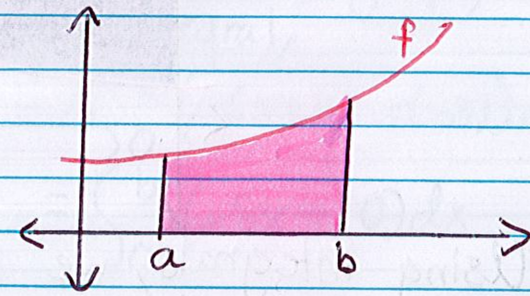
Lesson 9: Area Between Two Curves

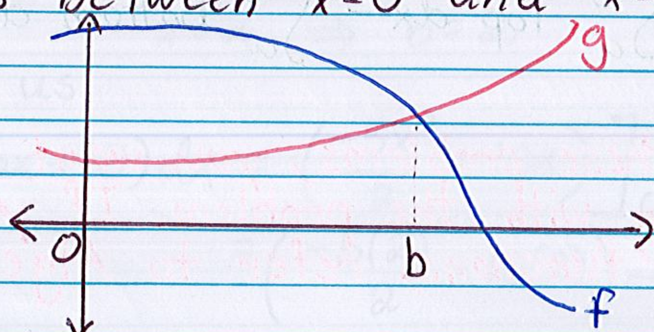
Recall from Calculus I that the definite integral has a geometric meaning, namely the area under a curve.

i.e. $\int_a^b f(x)dx \Rightarrow$

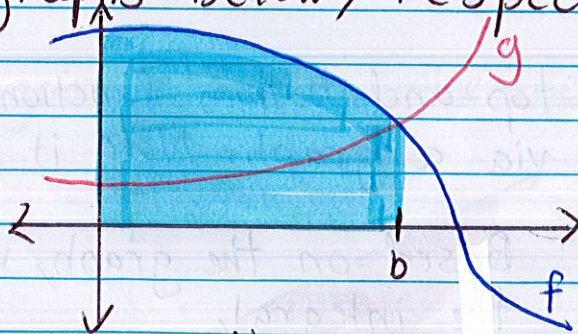


In this Lesson, we want the area BETWEEN 2 curves.

Consider the graphs of f and g, as shown below, and say we want to calculate the area bounded by the two curves between $x=0$ and $x=b$.

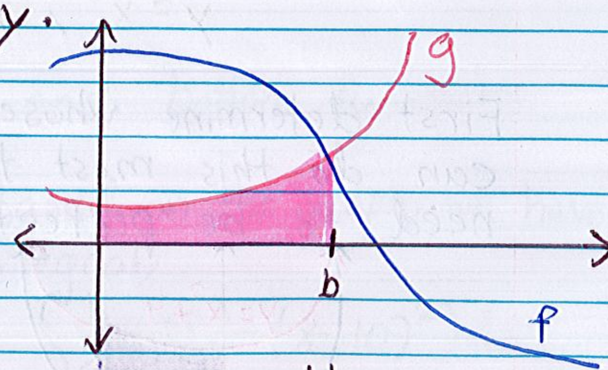


If we calculate the area under each curve separately we find the blue and red areas in the two graphs below, respectively.



$\Downarrow$

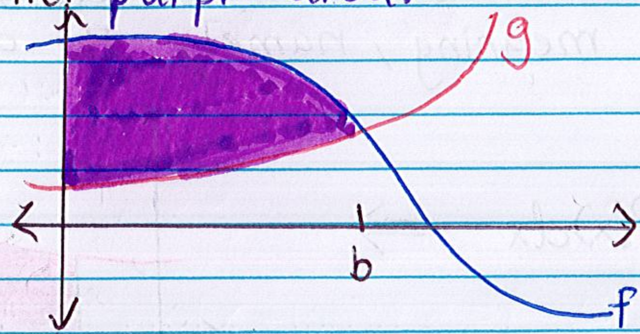
$\int_0^b f(x)dx$



$\Downarrow$

$\int_0^b g(x)dx$

Looking at these graphs we can see the graph on the right is what we don't want. So if we subtract the red area from the blue area, we get the area between the two curves. i.e. purple area.



Using integrals, we have

$$\text{Area} = \int_0^b f(x)dx - \int_0^b g(x)dx = \int_0^b (f(x)-g(x))dx$$

In this lesson, when finding the area between two curves on an interval $[a,b]$

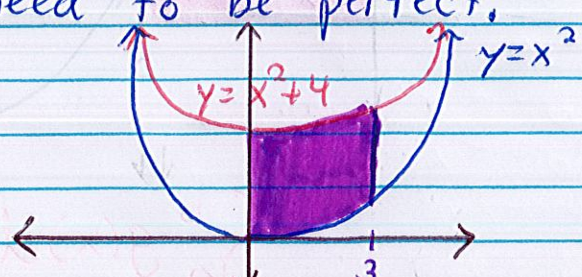
$$\text{Area} = \int_a^b \text{Top } dx - \int_a^b \text{Bottom } dx$$

With all of these problems, you want to draw the graph corresponding with the problem. If you need a refresher on graphing functions, refer to Algebra Review posted online.

Example 1: Find the area of the region bounded by

$$y = x^2, \; y = x^2+4, \; 0 \le x \le 3$$

First determine whose the top and bottom function. You can do this most times via a graph. Note it does not need to be perfect.



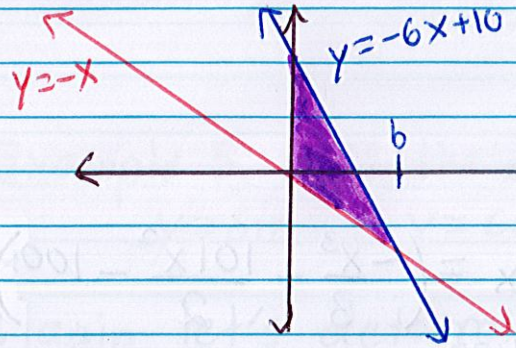
Based on the graph, we have the integral,

$$\int_0^3 (x^2+4-x^2)dx$$

$$= \int_0^3 4dx = 4x\Big]_0^3 = 12$$

Example 2: Find the area bounded by the curves $y=-x$, and $y=-6x+10$, and y-axis.

Again let's determine the top and bottom functions.



Based on the graph, we have the integral,

$$\int_0^b (-6x+10-(-x))\,dx$$
$$=\int_0^b (-5x+10)\,dx$$

But what is b?

In this question it's the point where both lines intersect.

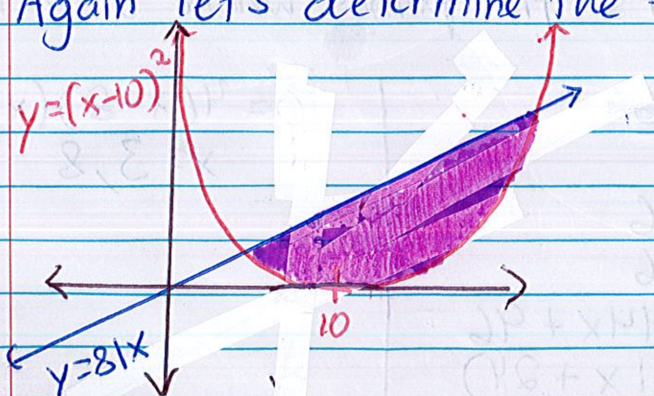
$$-x=-6x+10$$
$$+6x \quad +6x$$
$$5x=10$$
$$x=2 \Rightarrow b=2$$

Which gives us

$$\int_0^2 (-5x+10)\,dx = \left(\frac{-5x^2}{2}+10x\right)\Big]_0^2$$
$$=\left(-\frac{5(2)^2}{2}+10(2)\right)-\left(\frac{-5(0)^2}{2}+10(0)\right)=10$$

Example 3: Find the area bounded by the curves $y=(x-10)^2$, and $y=81x$

Again let's determine the top and bottom functions.



Based on the graph, we have the integral,

$$\int_a^b (81x-(x-10)^2)\,dx$$
$$=\int_a^b (81x-(x^2-20x+100))\,dx$$
$$=\int_a^b (-x^2+101x-100)\,dx$$

But what are a and b?

In this question it's the points where both lines intersect.

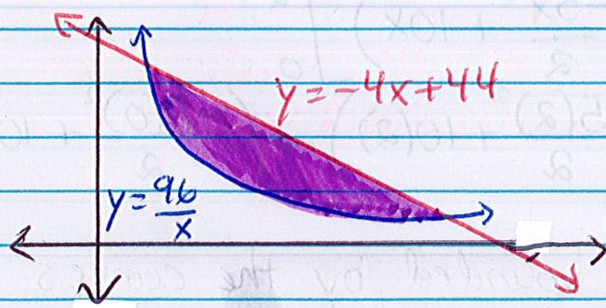
$$(x-10)^2 = 81x$$
$$x^2 - 20x + 100 = 81x$$
$$x^2 - 101x + 100 = 0$$
$$(x-1)(x-100) = 0$$
$$x = 1, 100$$

which gives us

$$\int_1^{100} (-x^2 + 101x - 100)\,dx = \left(-\frac{x^3}{3} + \frac{101x^2}{2} - 100x\right)\Big]_1^{100}$$
$$= \frac{323433}{2}$$

Example 4: Find the area of the region bounded by $y = \frac{96}{x}$, and $y = -4x + 44$

Again let's determine the top and bottom functions.



Based on the graph, we have the integral

$$\int_a^b \left(-4x + 44 - \left(\frac{96}{x}\right)\right)dx$$

But what are a and b?

In this question it's the points where both lines intersect.

$$\frac{-4x+44}{1} = \frac{96}{x}$$
$$(-4x + 44)x = 96$$
$$-4x^2 + 44x = 96$$
$$0 = 4x^2 - 44x + 96$$
$$0 = 4(x^2 - 11x + 24)$$

$$0 = 4(x-8)(x-3)$$
$$x = 3, 8$$

which gives us

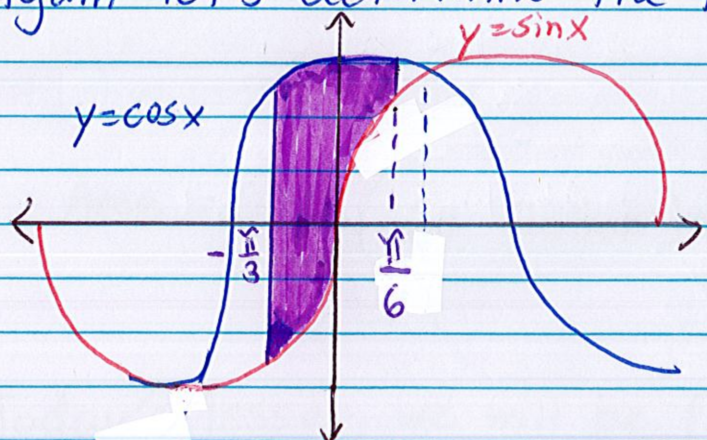
$$\int_3^8 \left(-4x+44-\frac{96}{x}\right)dx = \left(\frac{-4x^2}{2}+44x-96\ln|x|\right)\Bigg]_3^8$$

$$=\left(-2x^2+44x-96\ln|x|\right)\Bigg]_3^8$$

$$=110-96\ln(8)+96\ln(3)$$

Example 5: Find the area of the region bounded by $y=\sin x$, $y=\cos x$, $x=-\pi/3$, $x=\pi/6$

Again let's determine the top and bottom functions.



Based on the graph, we have the integral

$$\int_{-\pi/3}^{\pi/6}(\cos x-\sin x)\,dx$$

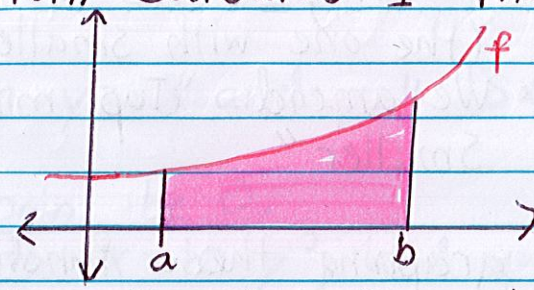
$$=\left(\sin x+\cos x\right)\Bigg]_{-\pi/3}^{\pi/6}$$

$$=\sin\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{6}\right)-\sin\left(\frac{-\pi}{3}\right)-\cos\left(\frac{-\pi}{3}\right)$$

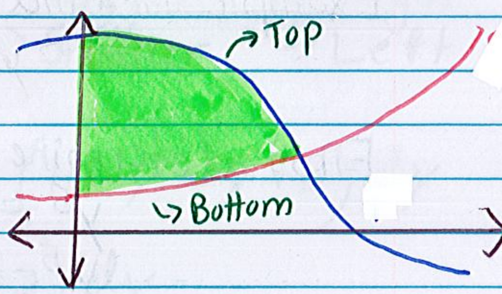
$$=\frac{1}{2}+\frac{\sqrt{3}}{2}-\left(\frac{-\sqrt{3}}{2}\right)-\frac{1}{2}=\sqrt{3}$$

Again, we recalled from Calculus I that

$$\int_{x=a}^{x=b} f(x)\,dx \Rightarrow$$



and with that interpretation found a formula for the area between two curves with respect to x, i.e.

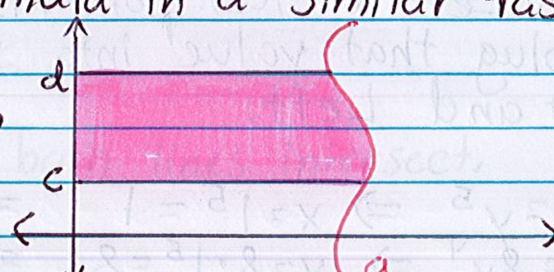


$$\text{Area} = \int_{x=a}^{x=b} (\text{Top} - \text{Bottom})\,dx$$

Today's lesson, we will be focussing on the area between two curves with respect to y.

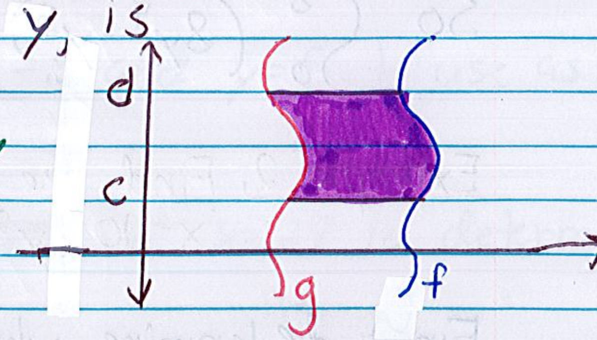
We will define the formula in a similar fashion. So

$$\int_{y=c}^{y=d} g(y)\,dy \Rightarrow$$



Hence we can say the formula for the area between two curves with respect to y, is

$$\text{Area} = \int_{y=c}^{y=d} (\text{Right} - \text{Left})\,dy$$



So what has changed?

- The roles of x and y switch.
- Given two curves which are in terms of y, we want to integrate (the one with larger x-values) minus (the one with smaller x-values).
- We amend "Top minus Bottom" to "Bigger minus Smaller"

Now graphing these functions can be quiet difficult so I'll be introducing a new way of doing this problems.

Example 6: Find the area of the region bounded by $x = y^5$ and $x = 8y^4$

First determine where both lines intersect.

$$y^5 = x = 8y^4$$
$$y^5 - 8y^4 = 0$$
$$y^4(y-8) = 0$$
$$y = 0, 8$$

Next choose a # between $y=0$, and $y=8$ to use as a test point.

ex. Let the test point be 1.

Now plug that value into $x = y^5$, and $x = 8y^4$ to determine Right and Left.

$x = y^5 \Rightarrow x = 1^5 = 1 \Rightarrow$ Smaller $\Rightarrow$ Left

$x = 8y^4 \Rightarrow x = 8 \cdot 1^5 = 8 \Rightarrow$ Bigger $\Rightarrow$ Right

So $\displaystyle\int_0^8 (8y^4 - y^5)\,dy = \left(\frac{8y^5}{5} - \frac{y^6}{6}\right)\Big]_0^8 = \frac{131072}{15}$

Example 7: Find the area of the region bounded by $x = 10 - y^2$, and $x = y - 2$

First determine where both lines intersect.

$$10-y^2 = y-2$$
$$0 = y^2+y-12$$
$$0 = (y+4)(y-3)$$
$$y = -3, 4$$

Next choose a # between $y=-3$ and $y=4$ to use as a test point.

<u>ex.</u> Let the test point be 0.

Now plug that value into $x=10-y^2$, and $x=y-2$ to determine Right and Left.

$x=10-y^2 \Rightarrow x=10-0^2=10 \Rightarrow$ Bigger $\Rightarrow$ Right

$x=y-2 \Rightarrow x=0-2=-2 \Rightarrow$ Smaller $\Rightarrow$ Left

So
$$\int_{-3}^{4} \left(10-y^2-(y-2)\right)dy = \int_{-3}^{4}\left(10-y^2-y+2\right)dy$$
$$= \int_{-3}^{4}\left(-y^2-y+12\right)dy$$
$$= \left(-\frac{y^3}{3}-\frac{y^2}{2}+12y\right)\Big]_{-3}^{4} = \frac{301}{6}$$

<u>Example 8</u>: Find the area of the region bounded by $x=y^2+2$ and $x=27$

First determine where both lines intersect.
$$y^2+2=27$$
$$y^2=25$$
$$y=\pm 5$$

Next choose a # between $y=-5$ and $y=5$ to use as a test point.

<u>ex.</u> Let the test point be 0.

Now plug that value into $x=y^2+2$ and $x=27$ to determine Right and Left

$x = y^2 + 2 \Rightarrow x = 0^2 + 2 = 2 \Rightarrow$ Smaller $\Rightarrow$ Left
$x = 27 \Rightarrow x = 27 \Rightarrow$ Bigger $\Rightarrow$ Right

So $\int_{-5}^{5} (27 - (y^2 + 2))\,dy = \int_{-5}^{5} (27 - y^2 - 2)\,dy$

$$= \int_{-5}^{5} (25 - y^2)\,dy$$

$$= \left(25y - \frac{y^3}{3}\right)\Big]_{-5}^{5} = \frac{500}{3}$$

Example 9: Find the area of the region bounded by $x = y^2 - 8$ and $x = 4 - y^2$

First determine where both lines intersect

$$y^2 - 8 = 4 - y^2$$
$$2y^2 = 12$$
$$y^2 = 6$$
$$y = \pm\sqrt{6}$$

Next choose a # between $y = -\sqrt{6}$ and $y = \sqrt{6}$ to use as a test point.

ex. Let the test point be 0.

Now plug that value into $x = y^2 - 8$ and $x = 4 - y^2$ to determine Right and Left.

$x = y^2 - 8 \Rightarrow x = 0^2 - 8 = -8 \Rightarrow$ Smaller $\Rightarrow$ Left
$x = 4 - y^2 \Rightarrow x = 4 - 0^2 = 4 \Rightarrow$ Bigger $\Rightarrow$ Right

So $\int_{-\sqrt{6}}^{\sqrt{6}} (4 - y^2 - (y^2 - 8))\,dy = \int_{-\sqrt{6}}^{\sqrt{6}} (4 - y^2 - y^2 + 8)\,dy$

$$= \int_{-\sqrt{6}}^{\sqrt{6}} (12 - 2y^2)\,dy$$

$$= \left(12y - \frac{2y^3}{3}\right)\Big]_{-\sqrt{6}}^{\sqrt{6}} = 16\sqrt{6}$$