MA 16020: Lesson 14
Volume By Revolution
Disk Method

By: Alexandra Cuadra

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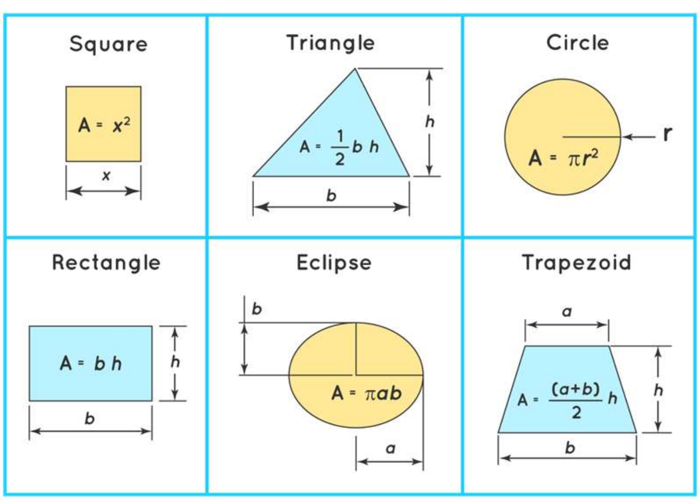


In Geometry,

When we first talked about the concept of area, we did this by going over all the formulas for the area of different polygons.

2

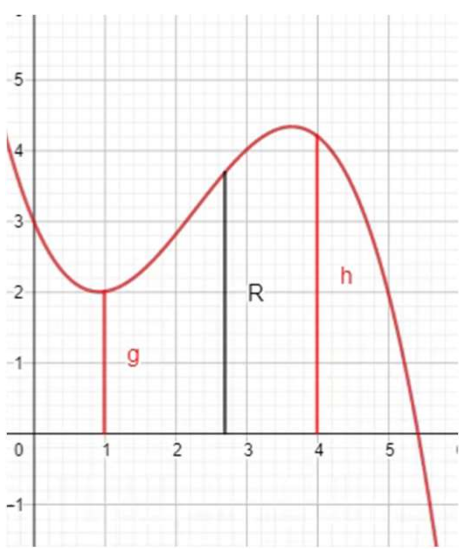
In Calculus I,

We learned about integration as a new technique for calculating area under a curve.

i.e. $\int_a^b f(x)\ dx = F(b) - F(a)$

3

Last Class,



https://www.geogebra.org/m/tgceabb2#material/tnnhu7gz

- We recapped how to take some region
  - Between a curve and an axis, or
  - 2 curves

  And find it's area by integration.

- Essentially, finding a length and sending it across the region.

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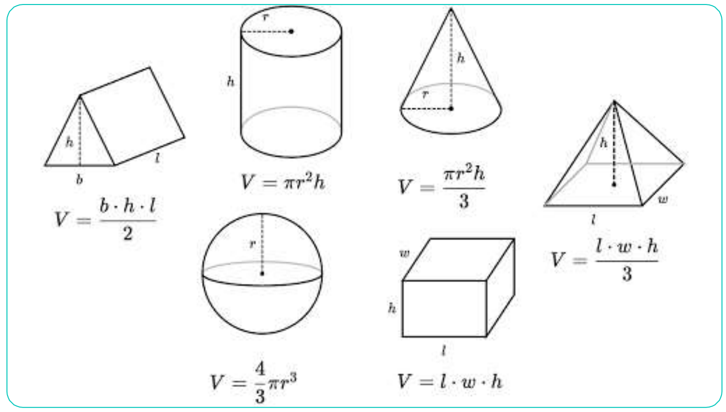
In Geometry,

- We also learned about 3-D figures, like cubes and prisms.
- We described the volume of these objects by the amount of 3-D space that they contained.
- We calculated the volume with formulas like the ones on the right.



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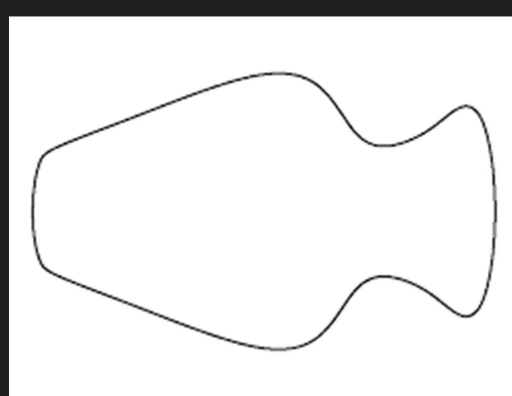
But once curves, like the one below, get involved all these formulas are **USELESS**.



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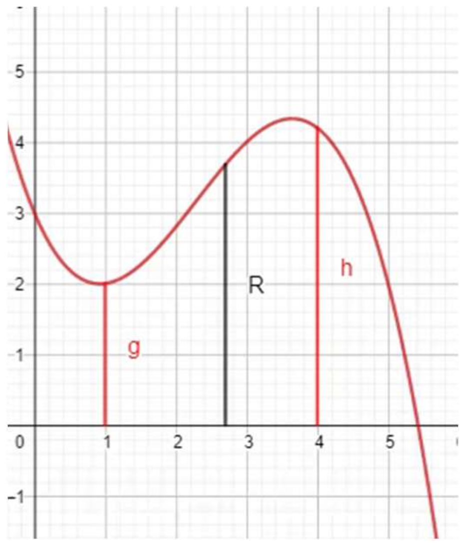

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https://www.geogebra.org/m/tgceabb2#material/tnnhu7gz

- In the same way, we run a line segment across a 2-D region to calculate its area,

  we can run a plane region, or a cross section, across a 3-D region to calculate its volume.
  - i.e. Running a 2-D plane across a 3-D volume.

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So we have integration again; just with an extra dimension.

- Instead of adding up tiny rectangles under a curve, we are adding up infinitely thin cross sections, which we can call
  - Disks (Lessons 14 + 16), or
  - Washers (Lessons 15 + 16), or
  - Shells (Lessons 17 + 18)
- Since each of these cross sections are 2-D, taking the integral of an area function will gives us volume.

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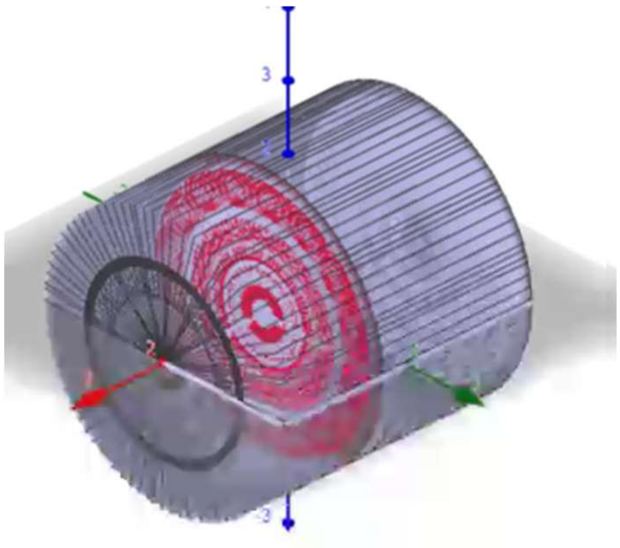

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Let's look at a cylinder.

Remember a cylinder is made up of many circles like the red circle.

So, we can think of our integral to be sum of all these circles.

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Volume of that Cylinder

- Geometry Way:
  - The formula for a Cylinder is

    $V = \pi\, r^2\, h$
  - Since our Cylinder has radius 2 and height 4,

    $V = \pi\, 2^2\, 4 = 16\,\pi$

- Calculus Way:

$$V = \int_{-2}^{2} 2^2\, \pi\, dx = 16\,\pi$$

where $\int_{-2}^{2}$ refers to the height, and

$2^2\,\pi$ refers to the area of a circle.

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Purpose of all of this…

- So in the case of a cylinder, this might be overkill.
- But this is the way we want to think of these questions.
  - Essentially find the cross section by graphing the lines given and apply the appropriate formula
    (found on the next slide.)

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Disk Method Formula(s)

For rotation around x-axis:

- If the volume of the solid is obtained by rotating $f(x)$ about the x-axis on the interval $a \leq x \leq b$ is given by

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

For rotation around y-axis:

- If the volume of the solid is obtained by rotating $g(y)$ about the y-axis on the interval $c \leq y \leq d$ is given by

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

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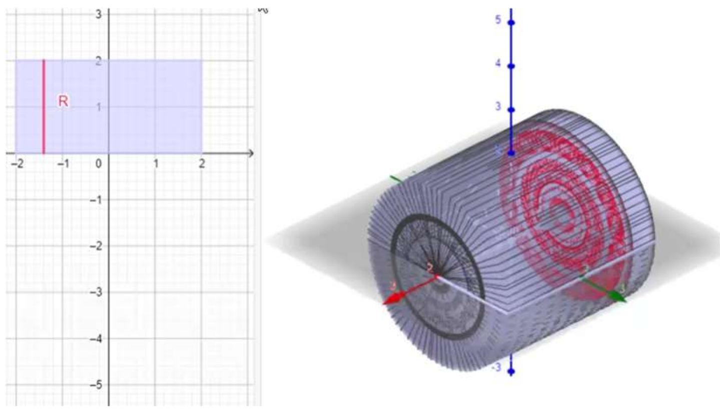

https://www.geogebra.org/m/tgceabb2#material/qcwutumt

Why π in the formula?

Note the π in both formulas comes from the fact we are playing with Disks.

So you can see the graph on the left shows the radius and the right shows the Disks.

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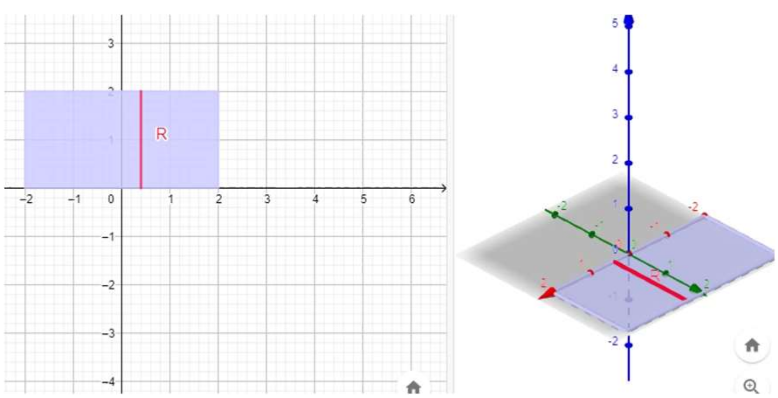

https://www.geogebra.org/m/tgceabb2#material/qcwutumt

But wait…

How is the left graph becoming the cylinder?

Answer: If you rotate the left graph, you get the cylinder.

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Examples

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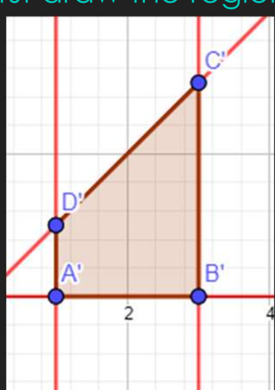
Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x, \qquad y = 0, \qquad x = 1, \qquad x = 3$$

About the x-axis.

First draw the region.



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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

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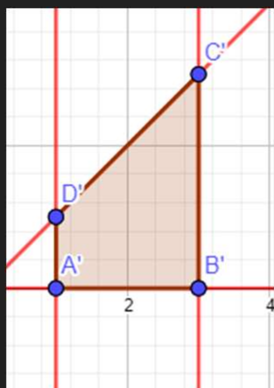
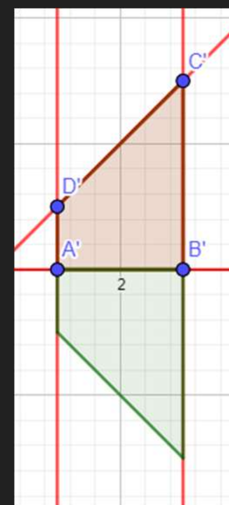
About the x-axis.



Rotation about x-axis



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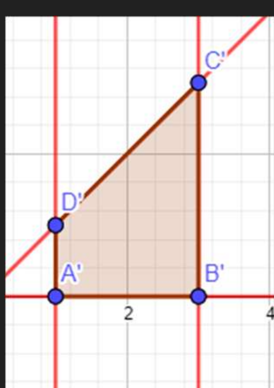
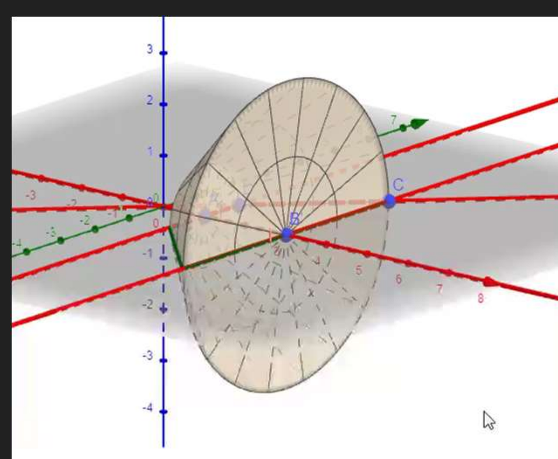
About the x-axis.



Furthermore, 3-D



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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

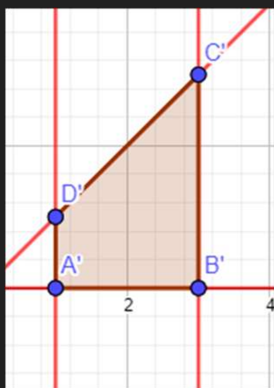
$$y = x, \qquad y = 0, \qquad x = 1, \qquad x = 3$$

About the x-axis.

$$V = \pi \int_1^3 (x)^2 dx$$

$$= \pi \left. \frac{x^3}{3} \right]_1^3$$

$$= \pi \left(\frac{3^3}{3} - \frac{1}{3} \right) = \frac{26}{3}\pi$$



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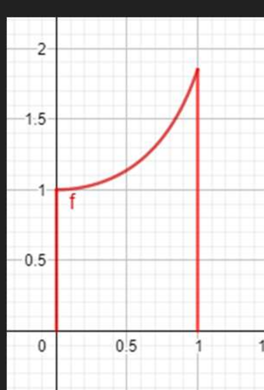
Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sec(x), \qquad y = 0, \qquad x = 0, \qquad x = 1$$

About the x-axis.

First draw the region.



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<u>Example 2:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

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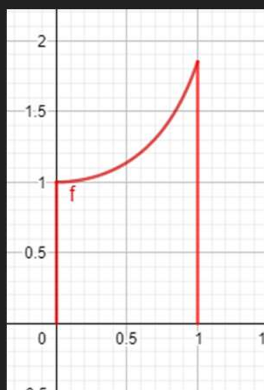
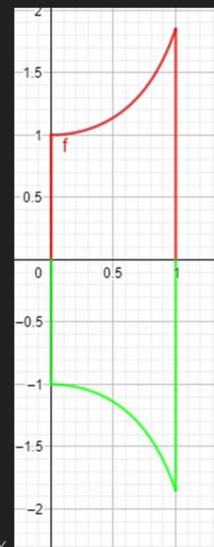
About the x-axis.



Rotation about x-axis



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<u>Example 2:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

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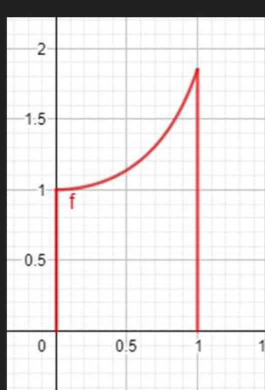
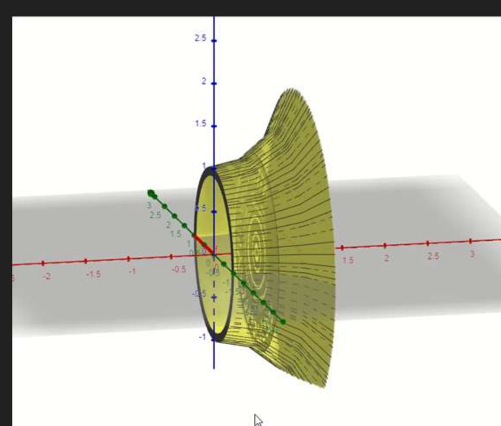
About the x-axis.



Furthermore, 3-D



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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

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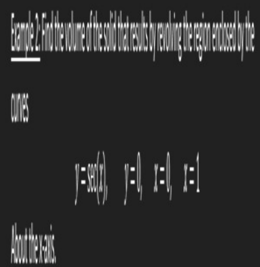
About the x-axis.


Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$y=\sec(x), \quad y=0, \quad x=0, \quad x=1$

About the x-axis.

$$V = \pi \int_0^1 (\sec x)^2\,dx$$

$$= \pi \tan x \Big]_0^1$$

$$= \pi(\tan(1) - \tan(0))$$

$$= \pi \tan(1)$$

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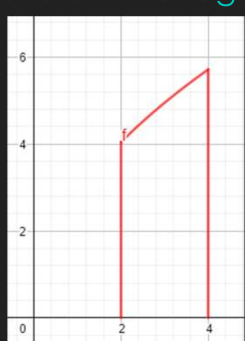
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{6x} + \sqrt{\frac{x}{6}}, \qquad x = 2\,, \qquad x = 4$$

About the x-axis.

First draw the region.



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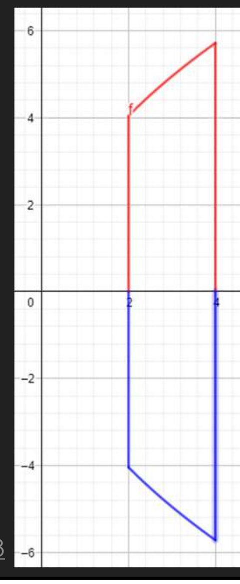
<u>Example 3:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{6x} + \sqrt{\frac{x}{6}}, \quad x = 2\ , \quad x = 4$$

About the x-axis.

Rotation about x-axis



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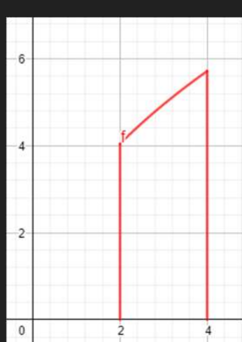
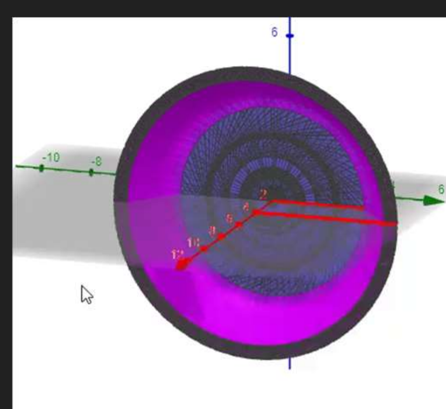
<u>Example 3:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{6x} + \sqrt{\frac{x}{6}}, \quad x = 2\ , \quad x = 4$$

About the x-axis.



Furthermore, 3-D



https://www.geogebra.org/m/tgceabb2#material/njkxvte3

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<u>Example 3:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

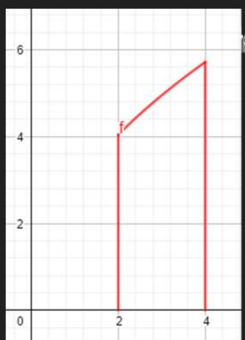
$$y=\sqrt{6x}+\sqrt{\frac{x}{6}},\quad x=2\ ,\quad x=4$$

About the x-axis.

$$V=\pi\int_2^4 \left(\sqrt{6x}+\sqrt{\frac{x}{6}}\right)^2 dx$$

$$=\pi\int_2^4 (\sqrt{6x})^2+2\sqrt{6x}\sqrt{\frac{x}{6}}+\left(\sqrt{\frac{x}{6}}\right)^2 dx$$

$$=\pi\int_2^4 \left(6x+2x+\frac{x}{6}\right)dx$$



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27

<u>Example 3:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

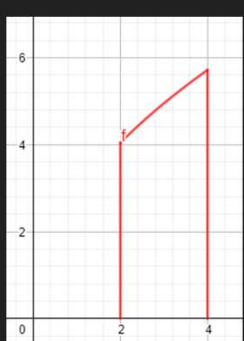
$$y=\sqrt{6x}+\sqrt{\frac{x}{6}},\quad x=2\ ,\quad x=4$$

About the x-axis.

$$V=\pi\int_2^4 \frac{49x}{6}\,dx$$

$$=\pi\,\frac{49}{6}\,\frac{x^2}{2}\Big]_2^4$$

$$=49\pi$$



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Example 4: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = 4x^2, \qquad x = 0, \qquad y = 4$$

About the y-axis.

First draw the region.

https://www.geogebra.org/m/tgceabb2#material/afywnvhr

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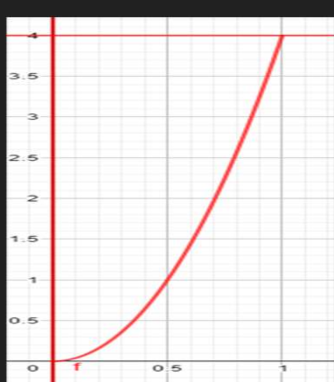
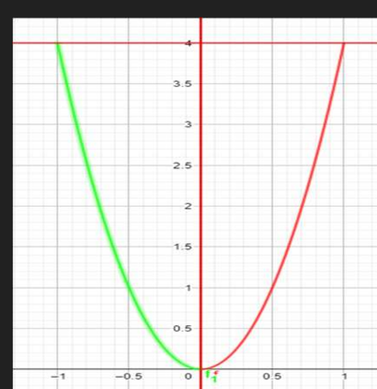
Example 4: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = 4x^2, \qquad x = 0, \qquad y = 4$$

About the y-axis.



Rotation about y-axis



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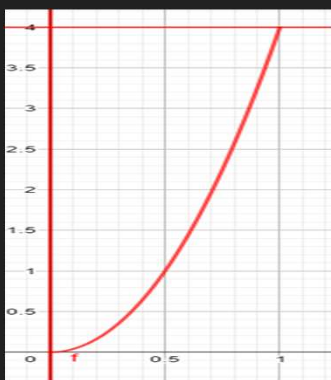
Example 4: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = 4x^2, \qquad x = 0, \qquad y = 4$$

About the y-axis.



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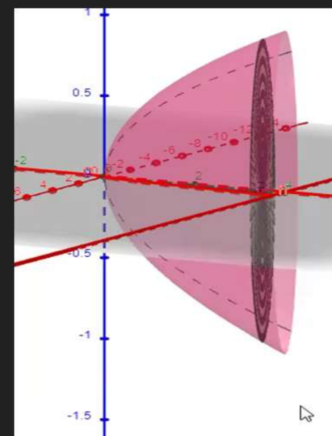

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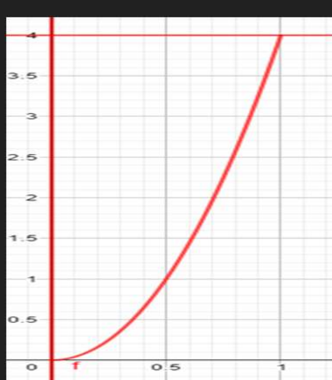
Example 4: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = 4x^2, \qquad x = 0, \qquad y = 4$$

About the y-axis.

First rewrite y as x =



$$\frac{y}{4} = x^2$$

$$\sqrt{\frac{y}{4}} = x$$

Note there is no negative b/c we are in the 1st quadrant

https://www.geogebra.org/m/tgceabb2#material/afywnvhr

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<u>Example 4:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = 4x^2, \qquad x = 0, \qquad y = 4$$

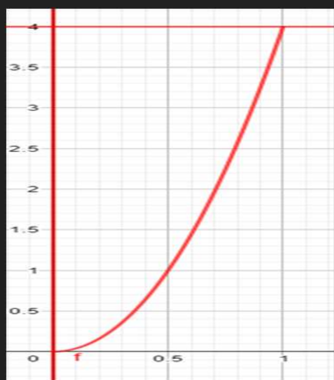
About the y-axis.

$$V = \pi \int_0^4 \left(\sqrt{\frac{y}{4}}\right)^2 dy$$

$$= \pi \int_0^4 \frac{y}{4}\, dy$$

$$= \pi \cdot \frac{1}{4} \frac{y^2}{2} \Big]_0^4$$

$$= 2\pi$$



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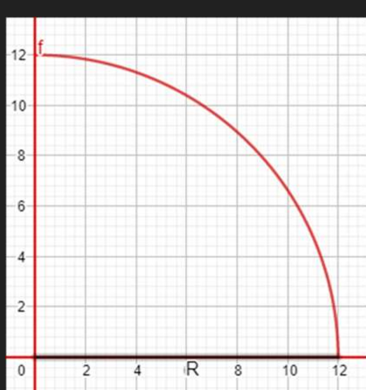
<u>Example 5:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{144 - x^2}, \qquad x = 0, \qquad y = 0$$

About the y-axis.

First draw the region.



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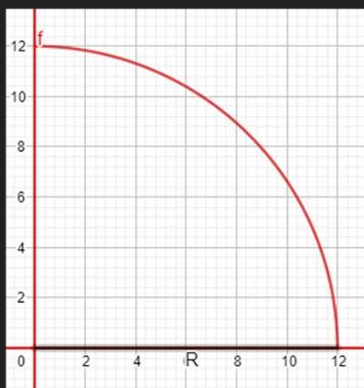
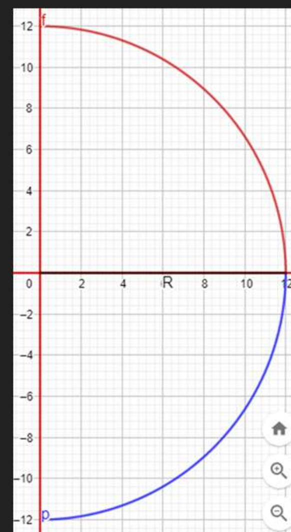
<u>Example 5:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{144 - x^2}, \qquad x = 0, \qquad y = 0$$

About the y-axis.



Rotation about y-axis



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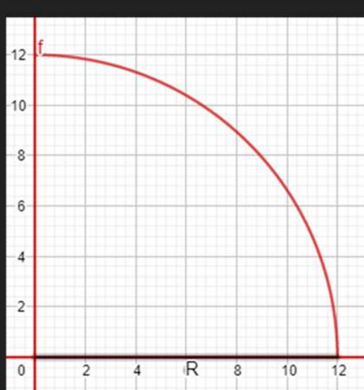
35

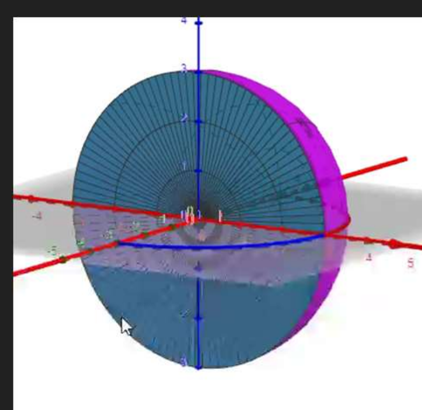
<u>Example 5:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{144 - x^2}, \qquad x = 0, \qquad y = 0$$

About the y-axis.



Furthermore, 3-D



https://www.geogebra.org/m/tgceabb2#material/a5s4n8u7

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<u>Example 5:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{144 - x^2}, \qquad x = 0, \qquad y = 0$$

About the y-axis.

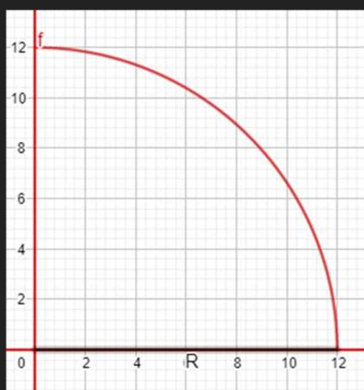
First rewrite y as x=

$y^2 = 144 - x^2$

$144 - y^2 = x^2$

$44 - y^2$

Note there is no negative b/c we are in the 1st quadrant



https://www.geogebra.org/m/tgceabb2#material/a5s4n8u7

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<u>Example 5:</u> Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \sqrt{144 - x^2}, \qquad x = 0, \qquad y = 0$$

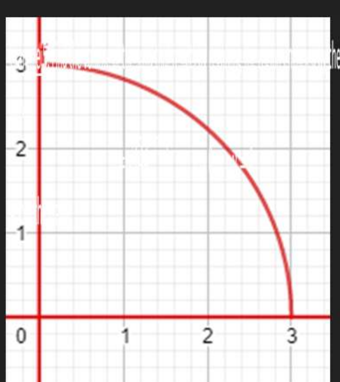
About the y-axis.

$$V = \pi \int_0^{12} \left(\sqrt{144 - x^2}\right)^2 dx$$

$$= \pi \int_0^{12} (144 - x^2)\, dx$$

$$= \pi \left(144x - \frac{x^3}{3}\right)\Big]_0^{12}$$

$$= 1152\pi$$



https://www.geogebra.org/m/tgceabb2#material/a5s4n8u7

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GeoGebra Link for Lesson 14

- https://www.geogebra.org/m/tgceabb2
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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