

Lesson 3: The Natural Logarithmic Function: Integration

The function $f(t) = 1/t$ is continuous on $(0, \infty)$.

By FTC, $f(t) = 1/t$ has an antiderivative on the interval with endpoints x and 1 when $x > 0$.

Definition: $\ln(x) = \int_1^x \frac{1}{t} dt$

But in general, $\ln|x| = \int \frac{dt}{t}$

Example 1: Let $f(x) = \ln(x^5 + 7x + 12)$. Compute $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{1}{x^5 + 7x + 12} \cdot (x^5 + 7x + 12)' \\ &= \frac{1}{x^5 + 7x + 12} \cdot (5x^4 + 7) \\ &= \frac{5x^4 + 7}{x^5 + 7x + 12} \end{aligned}$$

Example 2: Compute the following:

$$\textcircled{a} \int \frac{5x^4 + 7}{x^5 + 7x + 12} dx \quad \begin{array}{l} u = x^5 + 7x + 12 \\ du = (5x^4 + 7)dx \end{array} \quad \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|x^5 + 7x + 12| + C$$

$$\textcircled{b} \int \frac{10t}{9t^2 + 7} dt \quad \begin{array}{l} u = 9t^2 + 7 \\ du = 18t dt \\ \frac{du}{18t} = dt \end{array} \quad \int \frac{10t}{u} \cdot \frac{du}{18t} = \frac{5}{9} \int \frac{du}{u}$$
$$= \frac{5}{9} \ln|u| + C = \frac{5}{9} \ln|9t^2 + 7| + C$$

$$\textcircled{c} \int \frac{1000}{10+3x} dx \quad \begin{array}{l} u=10+3x \\ du=3dx \\ \frac{du}{3}=dx \end{array} \int \frac{1000}{u} \cdot \frac{du}{3} = \frac{1000}{3} \int \frac{du}{u}$$

$$= \frac{1000}{3} \ln|u| + C = \frac{1000}{3} \ln|10+3x| + C$$

$$\textcircled{d} \int \cot x dx = \int \frac{\cos x}{\sin x} dx \quad \begin{array}{l} u=\sin x \\ du=\cos x dx \end{array} \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sin x| + C$$

$$\textcircled{e} \int \tan x dx = \int \frac{\sin x}{\cos x} dx \quad \begin{array}{l} u=\cos x \\ du=-\sin x dx \\ -du=\sin x dx \end{array} \int \frac{-du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$\textcircled{f} \int \frac{8\sin x}{3+\cos x} dx \quad \begin{array}{l} u=3+\cos x \\ du=-\sin x dx \\ \frac{du}{-\sin x}=dx \end{array} \int \frac{8\sin x}{u} \cdot \frac{du}{-\sin x}$$

$$= -8 \int \frac{du}{u} = -8 \ln|u| + C = -8 \ln|3+\cos x| + C$$

Example 3: Compute the following:

$$\textcircled{a} \int \frac{dx}{5x(\ln(3x))^2} \quad \begin{array}{l} u=\ln(3x) \\ du=\frac{1}{3x} \cdot 3 dx \\ du=\frac{1}{x} dx \\ x du=dx \end{array} \int \frac{x du}{5x \cdot u^2} = \int \frac{du}{5u^2}$$

$$= \frac{1}{5} \int u^{-2} du = \frac{1}{5} \frac{u^{-1}}{-1} + C = \frac{1}{5u} + C$$

$$= \frac{1}{5 \ln(3x)} + C$$

$$\begin{aligned}
 \textcircled{b} \int \frac{dx}{x \ln(x^{19})} & \quad \begin{array}{l} u = \ln(x^{19}) \\ du = \frac{1}{x} \cdot 19x^{18} dx \\ du = \frac{19}{x} dx \\ \frac{x}{19} du = dx \end{array} \int \frac{1}{xu} \cdot \frac{x}{19} du = \frac{1}{19} \int \frac{du}{u} \\
 & = \frac{1}{19} \ln|u| + C = \frac{1}{19} \ln|\ln(x^{19})| + C
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{c} \int \frac{4(\ln x)^3}{x} dx & \quad \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \\ x du = dx \end{array} \int \frac{4u^3}{x} \cdot x du = \int 4u^3 du \\
 & = \frac{4u^4}{4} + C = u^4 + C = (\ln(x))^4 + C
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{d} \int \frac{\ln \sqrt{x}}{x} dx & = \int \frac{\ln(x^{1/2})}{x} dx = \int \frac{1/2 \ln x}{x} dx \quad \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \\ x du = dx \end{array} \\
 & = \frac{1}{2} \int \frac{u}{x} \cdot x du = \frac{1}{2} \int u du = \frac{1}{2} \cdot \frac{u^2}{2} + C \\
 & = \frac{1}{4} (\ln x)^2 + C
 \end{aligned}$$