

Lesson 7: Partial Fractions II

Recall the concept of adding fractions by getting a common denominator. For example,

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{2} \cdot \frac{3}{3} + \frac{1}{3} \cdot \frac{2}{2} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

So we can say that a partial fraction decomposition for $5/6$ is

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

This concept can be used with functions of x .

Example 1: Combine the following fractions

$$\begin{aligned} \frac{1}{x-2} + \frac{3}{x-5} &= \frac{1}{(x-2)} \cdot \frac{(x-5)}{(x-5)} + \frac{3}{(x-5)} \cdot \frac{(x-2)}{(x-2)} \\ &= \frac{(x-5) + 3(x-2)}{(x-2)(x-5)} \\ &= \frac{x-5+3x-6}{x^2-2x-5x+10} \\ &= \frac{4x-11}{x^2-7x+10} \end{aligned}$$

Why do we care about partial fraction decomposition?
It's because u-sub isn't enough.

Example 2: Evaluate $\int \frac{4x-11}{x^2-7x+10} dx$

Let's first try a u-sub.

$$\begin{aligned} u &= x^2 - 7x + 10 \\ du &= (2x - 7) dx \end{aligned} \quad \int \frac{4x-11}{u} \cdot \frac{du}{2x-7}$$

As you can see there is no way to eliminate the x's.

Now let's try partial fraction decomposition. Using Ex 1,

$$\int \frac{4x-11}{x^2-7x+10} dx = \int \frac{1}{x-2} dx + \int \frac{3}{x-5} dx$$

Now we know how to integrate these functions:

$$\bullet \int \frac{1}{x-2} dx \quad \begin{array}{l} u=x-2 \\ du=dx \end{array} \int \frac{1}{u} du = \ln|u| = \ln|x-2|$$

$$\bullet \int \frac{3}{x-5} dx \quad \begin{array}{l} u=x-5 \\ du=dx \end{array} \int \frac{3}{u} du = 3\ln|u| = 3\ln|x-5|$$

$$\text{So } \int \frac{4x-11}{x^2-7x+10} dx = \ln|x-2| + 3\ln|x-5| + C$$

Method of Decomposing into Partial Fractions.

Given a rational function $\frac{N(x)}{D(x)}$

① Factor the denominator as much as possible.

② Write the fraction into decomposition form.

ⓐ Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

ⓑ Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

ⓒ Distinct irreducible quadratic terms like x^2+a^2 decompose to

$$\frac{Ax+B}{x^2+a^2}$$

ⓓ Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{(x^2+a^2)^2}$$

③ Combine your decomposition from ② as 1 fraction.

④ Set the original numerator, $N(x)$, equal to the numerator from ③.

⑤ Equate the coefficients of the terms, to yields a system of equations. Then solve the constants, i.e. A, B, C .

⑥ Plug the values found in (5) in (2).

Example 3: Let $f(x) = \frac{6x+10}{x^2+5x}$

(a) Determine the partial fraction decomposition of $f(x)$.

(1) Factor x^2+5x completely.

$$x^2+5x = x(x+5)$$

(2) Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+5}$$

(3) Combine the fractions in (2).

$$\frac{A}{x} + \frac{B}{x+5} = \frac{A}{x} \cdot \frac{(x+5)}{(x+5)} + \frac{B}{x+5} \cdot \frac{x}{x}$$

$$= \frac{A(x+5) + Bx}{x(x+5)} = \frac{Ax + 5A + Bx}{x(x+5)} = \frac{(A+B)x + 5A}{x(x+5)}$$

(4) Set the old numerator = new numerator.

$$6x+10 = (A+B)x + 5A$$

(5) Create a system of equations from (4), and solve.

$$\begin{cases} A+B=6 & \text{(i)} \\ 5A=10 & \text{(ii)} \end{cases}$$

From (ii), we find $A=2$.

Plug $A=2$ into (i)

$$A+B=6$$

$$2+B=6$$

$$B=4$$

(6) Plug $A=2$ and $B=4$ into (2).

$$\frac{2}{x} + \frac{4}{x+5}$$

(b) Using (a), evaluate $\int f(x) dx$.

$$\int \frac{6x+10}{x^2+5x} dx = \int \frac{2}{x} dx + \int \frac{4}{x+5} dx$$

$$= 2 \ln|x| + 4 \ln|x+5| + C$$

Example 4: Let $f(x) = \frac{40}{x^2-16}$

(a) Determine the partial fraction decomposition of $f(x)$.

① Factor x^2-16 completely.

$$x^2-16 = (x-4)(x+4)$$

② Write the fraction into decomposition form,

$$\frac{A}{x-4} + \frac{B}{x+4}$$

③ Combine the fractions in ②,

$$\begin{aligned} \frac{A}{x-4} + \frac{B}{x+4} &= \frac{A}{x-4} \cdot \frac{(x+4)}{(x+4)} + \frac{B}{x+4} \cdot \frac{(x-4)}{(x-4)} \\ &= \frac{A(x+4) + B(x-4)}{(x-4)(x+4)} \end{aligned}$$

$$= \frac{Ax + 4A + Bx - 4B}{(x-4)(x+4)} = \frac{(A+B)x + (4A-4B)}{(x-4)(x+4)}$$

④ Set the old numerator = new numerator

$$0x + 40 = (A+B)x + (4A-4B)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B=0 \\ 4A-4B=40 \end{cases} \Rightarrow \begin{cases} B=-A & \textcircled{i} \\ A-B=10 & \textcircled{ii} \end{cases}$$

Plug ① into ②.

$$\begin{aligned} A-B &= 10 \\ A-(-A) &= 10 \\ A+A &= 10 \\ 2A &= 10 \\ A &= 5 \end{aligned}$$

Plug $A=5$ into ①

$$\begin{aligned} B &= -A \\ &= -5 \end{aligned}$$

⑥ Plug $A=5$ and $B=-5$ into ②,

$$\frac{5}{x-4} + \frac{-5}{x+4}$$

(b) Using (a), evaluate $\int f(x) dx$.

$$\int \frac{40}{x^2-16} dx = \int \frac{5}{x-4} dx + \int \frac{-5}{x+4} dx$$

$$= 5 \ln|x-4| - 5 \ln|x+4| + C$$

Example 5: Let $f(x) = \frac{x^2 + 2}{x^3 + 3x^2 + 2x}$

(a) Determine the partial fraction decomposition of $f(x)$.

① Factor $x^3 + 3x^2 + 2x$ completely.

$$\begin{aligned}x^3 + 3x^2 + 2x &= x(x^2 + 3x + 2) \\ &= x(x+1)(x+2)\end{aligned}$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

③ Combine the fractions in ②.

$$\begin{aligned}\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} &= \frac{A}{x} \cdot \frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{B}{x+1} \cdot \frac{x(x+2)}{x(x+2)} + \frac{C}{x+2} \cdot \frac{x(x+1)}{x(x+1)} \\ &= \frac{A(x^2 + 3x + 2) + B(x^2 + 2x) + C(x^2 + x)}{x(x+1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (3A+2B+C)x + (2A)}{x(x+1)(x+2)}\end{aligned}$$

④ Set the old numerator = new numerator

$$x^2 + 2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

$$x^2 + 0x + 2 = (A+B+C)x^2 + (3A+2B+C)x + (2A)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+B+C=1 & \text{(i)} \\ 3A+2B+C=0 & \text{(ii)} \\ 2A=2 & \text{(iii)} \end{cases}$$

From (iii), we find $A=1$. So we can plug that into (i) and (ii) yielding

$$\begin{cases} 1+B+C=1 \\ 3+2B+C=0 \end{cases} \Rightarrow \begin{cases} B+C=0 \\ 2B+C=-3 \end{cases} \Rightarrow \begin{cases} C=-B & \text{(i')} \\ 2B+C=-3 & \text{(ii')} \end{cases}$$

Plug (i') into (ii').

$$2B + C = -3$$

$$2B - B = -3$$

$$B = -3$$

Plug $B = -3$ into (i')

$$C = -B$$

$$= -(-3)$$

$$= 3$$

⑥ Plug $A=1$, $B=-3$, $C=3$ into ②.

$$\frac{1}{x} + \frac{-3}{x+1} + \frac{3}{x+2}$$

⑥ Using ②, evaluate $\int f(x) dx$

$$\int \frac{x^2+2}{x^3+3x^2+2x} dx = \int \frac{1}{x} dx + \int \frac{-3}{x+1} dx + \int \frac{3}{x+2} dx$$
$$= \ln|x| - 3\ln|x+1| + 3\ln|x+2| + C$$

Example 6: Let $f(x) = \frac{4x^2-4}{x^3-2x^2}$

① Determine the partial fraction decomposition of $f(x)$.

① Factor x^3-2x^2 completely.

$$x^3-2x^2 = x^2(x-2)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

③ Combine the fractions in ②.

Note the common denominator is $x^2(x-2)$.

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{Ax(x-2) + B(x-2) + Cx^2}{x^2(x-2)}$$
$$= \frac{Ax^2 - 2Ax + Bx - 2B + Cx^2}{x^2(x-2)}$$
$$= \frac{(A+C)x^2 + (B-2A)x - 2B}{x^3-2x^2}$$

④ Set the old numerator = new numerator

$$4x^2 - 4 = (A+C)x^2 + (B-2A)x - 2B$$

$$4x^2 + 0x - 4 = (A+C)x^2 + (B-2A)x - 2B$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A+C=4 & \text{(i)} \\ B-2A=0 & \text{(ii)} \\ -2B=-4 & \text{(iii)} \end{cases}$$

From (iii) $-2B = -4 \Rightarrow B = 2$ | Plug $A=1$ into (i).

Plug $B=2$ into (ii)

$$B-2A=0$$

$$2-2A=0$$

$$2=2A \Rightarrow A=1$$

$$A+C=4$$

$$1+C=4$$

$$C=3$$

⑥ Plug $A=1, B=2, C=3$ into ②.

$$\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x-2}$$

⑦ Using ⑥, evaluate $\int f(x) dx$.

$$\begin{aligned} \int \frac{4x^2-4}{x^3-2x^2} dx &= \int \frac{1}{x} dx + \int \frac{2}{x^2} dx + \int \frac{3}{x-2} dx \\ &= \int \frac{1}{x} dx + \int 2x^{-2} dx + \int \frac{3}{x-2} dx \\ &= \ln|x| + \frac{2x^{-1}}{-1} + 3 \ln|x-2| + C \\ &= \ln|x| - \frac{2}{x} + 3 \ln|x-2| + C \end{aligned}$$

Example 7: Let $f(x) = \frac{5x^2+9}{x^3+3x}$

① Determine the partial decomposition of $f(x)$.

① Factor x^3+3x completely.

$$x^3+3x = x(x^2+3)$$

② Write the fraction into decomposition form.

$$\frac{A}{x} + \frac{Bx+C}{x^2+3}$$

③ Combine the fractions in ②.

$$\frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3) + (Bx+C)x}{x(x^2+3)}$$

$$= \frac{Ax^2 + 3A + Bx^2 + Cx}{x^3+3x}$$

$$= \frac{(A+B)x^2 + Cx + 3A}{x^3+3x}$$

④ Set the old numerator = new numerator

$$5x^2+9 = (A+B)x^2 + Cx + 3A$$

$$5x^2+0x+9 = (A+B)x^2 + Cx + 3A$$

⑤ Create a system of equations from ④ and solve.

$$\begin{cases} 5 = A+B & \text{(i)} \\ 0 = C & \text{(ii)} \\ 9 = 3A & \text{(iii)} \end{cases}$$

Note (i) already gives us $C=0$.

From (ii) $9=3A \Rightarrow A=3$

Plug $A=3$ into (1).

$$5 = A + B$$

$$5 = 3 + B$$

$$2 = B$$

(6) Plug $A=3, B=2, C=0$ into (2).

$$\frac{3}{x} + \frac{2x+0}{x^2+3}$$

(b) Using (a), evaluate $\int f(x) dx$.

$$\int \frac{5x^2+9}{x^3+3x} dx = \int \frac{3}{x} dx + \int \frac{2x}{x^2+3} dx$$

u -sub $u = x^2+3$
 $du = 2x dx$

$$= \int \frac{3}{x} dx + \int \frac{du}{u}$$

$$= 3 \ln|x| + \ln|u| + C$$

$$= 3 \ln|x| + \ln|x^2+3| + C$$

Example 8: Let $f(x) = \frac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

(a) Determine the partial fraction decomposition of $f(x)$.

(1) Factor $(x^2+x+1)^2$ completely.

Done!

(2) Write the fraction into decomposition form.

$$\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$$

(3) Combine the fractions in (2).

Note the common denominator is $(x^2+x+1)^2$

$$\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} = \frac{(Ax+B)(x^2+x+1) + Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3+Bx^2+Ax^2+Bx+Ax+B+Cx+D}{(x^2+x+1)^2}$$

$$= \frac{Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)}{(x^2+x+1)^2}$$

④ Set the old numerator = new numerator

$$2x^3 + 3x^2 + 5x + 2 = Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D)$$

⑤ Create a system of equations from ④, and solve.

$$\begin{cases} A=2 & \textcircled{i} \\ A+B=3 & \textcircled{ii} \\ A+B+C=5 & \textcircled{iii} \\ B+D=2 & \textcircled{iv} \end{cases}$$

From ①, we have $A=2$.

Plug $A=2$ into ②

$$A+B=3$$

$$2+B=3$$

$$B=1$$

Plug $A=2, B=1$ into ③

$$A+B+C=5$$

$$2+1+C=5$$

$$3+C=5$$

$$C=2$$

Plug $B=1$ into ④

$$B+D=2$$

$$1+D=2$$

$$D=1$$

⑥ Plug $A=2, B=1, C=2, D=1$ into ②,

$$\frac{2x+1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$$

⑦ Using ⑥, evaluate $\int f(x) dx$.

$$\int \frac{2x^3 + 3x^2 + 5x + 2}{(x^2+x+1)^2} dx = \int \frac{2x+1}{x^2+x+1} dx + \int \frac{2x+1}{(x^2+x+1)^2} dx$$

Note both integrals have a u -sub.

This time only it's the same u .

$$\begin{aligned} u &= x^2+x+1 \\ du &= (2x+1)dx \\ \int \frac{du}{u} + \int \frac{du}{u^2} &= \int \frac{du}{u} + \int u^{-2} du \\ &= \ln|u| - u^{-1} + C = \ln|u| - \frac{1}{u} + C \end{aligned}$$

$$= \ln|x^2+x+1| - \frac{1}{x^2+x+1} + C$$