Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided.

Name: Solutions

1. Which derivative rule is undone by integration by substitution?
   (A) Power Rule
   (B) Quotient Rule
   (C) Product Rule
   (D) Chain Rule
   (E) Constant Rule
   (F) None of these

2. Which derivative rule is undone by integration by parts?
   (A) Power Rule
   (B) Quotient Rule
   (C) Product Rule
   (D) Chain Rule
   (E) Constant Rule
   (F) None of these

3. What would be the best substitution to make the solve the given integral?

$$\int e^{2x}\cos(e^{2x})\sin^3(e^{2x})\,dx$$

$$\int e^{2x}\cos(e^{2x})[\sin(e^{2x})]^3\,dx$$

$u = \boxed{\sin(e^{2x})}$

Always check du is in integral

4. What would be the best substitution to make the solve the given integral?

$$\int \sec^2(5x)e^{\tan(5x)}\,dx$$

$u = \boxed{\tan(5x)}$

Always check du is in integral

5. What would be the best substitution to make the solve the given integral?

$$\int \tan(5x)\sec(5x)e^{\sec(5x)}\,dx$$

$u = \boxed{\sec(5x)}$

Always check du is in integral

6. Find the area under the curve $y = 14e^{7x}$ for $0 \leq x \leq 4$.

$$A = \int_0^4 14e^{7x}\,dx \quad \frac{u = 7x}{du = 7dx} \quad \int 2e^u\,du$$

$$= 2e^u = 2e^{7x}\Big]_0^4 = 2e^{28} - 2$$

Area = $\boxed{2e^{28} - 2}$

7. Evaluate the definite integral.

$$\int_0^2 (5e^{2x} + 8)\,dx$$

$$\underbrace{\int_0^2 5e^{2x}\,dx}_{\text{u-sub}} + \int_0^2 8\,dx = \frac{5}{2}e^{2x}\Big]_0^2 + 8x\Big]_0^2$$

$$= \frac{5}{2}(e^4 - e^0) + 8(2-0)$$

$$= \frac{5}{2}e^4 - \frac{5}{2} + 16$$

$$= \frac{5}{2}e^4 - \frac{27}{2}$$

$\int_0^2 (5e^{2x} + 8)\,dx =$ $\boxed{\frac{5}{2}e^4 + \frac{27}{2}}$

8. Evaluate the indefinite integral.

$$\int 64x^7 \sin(x^8)\, dx$$

$u = x^8$

$du = 8x^7 dx$

$\frac{du}{8x^7} = dx$

$\int \cancel{64x^7}^{\,8} \sin(u) \frac{du}{\cancel{8x^7}} = \int 8 \sin(u) du$

$= -8\cos(u) + C$

$= -8\cos(x^8) + C$

$$\int 64x^7 \sin(x^8)\, dx = \boxed{-8\cos(x^8) + C}$$

9. Evaluate the indefinite integral.

$$\int 9x^3 e^{-x^4}\, dx$$

$u$

$u = -x^4$

$du = -4x^3 dx$

$\frac{du}{-4x^3} = dx$

$\int 9x^3 e^u \frac{du}{-4x^3} = -\frac{9}{4}\int e^u du$

$= -\frac{9}{4} e^u = -\frac{9}{4} e^{-x^4} + C$

$$\int 9x^3 e^{-x^4}\, dx = \boxed{-\frac{9}{4} e^{-x^4} + C}$$

10. Evaluate the indefinite integral.

$$\int \frac{28x}{x^2+11}\,dx$$

$u = x^2+11$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$\int \frac{\overset{14}{\cancel{28}}x}{u} \cdot \frac{du}{\cancel{2x}} = \int \frac{14}{u}\,du$

$= 14\ln|u| + C$

$= 14\ln|x^2+11| + C$

$$\int \frac{28x}{x^2+11}\,dx = \boxed{14\ln|x^2+11| + C}$$

11. Evaluate the indefinite integral

$$\int \frac{\ln(5x)}{x}\,dx$$

$u = \ln(5x)$

$du = \frac{1}{5x} \cdot 5\,dx$

$du = \frac{1}{x}\,dx$

$\int u\,du = \frac{u^2}{2} = \frac{(\ln(5x))^2}{2} + C$

$$\int \frac{\ln(5x)}{x}\,dx = \boxed{\frac{(\ln(5x))^2}{2} + C}$$

12. Evaluate

$$\int_1^e \frac{\ln(x^4)}{x}\,dx$$

Rewrite $\int_1^e \frac{4\ln x}{x}\,dx$ $\quad \frac{u=\ln x}{du=\frac{1}{x}dx}$ $\quad \int 4u\,du = \frac{4u^2}{2} = 2u^2 = 2(\ln x)^2\Big]_1^e$

$= \underbrace{2(\ln e)^2}_{2} - \underbrace{2(\ln 1)^2}_{0}$

$= 2$

$$\int_1^e \frac{\ln(x^4)}{x}\,dx = \boxed{2}$$

13. After an oil spill, a company uses oil-eating bacteria to help clean up. It is estimated that $t$ hours after being placed in the spill, the bacteria will eat the oil at a rate of

$$L'(t) = \sqrt{3t+2} \quad \text{gallows per hour.}$$

How many gallons of oil will the bacteria eat in the first 4 hours? Round to 4 decimal places.

i.e. $\int_0^4 (3t+2)^{1/2}\,dt$ $\quad \frac{u=3t+2}{du=3dt}$ $\quad \int u^{1/2}\frac{du}{3}$

$\frac{du}{3} = dt$

$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} = \frac{2}{9}(3t+2)^{3/2}\Big]_0^4$

$\approx 11.0122$

Answer: $\boxed{11.0122}$

14. It is estimated that $t$-weeks into a semester, the average amount of sleep a college math student gets per day $S(t)$ changes at a rate of

$$\frac{-4t}{e^{t^2}} = -4te^{-t^2}$$

hours per day. When the semester begins, math students sleep an average of 8.2 hours per day. What is $S(t)$, 2 weeks into the semester?

① $\int -4te^{-t^2}\,dt$ $\quad \dfrac{u=-t^2}{du=-2t\,dt}$ $\quad \int \cancel{-4t}^{\,2} e^u \dfrac{du}{\cancel{-2t}}$

$\dfrac{du}{-2t} = dt$

$= \int 2e^u\,du = 2e^u + C$

$= 2e^{-t^2} + C$

② $S(0) = 8.2$ Find $C$.

$8.2 = 2e^0 + C$

$8.2 = 2 + C$

$C = 6.2$

③ $S(t) = 2e^{-t^2} + 6.2$

$S(2) = 2e^{-4} + 6.2$

$\approx 6.237$

Answer: $\boxed{6.237}$

15. A biologist determines that, $t$ hours after a bacterial colony was established, the population of bacteria in the colony is changing at a rate given by

$$P'(t) = \frac{5e^t}{1+e^t}$$

million bacteria per hour, $0 \le t \le 5$.

If the bacterial colony started with a population of 1 million, how many bacteria, in millions are present in the colony after the 5-hour experiment?

① $\int \frac{5e^t}{1+e^t}\,dt$ $\quad \underline{u = 1+e^t}$ $\quad \int \frac{5\cancel{e^t}}{u}\frac{du}{\cancel{e^t}} = \int \frac{5}{u}\,du$

$du = e^t\,dt$

$\frac{du}{e^t} = dt$

$= 5\ln|u| + c$

$= 5\ln|1+e^t| + c$

② $P(0) = 1$ Find $c$.

$1 = 5\ln|1+e^0| + c$

$1 = 5\ln|1+1| + c$

$1 = 5\ln 2 + c$

$1 - 5\ln 2 = c$

③ $P(t) = 5\ln|1+e^t| + 1 - 5\ln 2$

$P(5) = 5\ln|1+e^5| + 1 - 5\ln 2$

$\approx 22.57$

Answer: $\boxed{22.57}$

16. Evaluate

$$\int 3x \ln(x^7)\, dx$$

Rewrite $\int 3x(7\ln(x))\,dx = \int 21x\ln x\,dx$

$u = 21\ln(x)$ $\quad dv = x\,dx$ $\quad uv - \int v\,du$

$du = \frac{21}{x}\,dx$ $\quad v = \frac{x^2}{2}$

$= \frac{21x^2\ln x}{2} - \int \frac{x^2}{2}\cdot\frac{21}{x}\,dx$

$= \frac{21x^2\ln x}{2} - \int \frac{21}{2}x\,dx$

$= \frac{21x^2\ln x}{2} - \frac{21}{2}\cdot\frac{x^2}{2} + C$

$= \frac{21x^2\ln x}{2} - \frac{21x^2}{4} + C$

$\int 3x\ln(x^7)\,dx =$ $\boxed{\frac{21x^2\ln x}{2} - \frac{21x^2}{4} + C}$

17. Evaluate the indefinite integral.

$$\int 4x \sin(7x)\, dx$$

$u = 4x$ $\quad dv = \sin(7x)\,dx$ $\quad uv - \int v\,du$

$du = 4\,dx$ $\quad v = -\frac{\cos(7x)}{7}$

$= -\frac{4}{7}x\cos(7x) + \int \frac{4}{7}(+\cos(7x))\,dx$

$= -\frac{4}{7}x\cos(7x) + \frac{4}{7}\int\cos(7x)\,dx$

$= -\frac{4}{7}x\cos(7x) + \frac{4}{7}\,\frac{\sin(7x)}{7} + C$

$\int 4x\sin(7x)\,dx =$

$\boxed{-\frac{4}{7}x\cos(7x) + \frac{4}{7}\,\frac{\sin(7x)}{7} + C}$

18. The velocity of a cyclist during an hour-long race is given by the function

$$v(t) = 166te^{-2.2t}\text{mi/hr}, \qquad 0 \le t \le 1$$

Assuming the cyclist starts from rest, what is the distance in miles he traveled during the first hour of the race?

① $\int 166te^{-2.2t}\,dt$

$$\frac{u = 166t}{du = 166\,dt} \qquad \frac{dv = e^{-2.2t}\,dt}{v = \frac{e^{-2.2t}}{-2.2}} \qquad uv - \int v\,du$$

$$= \frac{166te^{-2.2t}}{-2.2} + \int \frac{e^{-2.2t}}{+2.2}\cdot 166\,dt$$

$$= -\frac{166te^{-2.2t}}{2.2} + \frac{166}{2.2}\cdot\frac{e^{-2.2t}}{-2.2} + C$$

$$= \frac{-166te^{-2.2t}}{2.2} - \frac{166e^{-2.2t}}{(2.2)^2} + C$$

② $s(0) = 0$. Find $C$.

$$0 = 0 - \frac{166}{(2.2)^2} + C \longrightarrow C = \frac{166}{(2.2)^2}$$

③ $$s(t) = \frac{-166te^{-2.2t}}{2.2} - \frac{166e^{-2.2t}}{(2.2)^2} + \frac{166}{(2.2)^2}$$

$$s(1) = -\frac{166}{22}e^{-2.2} - \frac{166}{(2.2)^2}e^{-2.2} + \frac{166}{(2.2)^2}$$

$\approx 22.137$

Answer: $\boxed{22.137}$

19. Which of the following is a partial fraction decomposition of the rational expression show? Do not explicitly solve for the constant.

$$f(x) = \frac{3x+1}{x^2(x+1)^2(x^2+1)}$$

(A)

$$\frac{A}{x^2} + \frac{B}{(x+1)^2} + \frac{C}{x^2+1}$$

(B)

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{x^2+1}$$

(C)

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}$$

(D)

$$\frac{A}{x} + \frac{Bx+C}{x^2} + \frac{D}{x+1} + \frac{Ex+F}{(x+1)^2} + \frac{Gx+H}{x^2+1}$$

(E)

$$\frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x^2+1}$$

20. Determine the partial fraction decomposition of

$$\frac{7x^2+9}{x^3+3x}$$

$x^3+3x = x(x^2+3)$

So partial fractions are

$$\frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3)+x(Bx+C)}{x(x^2+3)}$$

$$= \frac{Ax^2+3A+Bx^2+Cx}{x(x^2+3)}$$

$$= \frac{(A+B)x^2+Cx+3A}{x(x^2+3)}$$

$(A+B)x^2+Cx+3A = 7x^2+0x+9$

$$\begin{cases} A+B=7 \\ C=0 \\ 3A=9 \to A=3 \end{cases}$$

So $B=4$

Answer: $\boxed{\dfrac{3}{x} + \dfrac{4x}{x^2+3}}$

21. Determine the partial fraction decomposition of

$$\frac{4x-11}{x^2-7x+10}$$

Factor $x^2-7x+10=(x-2)(x-5)$

$$\frac{4x-11}{(x-2)(x-5)}=\frac{A}{x-2}+\frac{B}{x-5}$$

$$=\frac{A(x-5)+B(x-2)}{(x-2)(x-5)}$$

$$=\frac{(A+B)x+(-5A-2B)}{(x-2)(x-5)}$$

So $4x-11=(A+B)x+(-5A-2B)$

$$\begin{cases} 4=A+B & \text{(i)} \\ -11=-5A-2B & \text{(ii)} \end{cases}$$

Multiply (i) by 5 and add (i) + (ii).

$$\begin{array}{rl} & 20=\cancel{5A}+5B \\ + & -11=-\cancel{5A}-2B \\ \hline & 9=3B \end{array}$$

$B=3$

Plug $B=3$ into (i)

$4=A+B$

$4=A+3$

$A=1$

Answer: $\boxed{\dfrac{1}{x-2}+\dfrac{3}{x-5}}$

22. Evaluate $\int \frac{5x^2+9}{x^3+3x^2}\,dx$

Factor $x^3+3x^2 = x^2(x+3)$. So,

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3} = \frac{Ax(x+3)+B(x+3)+Cx^2}{x^2(x+3)}$$

$$= \frac{Ax^2+3Ax+Bx+3B+Cx^2}{x^2(x+3)}$$

$$= \frac{(A+C)x^2+(3A+B)x+3B}{x^2(x+3)}$$

$$(A+C)x^2+(3A+B)x+3B = 5x^2+0x+9$$

$$\begin{cases} A+C=5 \\ 3A+B=0 \\ 3B=9 \to B=3 \end{cases}$$

$3A+B=0$
$3A+3=0$
$3A=-3$
$A=-1$

$A+C=5$
$-1+C=5$
$C=6$

$$\int -\frac{1}{x}\,dx + \int \frac{3}{x^2}\,dx + \int \frac{6}{x+3}\,dx =$$

$\int \frac{5x^2+9}{x^3+3x^2)}\,dx = \boxed{-\ln|x| - \frac{3}{x} + 6\ln|x+3| + c}$

23. Evaluate $\int \frac{x^2+2}{x^3+3x^2+2x}\,dx$

Factor $x^3+3x^2+2x = x(x^2+3x+2) = x(x+1)(x+2)$

So $\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+2} = \frac{A(x+1)(x+2)+Bx(x+2)+Cx(x+1)}{x(x+1)(x+2)}$

$= \frac{A(x^2+3x+2)+B(x^2+2x)+C(x^2+x)}{x(x+1)(x+2)}$

$= \frac{(A+B+C)x^2+(3A+2B+C)x+2A}{x(x+1)(x+2)}$

So $x^2+2 = (A+B+C)x^2+(3A+2B+C)x+2A$

$1\cdot x^2+0x+2 = (A+B+C)x^2+(3A+2B+C)x+2A$

$$\begin{cases} 1 = A+B+C & \text{(i)} \\ 0 = 3A+2B+C & \text{(ii)} \\ 2 = 2A & \text{(iii)} \end{cases}$$

Solve (iii).

$2 = 2A$

$A = 1$

Plug $A=1$ into (i) and (ii).

$$\begin{cases} 1 = 1+B+C & \text{(i)} \\ 0 = 3+2B+C & \text{(ii)} \end{cases}$$

Subtract the eqns.

$1 = 1+B+\cancel{C}$

$-(0 = 3+2B+\cancel{C})$

$1 = -2-B$

$+2 \quad +2$

$3 = -B$

$B = -3$

Plug $B=-3$ into (i).

$1 = 1+B+C$

$1 = 1-3+C$

$1 = -2+C$

$3 = C$

Plug $A=1$, $B=-3$, $C=3$ into decomposition.

$\frac{1}{x}+\frac{-3}{x+1}+\frac{3}{x+2}$

So $\int \frac{1}{x}\,dx + \int \frac{-3}{x+1}\,dx + \int \frac{3}{x+2}\,dx$

$\int \frac{x^2+2}{x^3+3x^2+2x}\,dx = \boxed{\ln|x| - 3\ln|x+1| + 3\ln|x+2| + C}$