Lesson 4: Integration by Parts I

Recall the Product Rule,

$$(uv)' = u'v + uv'$$

what if we integrate both sides with respect to x.

$$\int (uv)' dx = \int (u'v + uv') dx$$

$$\int (uv)' dx = \int u'v\,dx + \int uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

Remember $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$. So

$$uv = \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx$$

$$uv = \int v\,du + \int u\,dv$$

$$uv - \int v\,du = \int u\,dv$$

Interesting enough is that for many cases an integral will have the form

$$\int u\,dv$$

This technique of turning one integral into another is called Integration by Parts.

It's formula is: $\int u\,dv = uv - \int v\,du$

To use this technique

- Choose u to be the one to differentiate
- Choose dv to be integrated

Remark: Sometimes picking u and dv can be tricking. There is an acronym that makes picking u easier. Think of it as a sort of order of operation for choosing u.

- L - Logarithmic
- A - Algebraic
- T - Trigonometric
- E - Exponential

Example 1: Evaluate

ⓐ $\int x \ln x \, dx$ $\quad \frac{u = \ln x}{du = \frac{1}{x} dx} \quad \frac{dv = x\,dx}{v = \frac{x^2}{2}} \quad uv - \int v\,du$

$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$

$= \frac{x^3}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

ⓑ $\int x \cos x \, dx$ $\quad \frac{u = x}{du = dx} \quad \frac{dv = \cos x\,dx}{v = \sin x} \quad uv - \int v\,du$

$= x \sin x - \int \sin x \, dx = x \sin x - (-\cos x) + C$

$= x \sin x + \cos x + C$

ⓒ $\int x e^{2x} dx$ $\quad \frac{u = x}{du = dx} \quad \frac{dv = e^{2x} dx}{v = \frac{1}{2} e^{2x}} \quad uv - \int v\,du$

$= \frac{x}{2} e^{2x} - \underbrace{\int \frac{1}{2} e^{2x} dx}_{\text{found by u-sub}} = \frac{x}{2} e^{2x} - \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C$

$= \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C$

Example 2: Evaluate

(a) $\int_0^{\pi/2} 8x\sin x\,dx$ $\quad \dfrac{u=8x}{du=8dx} \quad \dfrac{dv=\sin x\,dx}{v=-\cos x}$

$= -8x\cos x\Big]_0^{\pi/2} - \int_0^{\pi/2} -8\cos x\,dx$

$= -8x\cos x\Big]_0^{\pi/2} + \int_0^{\pi/2} 8\cos x\,dx$

$= -8x\cos x\Big]_0^{\pi/2} + 8\sin x\Big]_0^{\pi/2}$

$= -\dfrac{8\pi}{2}\cos\left(\dfrac{\pi}{2}\right) - \left(-8(0)\cos(0)\right) + 8\sin\left(\dfrac{\pi}{2}\right) - 8\sin(0)$

$= 0 - 0 + 8 - 0 = 8$

(b) $\int_1^e x\ln\sqrt[3]{x}\,dx = \int_1^e x\ln(x^{1/3})\,dx = \int_1^e \dfrac{1}{3}x\ln x\,dx$

$\dfrac{u=\ln x}{du=\frac{1}{x}dx} \quad \dfrac{dv=\frac{1}{3}x\,dx}{v=\frac{1}{3}\cdot\frac{x^2}{2}} \quad \dfrac{x^2}{6}\ln x\Big]_1^e - \int_1^e \dfrac{x^2}{6}\cdot\dfrac{1}{x}\,dx$

$= \dfrac{x^2}{6}\ln x\Big]_1^e - \int_1^e \dfrac{x}{6}\,dx = \dfrac{x^2}{6}\ln x\Big]_1^e - \dfrac{x^2}{12}\Big]_1^e$

$= \dfrac{e^2}{6}\ln e - \dfrac{1}{6}\ln(1) - \left(\dfrac{e^2}{12} - \dfrac{1}{12}\right)$

$= \dfrac{e^2}{6} - 0 - \dfrac{e^2}{12} + \dfrac{1}{12}$

(c) $\int_1^e \dfrac{\ln x}{x^4}\,dx = \int_1^e x^{-4}\ln x\,dx$ $\quad \dfrac{u=\ln x}{du=\frac{1}{x}dx} \quad \dfrac{dv=x^{-4}dx}{v=\frac{x^{-3}}{-3}=\frac{-1}{3x^3}}$

$= \dfrac{-1}{3x^3}\ln x\Big]_1^e - \int_1^e \dfrac{-1}{3x^3}\cdot\dfrac{1}{x}\,dx = -\dfrac{\ln x}{3x^3}\Big]_1^e + \int_1^e \dfrac{1}{3x^4}\,dx$

$= -\dfrac{\ln x}{3x^3}\Big]_1^e + \dfrac{1}{3}\int_1^e x^{-4}\,dx = -\dfrac{\ln x}{3x^3}\Big]_1^e + \dfrac{1}{3}\,\dfrac{x^{-3}}{-3}\Big]_1^e$

$= -\dfrac{\ln e}{3e^3} - \left(\dfrac{-\ln(1)}{3\cdot 1}\right) - \dfrac{1}{9}\left((e)^{-3} - 1^{-3}\right)$

$= \dfrac{-1}{3e^3} - 0 - \dfrac{1}{9}\left(\dfrac{1}{e^3} - 1\right) = \dfrac{-1}{3e^3} - \dfrac{1}{9e^3} + \dfrac{1}{9}$