

Lesson 6: Partial Fractions I

Recall the concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{2} \cdot \frac{3}{3} + \frac{1}{3} \cdot \frac{2}{2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

So we can say that a partial fraction decomposition for $5/6$ is

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

This concept can be used with functions of x .

Example 1: Combine the following fractions

$$\begin{aligned} \frac{1}{x-2} + \frac{3}{x-5} &= \frac{1}{(x-2)} \cdot \frac{(x-5)}{(x-5)} + \frac{3}{(x-5)} \cdot \frac{(x-2)}{(x-2)} \\ &= \frac{(x-5) + 3(x-2)}{(x-2)(x-5)} \\ &= \frac{x-5+3x-6}{x^2-2x-5x+10} \\ &= \frac{4x-11}{x^2-7x+10} \end{aligned}$$

Why do we care about partial fraction decomposition?
It's because u-sub isn't enough.

Example 2: Evaluate $\int \frac{4x-11}{x^2-7x+10} dx$

Let's first try a u-sub.

$$\begin{aligned} u &= x^2 - 7x + 10 \\ du &= (2x - 7) dx \end{aligned} \quad \int \frac{4x-11}{u} \cdot \frac{du}{2x-7}$$

As you can see there is no way to eliminate the x 's.

Now let's try partial fraction decomposition. Using Ex 1,

$$\int \frac{4x-11}{x^2-7x+10} dx = \int \frac{1}{x-2} dx + \int \frac{3}{x-5} dx$$

Now we know how to integrate these functions:

$$\bullet \int \frac{1}{x-2} dx \quad \begin{array}{l} u=x-2 \\ du=dx \end{array} \int \frac{1}{u} du = \ln|u| = \ln|x-2|$$

$$\bullet \int \frac{3}{x-5} dx \quad \begin{array}{l} u=x-5 \\ du=dx \end{array} \int \frac{3}{u} du = 3\ln|u| = 3\ln|x-5|$$

$$\text{So } \int \frac{4x-11}{x^2-7x+10} dx = \ln|x-2| + 3\ln|x-5| + C$$

Method of Decomposing into Partial Fractions.

Given a rational function $\frac{N(x)}{D(x)}$

① Factor the denominator as much as possible.

② Write the fraction into decomposition form.

Ⓐ Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

Ⓑ Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

Ⓒ Distinct irreducible quadratic terms like x^2+a^2 decompose to

$$\frac{Ax+B}{x^2+a^2}$$

Ⓓ Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

③ Combine your decomposition from ② as 1 fraction.

④ Set the original numerator, $N(x)$, equal to the numerator from ③.

⑤ Equate the coefficients of the terms, to yields a system of equations. Then solve the constants, i.e. A, B, C .

⑥ Plug the values found in (5) in (2).

In today's class, we will be focussing on this method.

Example 3; Determine the partial fraction decomposition without explicitly solving for the constants,

i.e. Do Steps 1+2 of the Method of Decomposing into Partial Fractions. Handout

(a) $f(x) = \frac{6x+10}{x^2+5x}$

① Factor the denominator, x^2+5x , completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.

Note x and $x+5$ are distinct linear terms. So by (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{6x+10}{x^2+5x} = \frac{6x+10}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}$$

(b) $f(x) = \frac{40}{x^2-16}$

① Factor the denominator, x^2-16 , completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form.

Note $x-4$ and $x+4$ are distinct linear terms. So by (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{40}{x^2-16} = \frac{40}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4}$$

(c) $f(x) = \frac{x^2+2}{x^3+3x^2+2x}$

① Factor the denominator, x^3+3x^2+2x , completely.

$$x^3+3x^2+2x = x(x^2+3x+2) = x(x+1)(x+2)$$

- (2) Write the fraction into decomposition form.
 Note x , $x+1$, and $x+2$ are distinct linear terms.
 So by (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{x^2+2}{x^3+3x^2+2x} = \frac{x^2+2}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

(d) $f(x) = \frac{4x^2-4}{x^3-2x^2}$

- (1) Factor the denominator, x^3-2x^2 , completely.
 $x^3-2x^2 = x^2(x-2) = (x-0)^2(x-2)$

- (2) Write the fraction into decomposition form.
 Note $(x-0)^2$ is a repeated linear term, and
 $x-2$ is a distinct linear term.

So by (2b) and (2a) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{4x^2-4}{x^3-2x^2} = \frac{4x^2-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

(e) $f(x) = \frac{31}{(x^2-81)^2}$

- (1) Factor the denominator, $(x^2-81)^2$, completely.
 $(x^2-81)^2 = [(x-9)(x+9)]^2 = (x-9)^2(x+9)^2$

- (2) Write the fraction into decomposition form.
 Note both $(x-9)^2$ and $(x+9)^2$ are repeated linear terms. So by (2b) in the Method of Decomposing Into Partial Fractions Handout,

$$\begin{aligned} \frac{31}{(x^2-81)^2} &= \frac{31}{(x-9)^2(x+9)^2} \\ &= \frac{A}{x-9} + \frac{B}{(x-9)^2} + \frac{C}{x+9} + \frac{D}{(x+9)^2} \end{aligned}$$

$$f) f(x) = \frac{5x^2 + 9}{x^3 + 3x}$$

① Factor the denominator, $x^3 + 3x$, completely.
 $x^3 + 3x = x(x^2 + 3)$

② Write the fraction into decomposition form.

Note x is a distinct linear term, and

$x^2 + 3$ is a distinct irreducible quadratic term.

So by (2a) and (2c) in the Method of Decomposing Into Partial Fractions Handout,

$$\frac{5x^2 + 9}{x^3 + 3x} = \frac{5x^2 + 9}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

$$g) f(x) = \frac{27x - 4}{(x^2 - 25)(x^2 + 64)}$$

① Factor the denominator, $(x^2 - 25)(x^2 + 64)$, completely.

$$\underbrace{(x^2 - 25)}_{\text{Factorable}} \underbrace{(x^2 + 64)}_{\text{Not}} = (x - 5)(x + 5)(x^2 + 64)$$

② Write the fraction into decomposition form.

Note $x - 5$ and $x + 5$ are distinct linear terms, and

$x^2 + 64$ is a distinct irreducible quadratic term.

So by (2a) and (2c) in the Method of Decomposing Into Partial Fractions Handout,

$$\begin{aligned} \frac{27x - 4}{(x^2 - 25)(x^2 + 64)} &= \frac{27x - 4}{(x - 5)(x + 5)(x^2 + 64)} \\ &= \frac{A}{x - 5} + \frac{B}{x + 5} + \frac{Cx + D}{x^2 + 64} \end{aligned}$$

$$h) f(x) = \frac{2x^3 + 3x^2 + 5x + 2}{(x^2 + x + 1)^2}$$

① Factor the denominator, $(x^2 + x + 1)^2$, completely.
 Done!

② Write the fraction into decomposition form.

Note $(x^2 + x + 1)^2$ is a repeated irreducible quadratic term

So by (2d) in the Method of Decomposing Into Partial Fractions Handout

$$f(x) = \frac{2x^3 + 3x^2 + 5x + 2}{(x^2 + x + 1)^2} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + x + 1)^2}$$

Now let's find those constants.

Example 4: Let $f(x) = \frac{6x+10}{x^2+5x}$

Determine the partial fraction decomposition of $f(x)$.

① Factor x^2+5x completely.

$$x^2+5x = x(x+5)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+5}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{B}{x+5} = \frac{A}{x} \cdot \frac{(x+5)}{(x+5)} + \frac{B}{x+5} \cdot \frac{x}{x}$$

$$= \frac{A(x+5) + Bx}{x(x+5)} = \frac{Ax + 5A + Bx}{x(x+5)} = \frac{(A+B)x + 5A}{x(x+5)}$$

④ Set the old numerator = new numerator.

$$6x+10 = (A+B)x + 5A$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=6 & \textcircled{i} \\ 5A=10 & \textcircled{ii} \end{cases}$$

From ②, we find $A=2$.

Plug $A=2$ into ①

$$A+B=6$$

$$2+B=6$$

$$B=4$$

⑥ Plug $A=2$ and $B=4$ into ②.

$$\frac{2}{x} + \frac{4}{x+5}$$

Example 5: Let $f(x) = \frac{40}{x^2-16}$

Determine the partial fraction decomposition of $f(x)$.

① Factor x^2-16 completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form,

$$\frac{A}{x-4} + \frac{B}{x+4}$$

③ Combine the fractions in ②,

$$\frac{A}{x-4} + \frac{B}{x+4} = \frac{A}{x-4} \cdot \frac{(x+4)}{(x+4)} + \frac{B}{x+4} \cdot \frac{(x-4)}{(x-4)}$$

$$= \frac{A(x+4) + B(x-4)}{(x-4)(x+4)}$$

$$= \frac{Ax + 4A + Bx - 4B}{(x-4)(x+4)} = \frac{(A+B)x + (4A-4B)}{(x-4)(x+4)}$$

④ Set the old numerator = new numerator

$$0x + 40 = (A+B)x + (4A-4B)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=0 \\ 4A-4B=40 \end{cases} \Rightarrow \begin{cases} B=-A & \textcircled{i} \\ A-B=10 & \textcircled{ii} \end{cases}$$

Plug ① into ②.

$$A-B=10$$

$$A-(-A)=10$$

$$A+A=10$$

$$2A=10$$

$$A=5$$

Plug $A=5$ into ①

$$B=-A$$

$$B=-5$$

⑥ Plug $A=5$ and $B=-5$ into ②,

$$\frac{5}{x-4} + \frac{-5}{x+4}$$

Example 6: Let $f(x) = \frac{x^2 + 2}{x^3 + 3x^2 + 2x}$

Determine the partial fraction decomposition of $f(x)$.

① Factor $x^3 + 3x^2 + 2x$ completely.

$$\begin{aligned} x^3 + 3x^2 + 2x &= x(x^2 + 3x + 2) \\ &= x(x+1)(x+2) \end{aligned}$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

③ Combine the fractions in ②.

$$\begin{aligned} \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} &= \frac{A}{x} \cdot \frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{B}{x+1} \cdot \frac{x(x+2)}{x(x+2)} + \frac{C}{x+2} \cdot \frac{x(x+1)}{x(x+1)} \\ &= \frac{A(x^2 + 3x + 2) + B(x^2 + 2x) + C(x^2 + x)}{x(x+1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (3A+2B+C)x + (2A)}{x(x+1)(x+2)} \end{aligned}$$

④ Set the old numerator = new numerator

$$x^2 + 2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

$$x^2 + 0x + 2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B+C=1 & \text{(i)} \\ 3A+2B+C=0 & \text{(ii)} \\ 2A=2 & \text{(iii)} \end{cases}$$

From (iii), we find $A=1$. So we can plug that into (i) and (ii) yielding

$$\begin{cases} 1+B+C=1 \\ 3+2B+C=0 \end{cases} \Rightarrow \begin{cases} B+C=0 \\ 2B+C=-3 \end{cases} \Rightarrow \begin{cases} C=-B & \text{(i')} \\ 2B+C=-3 & \text{(ii')} \end{cases}$$

Plug (i') into (ii').

$$2B + C = -3$$

$$2B - B = -3$$

$$B = -3$$

Plug $B = -3$ into (i')

$$C = -B$$

$$= -(-3)$$

$$= 3$$

⑥ Plug $A=1$, $B=-3$, $C=3$ into ②.

$$\frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$$

Example 7: Let $f(x) = \frac{4x^2 - 4}{x^3 - 2x^2}$

Determine the partial fraction decomposition of $f(x)$.

① Factor $x^3 - 2x^2$ completely.

$$x^3 - 2x^2 = x^2(x - 2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

③ Combine the fractions in ②.

Note the common denominator is $x^2(x-2)$.

$$\begin{aligned} \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} &= \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)} \\ &= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)} \\ &= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^3 - 2x^2} \end{aligned}$$

④ Set the old numerator = new numerator

$$4x^2 - 4 = (A+C)x^2 + (B-2A)x - 2B$$

$$4x^2 + 0x - 4 = (A+C)x^2 + (B-2A)x - 2B$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+C=4 & \text{(i)} \\ B-2A=0 & \text{(ii)} \\ -2B=-4 & \text{(iii)} \end{cases}$$

From (iii) $-2B = -4 \Rightarrow B = 2$ | Plug $A=1$ into (i).

Plug $B=2$ into (ii)

$$B - 2A = 0$$

$$2 - 2A = 0$$

$$2 = 2A \Rightarrow A = 1$$

$$A + C = 4$$

$$1 + C = 4$$

$$C = 3$$

⑥ Plug $A=1, B=2, C=3$ into ②.

$$\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x-2}$$

Example 8: Let $f(x) = \frac{5x^2+9}{x^3+3x}$

Determine the partial decomposition of $f(x)$.

① Factor x^3+3x completely.

$$x^3+3x = x(x^2+3)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{Bx+C}{x^2+3}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3) + (Bx+C)x}{x(x^2+3)}$$

$$= \frac{Ax^2 + 3A + Bx^2 + Cx}{x^3+3x}$$

$$= \frac{(A+B)x^2 + Cx + 3A}{x^3+3x}$$

④ Set the old numerator = new numerator

$$5x^2+9 = (A+B)x^2 + Cx + 3A$$

$$5x^2+0x+9 = (A+B)x^2 + Cx + 3A$$

⑤ Create a system of equations from ④ and solve.

$$\begin{cases} 5 = A+B & \text{(i)} \\ 0 = C & \text{(ii)} \\ 9 = 3A & \text{(iii)} \end{cases}$$

Note (i) already gives us $C=0$.

From (ii) $9=3A \Rightarrow A=3$

Plug $A=3$ into (1).

$$5 = A + B$$

$$5 = 3 + B$$

$$2 = B$$

(6) Plug $A=3, B=2, C=0$ into (2).

$$\frac{3}{x} + \frac{2x+0}{x^2+3}$$

Example 9: Let $f(x) = \frac{2x^3 + 3x^2 + 5x + 2}{(x^2 + x + 1)^2}$

Determine the partial fraction decomposition of $f(x)$.

(1) Factor $(x^2 + x + 1)^2$ completely.

Done!

(2) Write the fraction into decomposition form.

$$\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$$

(3) Combine the fractions in (2).

Note the common denominator is $(x^2 + x + 1)^2$

$$\begin{aligned} \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} &= \frac{(Ax+B)(x^2+x+1) + Cx+D}{(x^2+x+1)^2} \\ &= \frac{Ax^3 + Bx^2 + Ax^2 + Bx + Ax + B + Cx + D}{(x^2+x+1)^2} \end{aligned}$$

$$= \frac{Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)}{(x^2+x+1)^2}$$

④ Set the old numerator = new numerator

$$2x^3 + 3x^2 + 5x + 2 = Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A = 2 & \text{(i)} \\ A + B = 3 & \text{(ii)} \\ A + B + C = 5 & \text{(iii)} \\ B + D = 2 & \text{(iv)} \end{cases}$$

From ①, we have $A = 2$.

Plug $A = 2$ into ②

$$A + B = 3$$

$$2 + B = 3$$

$$B = 1$$

Plug $A = 2, B = 1$ into ③

$$A + B + C = 5$$

$$2 + 1 + C = 5$$

$$3 + C = 5$$

$$C = 2$$

Plug $B = 1$ into ④

$$B + D = 2$$

$$1 + D = 2$$

$$D = 1$$

⑥ Plug $A = 2, B = 1, C = 2, D = 1$ into ②,

$$\frac{2x+1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$$