^{1/2} + 28(x+7)^{-1/2} + C$$

Now let's do integration by substitution for definite integrals.

Example 3: Compute

$$\int_2^4 x \sin(x^2) dx \quad \frac{u=x^2}{\frac{du}{dx}=2x} \quad \int \cancel{x} \sin(u) \frac{du}{2\cancel{x}} = \int \frac{1}{2} \sin(u) du$$

Issue: What do I do with the limit values? i.e. S_2^4

$$\text{Well } \int_2^4 x \sin(x^2) dx = \int_{x=2}^{x=4} x \sin(x^2) dx$$

Method 1: Changing $x=2$ and $x=4$ using $u=x^2$

$$\begin{aligned} \int_{x=2}^{x=4} x \sin(x^2) dx & \xrightarrow[u=4]{u=x^2} \int_{u=4}^{u=16} \sin(u) \frac{du}{2} \\ & = -\frac{\cos(u)}{2} \Big|_{u=4}^{u=16} = -\frac{1}{2} (\cos(16) - \cos(4)) \end{aligned}$$

Method 2: To treat the integral after the u -sub as indefinite and when you plug u back evaluate with original bounds.

$$\begin{aligned} \int_{x=2}^{x=4} x \sin(x^2) dx & = \frac{u=x^2}{du=dx} \int \frac{\sin(u)}{2} du \\ & = -\frac{\cos(u)}{2} = -\frac{\cos(x^2)}{2} \Big|_{x=2}^{x=4} \\ & = -\frac{1}{2} (\cos(16) - \cos(4)) \end{aligned}$$

Example 4: IF the area of the region under the curve $y = \frac{1}{\sqrt{9x+4}}$ over the interval $0 \leq x \leq a$ is 6, then what is a ?

i.e. Solve $\int_0^a \frac{1}{\sqrt{9x+4}} dx = 6$ for a ,

$$\begin{aligned} 6 & = \int_0^a \frac{1}{\sqrt{9x+4}} dx \xrightarrow[du=9dx]{u=9x+4} \int \frac{1}{\sqrt{u}} \cdot \frac{du}{9} = \frac{1}{9} \int u^{-1/2} du \\ & = \frac{1}{9} \cdot \frac{2}{1} u^{1/2} = \frac{2}{9} (9x+4)^{1/2} \Big|_0^a = \frac{2}{9} (9a+4)^{1/2} - \frac{2}{9} (4)^{1/2} \\ & = \frac{2}{9} (9a+4)^{1/2} - \frac{4}{9} \end{aligned}$$

$$\frac{58}{9} = \frac{2}{9} (9a+4)^{1/2}$$

$$\frac{9}{2} \cdot \frac{58}{9} = \frac{9}{2} \cdot \frac{2}{9} (9a+4)^{1/2}$$

$$29 = (9a+4)^{1/2}$$

$$(29)^2 = 9a+4$$

$$841 = 9a+4$$

$$837 = 9a$$

$$a = \frac{837}{9} = 93$$

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Example 5: It is estimated that t hours after 8:00 am, the population of a certain bacterial sample will be changing at a rate of:

$$N'(t) = \frac{3t}{\sqrt{t+4}} \quad \text{bacteria per hour.}$$

Find the increase in the bacteria population from 11:00 am to 1:00 pm.

$$\left. \begin{array}{l} 11:00 \text{ am} \Rightarrow t=3 \\ 1:00 \text{ pm} \Rightarrow t=5 \end{array} \right\} \Rightarrow \int_3^5 \frac{3t}{\sqrt{t+4}} dt$$

$$\text{So } \underbrace{u=t+4 \Leftrightarrow u-4=t}_{du=dt} \quad \int \frac{3(u-4)}{u^{1/2}} du = \int (3u^{1/2} - 12u^{-1/2}) du$$

$$= 3 \cdot \frac{2}{3} u^{3/2} - 12 \cdot \frac{2}{1} u^{1/2} = \left(2(t+4)^{3/2} - 24(t+4)^{1/2} \right) \Big|_3^5$$

$$= \left(2(5+4)^{3/2} - 24(5+4)^{1/2} \right) - \left(2(3+4)^{3/2} - 24(3+4)^{1/2} \right)$$

$$\approx 8.458$$

Example 6: It is estimated that t – weeks into a semester, the average amount of sleep a college math student gets per day $S(t)$ at a rate of

$$-\frac{6t}{e^{t^2}} \quad \text{hours per day.}$$

When the semester begins, math students sleep on average of 8.1 hours per day. What is $S(t)$, 10 week(s) into the semester?

$$S(t) = \int -\frac{6t}{e^{t^2}} dt = \int -6t e^{-t^2} dt \quad \begin{array}{l} u = -t^2 \\ du = -2t dt \\ \frac{du}{-2t} = dt \end{array} \quad \int -6t e^u \cdot \frac{du}{-2t}$$

$$= \int 3e^u du = 3e^u + C = 3e^{-t^2} + C$$

Now find C with $S(0) = 8.1$

$$8.1 = S(0) = 3e^{-0^2} + C$$

$$8.1 = 3 + C$$

$$5.1 = C$$

$$\text{So } S(t) = 3e^{-t^2} + 5.1$$

$$S(10) = 3e^{-100} + 5.1 = 5.1$$