MA 16020: Lesson 15
Volume By Revolution
Washer Method

By: Alexandra Cuadra

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Last Time, we talked about…

- How Geometry gave us formulas for simple shapes and solids to find their area or volume, and
- How Integration can allow us to find area or volume of ANYTHING !

  How?

  - We introduced this notion of cross-sections which can be of the form of
    - Disks (Lessons 14 + 16), or
    - Washers (Lessons 15 + 16), or
    - Shells (Lessons 17 + 18)

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Geometry: How to Calculate The Area of a Shaded Region

Suppose we are asked to find the area of a rectangle with a triangle missing from the middle.

How do we calculate that area?

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Geometry: How to Calculate The Area of a Shaded Region

First, we would find the area of the rectangle and the area of the triangle separately.

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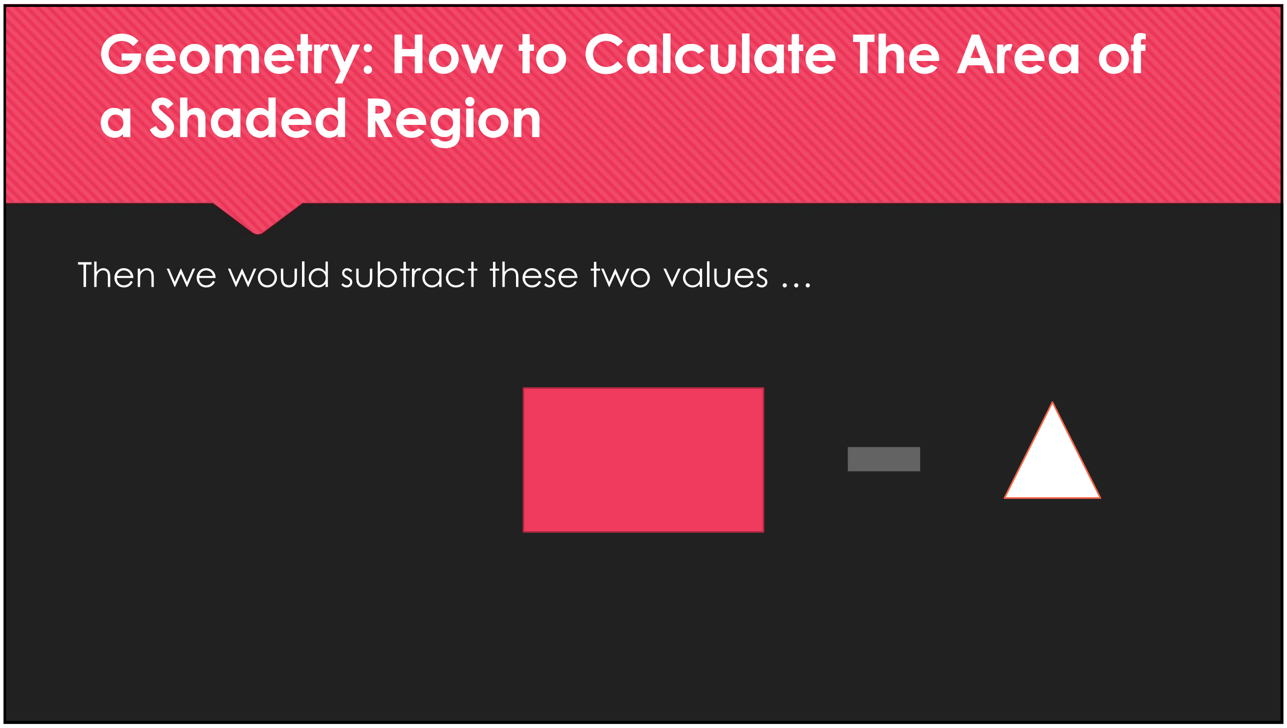
Geometry: How to Calculate The Area of a Shaded Region
Then we would subtract these two values …

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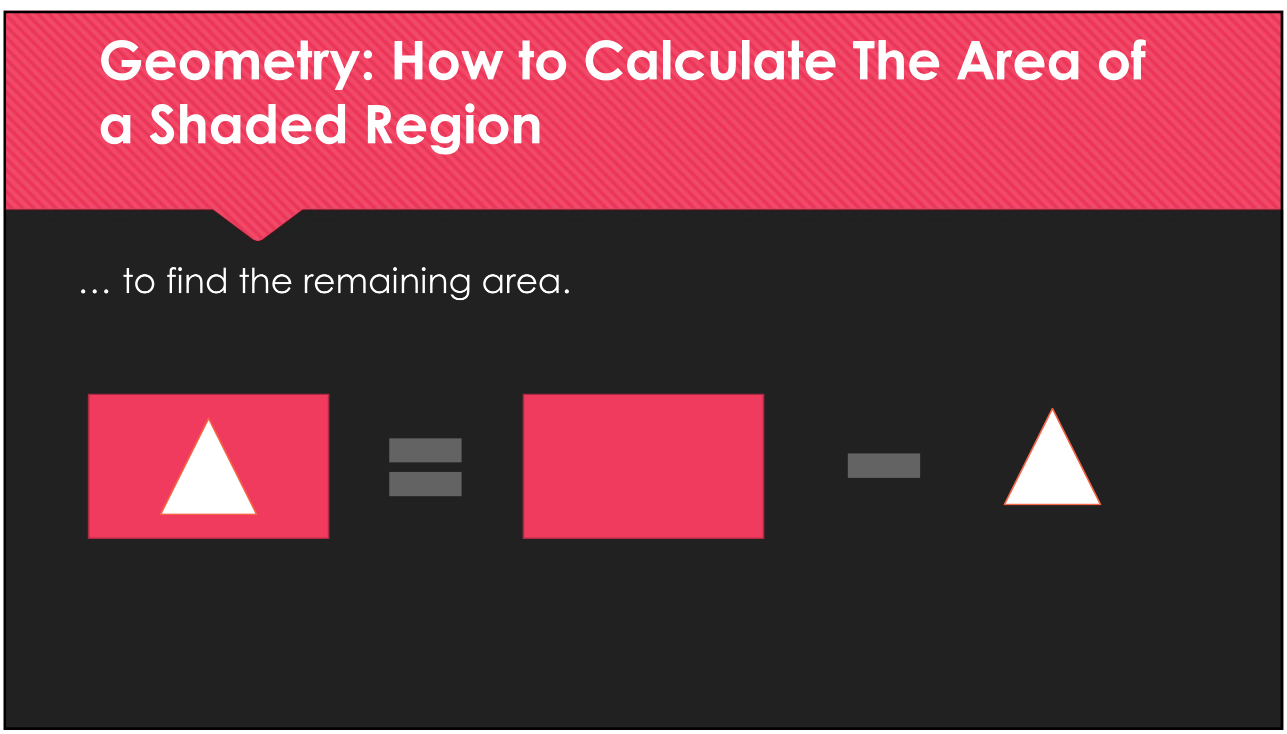
Geometry: How to Calculate The Area of a Shaded Region
… to find the remaining area.

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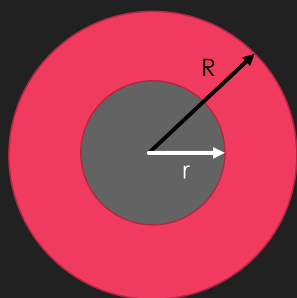
What if we did this with disks?

Let's find the area of the red annulus.



The area of the red circle is $\pi\, R^2$, and the area of the gray circle is $\pi\, r^2$.

So if we subtract the two, we get

$$\pi\, R^2 - \pi\, r^2 = \pi\,(R^2 - r^2)$$

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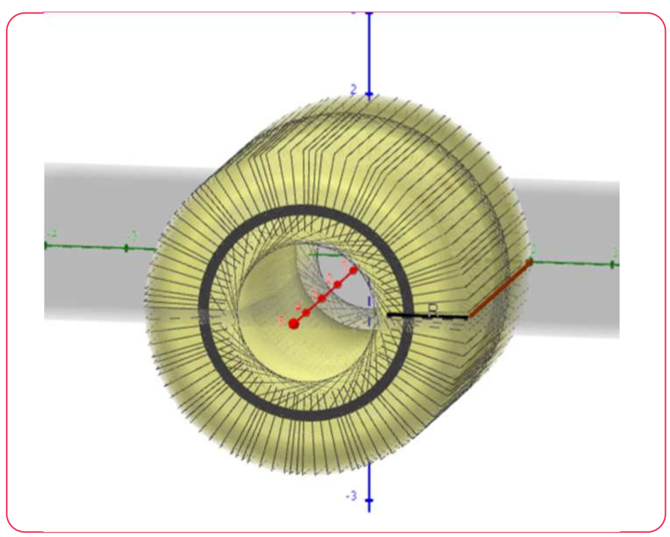
Today's Lecture

- In this lesson, we are going to play with disks, but remove a portion of it.
- This method is called the washer method.



https://www.geogebra.org/m/uym6dwyd

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Washer Method Formula

Since we are just cutting out the middle of the solid, we choose dx or dy in the same way as the disk method.

- Rotating around x-axis ⇒ " dx " problem
- Rotating around y-axis ⇒ " dy " problem

$$V = \pi \int_a^b (R^2 - r^2)\, dx$$

where a and b are bounds of the region we are rotating.

- R is the farthest from the axis rotation
- r is the closest

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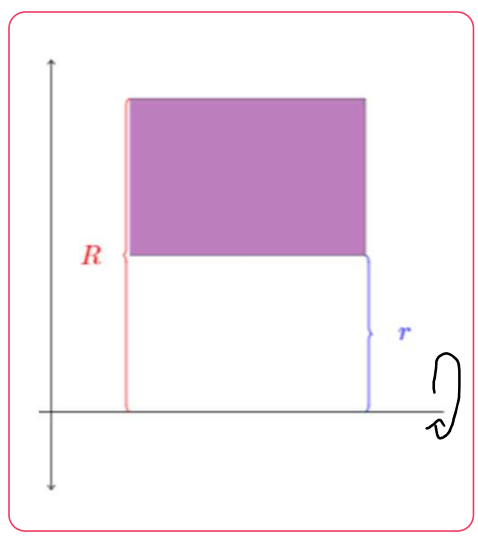
Let's talk a bit more about R and r

- Recall Lessons 12+13 which were about finding the area between 2 curves.
- The same principle applies here.
- For rotation around the x-axis,
  - R is the "Top" Function
  - r is the "Bottom" Function
- Just remember the formula is

$$V = \pi \int_a^b (R^2 - r^2)\, dx$$



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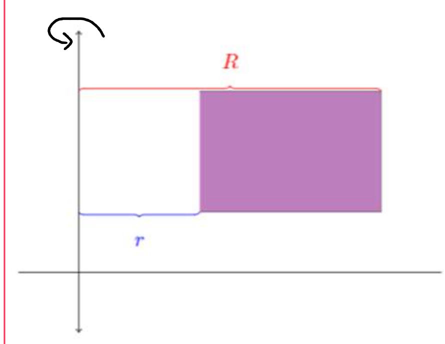
Let's talk a bit more about R and r

- For rotation around the y-axis,
  - R is the "Right" Function
  - r is the "Left" Function
- Just remember the formula is

$$V = \pi \int_c^d (R^2 - r^2)\, dy$$



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How to Proceed with Washer Problems

1. Draw the region
2. Determine which axis you are rotating on
   a. If x - axis: Determine Top and Bottom Function
      i. R is Top
      ii. r is Bottom
   b. If y - axis: Determine Right and Left Function
      i. R is Right
      ii. r is Left
3. Finally, apply the washer formula

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Examples

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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \qquad y = 3x, \qquad \text{and} \qquad x = 2$$

About the x-axis.

https://www.geogebra.org/m/m2p2kdmp

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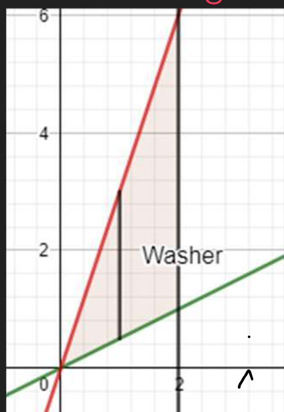
Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \qquad y = 3x, \qquad \text{and} \qquad x = 2$$

About the x-axis.

First draw the region.



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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

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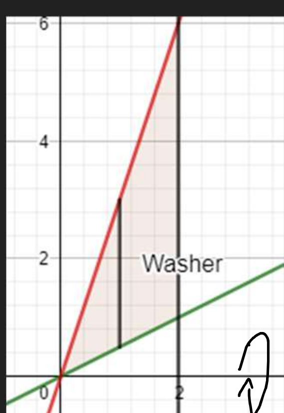
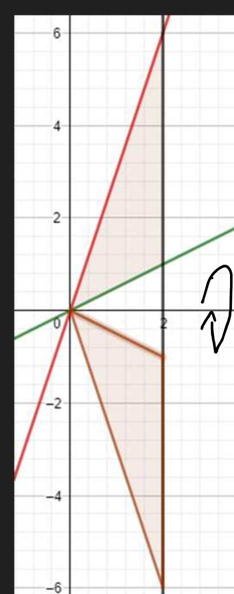
About the x-axis.



Rotation about x-axis



https://www.geogebra.org/m/m2p2kdmp

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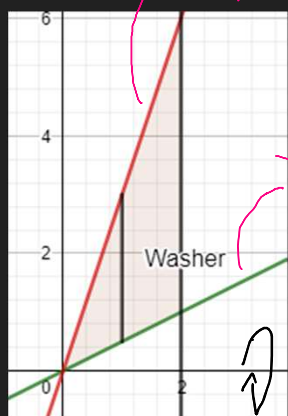
Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = \frac{x}{2}, \qquad y = 3x, \qquad \text{and} \qquad x = 2$$

About the x-axis.

$y = 3x \to$ Top



$y = \frac{x}{2} \to$ Bottom

$$V = \pi \int_0^2 (3x)^2 - \left(\frac{x}{2}\right)^2 dx$$

https://www.geogebra.org/m/m2p2kdmp

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Example 1: Find the volume of the solid that results by revolving the region enclosed by the curves

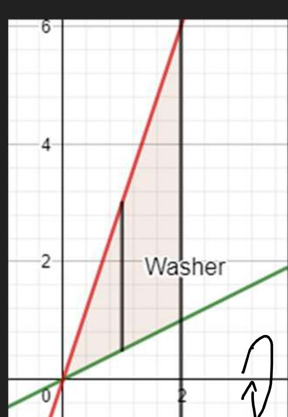
$$y = \frac{x}{2}, \qquad y = 3x, \qquad \text{and} \qquad x = 2$$

About the x-axis.

$$V = \pi \int_0^2 \left(9x^2 - \frac{x^2}{4}\right) dx$$

$$= \pi \int_0^2 \frac{35}{4} x^2 \, dx$$

$$= \frac{35\pi}{4} \frac{x^3}{3} \Big]_0^2 = \frac{70\pi}{3}$$



https://www.geogebra.org/m/m2p2kdmp

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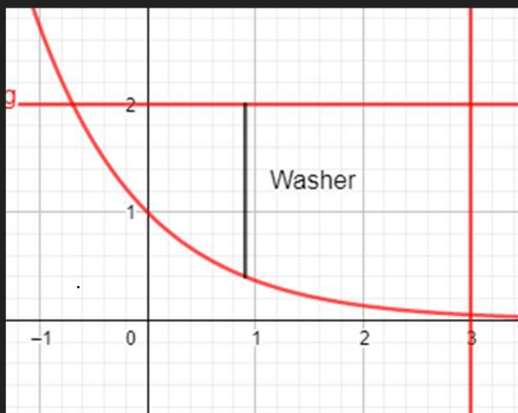
Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \text{ and } \quad x = 3$$

About the x-axis.

First draw the region.



https://www.geogebra.org/m/jfta4b52

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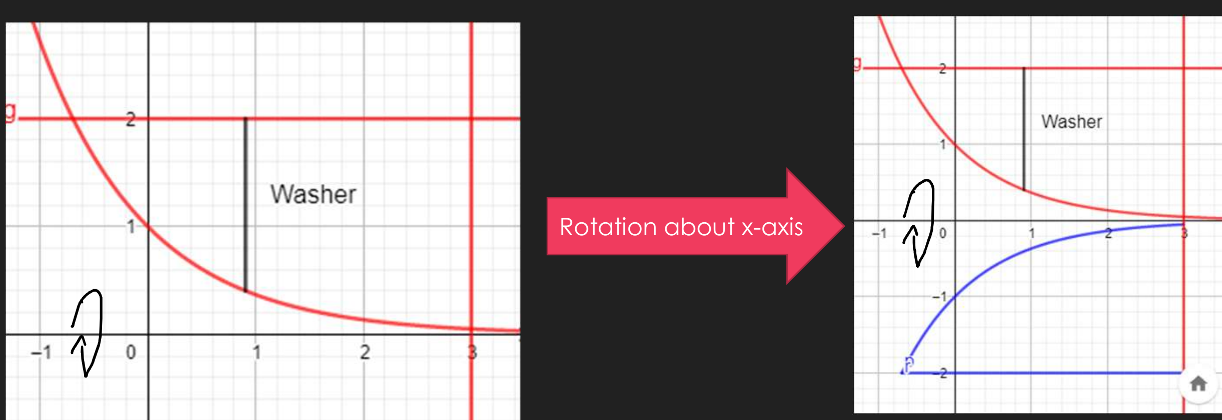

https://www.geogebra.org/m/jfta4b52

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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \text{ and } \quad x = 3$$

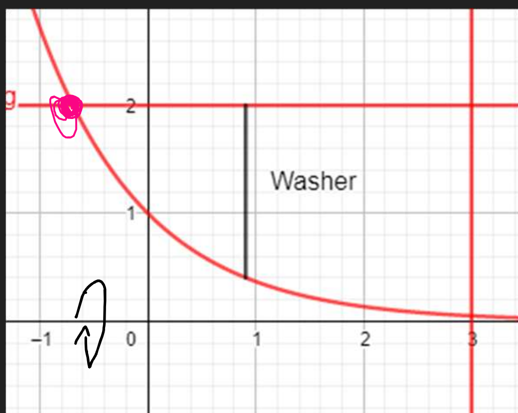
About the x-axis.

See $x=3$ is one bound, we need to find the other

$2 = e^{-x}$

$\ln 2 = -x$

$-\ln 2 = x$



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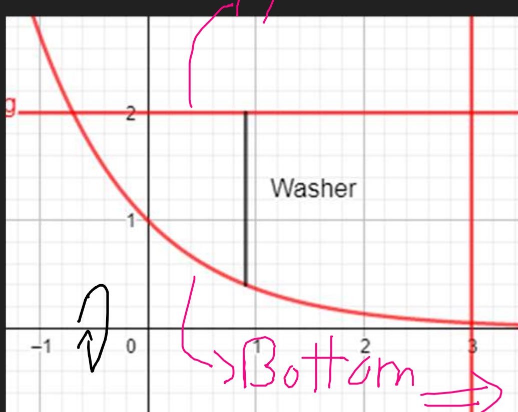
Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \text{ and } \quad x = 3$$

About the x-axis.

$\to y = 2 \to$ Top



$\to$ Bottom $\Rightarrow y = e^{-x}$

So

$$V = \pi \int_{-\ln 2}^{3} 2^2 - (e^{-x})^2 \, dx$$

https://www.geogebra.org/m/jfta4b52

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Example 2: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = e^{-x}, \quad y = 2, \text{ and } \quad x = 3$$

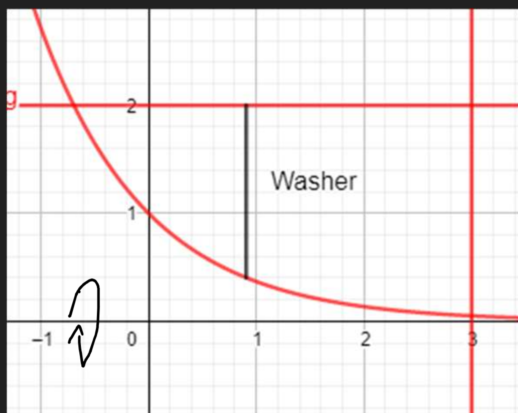
About the x-axis.

$V = \pi \int_{-\ln 2}^{3} 4 - e^{-2x}\, dx$

$= \pi \left(4x - \frac{e^{-2x}}{-2}\right)\Big]_{-\ln 2}^{3}$

$= \pi \left(4x + \frac{e^{-2x}}{2}\right)\Big]_{-\ln 2}^{3}$

≈ 40.13



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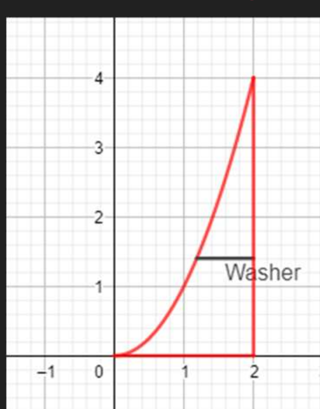
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \text{ and } \quad y = 0$$

About the y-axis.

First draw the region.



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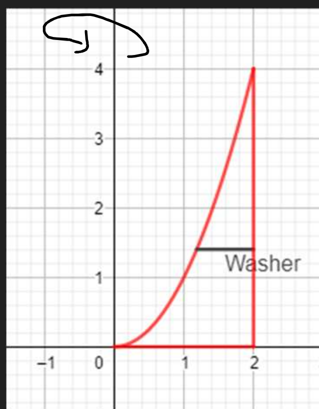
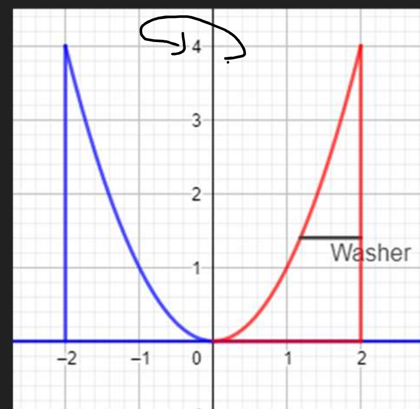
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \text{ and } \quad y = 0$$

About the y-axis.



Rotation about y-axis



https://www.geogebra.org/m/znzmhqq7

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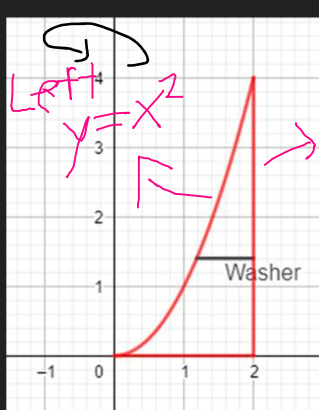
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \text{ and } \quad y = 0$$

About the y-axis.

Rewrite $y=x^2$ to be $x=$

$x=\sqrt{y}$



→ x=2
→ Right

Note no − b/c we are in the first quadrant.

https://www.geogebra.org/m/znzmhqq7

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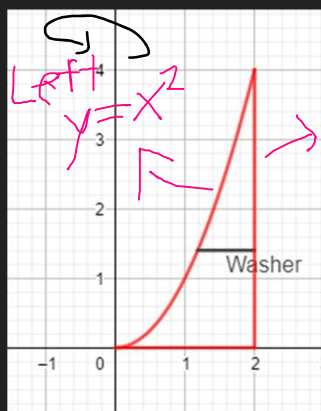
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \text{ and } \quad y = 0$$

About the y-axis.

So $V = \pi \int (2)^2 - (\sqrt{y})^2 \, dy$



$x = 2$
→ Right

$= \pi \int (4 - y) \, dy$

From the graph, we see
$y = 0$ to $y = 4$.

https://www.geogebra.org/m/znzmhqq7

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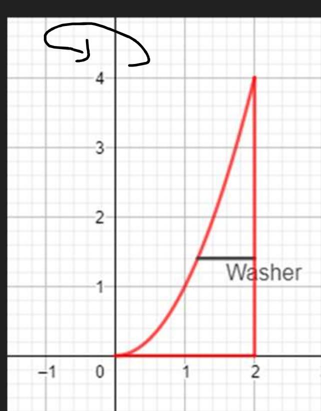
Example 3: Find the volume of the solid that results by revolving the region enclosed by the curves

$$y = x^2, \quad x = 2, \text{ and } \quad y = 0$$

About the y-axis.



$V = \pi \int_0^4 (4 - y) \, dy$

$= \pi \left(4y - \frac{y^2}{2} \right) \Big]_0^4$

$= 8\pi$

https://www.geogebra.org/m/znzmhqq7

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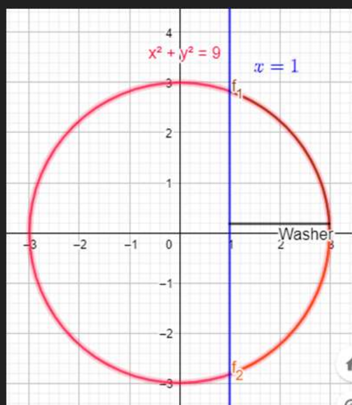
Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

First draw the region.



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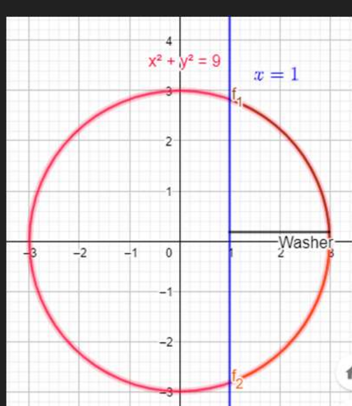
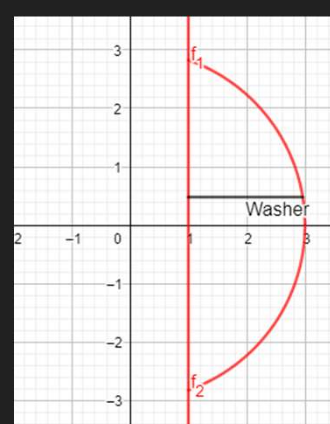
Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

First draw the region.



Note we are only looking at the right of x=1



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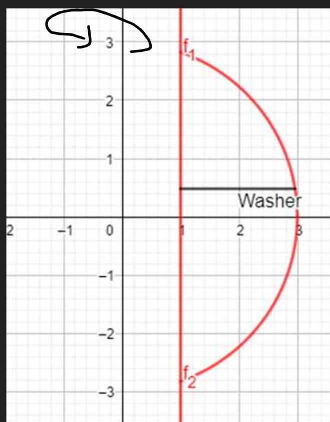
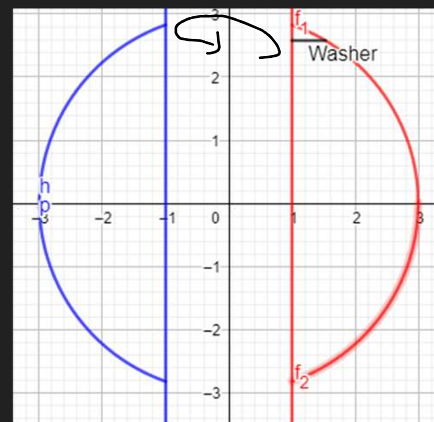


Rotation about y-axis



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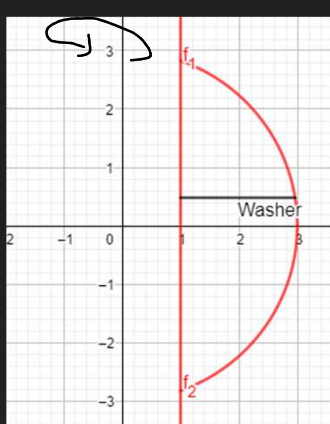
Example 4: Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

Rewrite $x^2+y^2=9$ to be $x=$

$x^2=9-y^2$

$x=\sqrt{9-y^2}$



Note no – b/c we are looking at the right of $x=1$.

https://www.geogebra.org/m/c2wzbrbf

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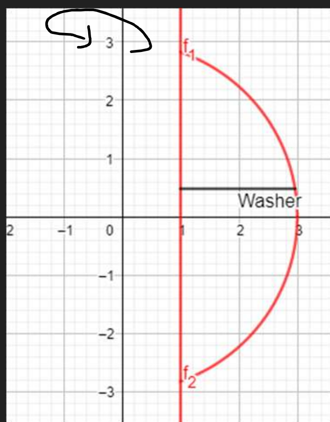
<u>Example 4:</u> Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

To find the bounds, plug $x=1$ into



$x^2+y^2=9$

$1^2+y^2=9$

$1+y^2=9$

$y^2=8$

$y=\pm\sqrt{8}$

https://www.geogebra.org/m/c2wzbrbf

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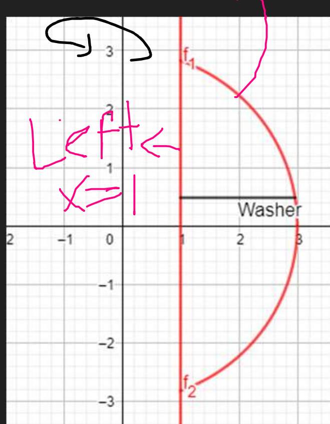
<u>Example 4:</u> Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

$x=\sqrt{9-y^2}$



$V=\pi\int_{-\sqrt{8}}^{\sqrt{8}} \left(\sqrt{9-y^2}\right)^2-1^2\,dy$

$=\pi\int_{-\sqrt{8}}^{\sqrt{8}} (9-y^2-1)\,dy$

$=\pi\int_{-\sqrt{8}}^{\sqrt{8}} (8-y^2)\,dy$

https://www.geogebra.org/m/c2wzbrbf

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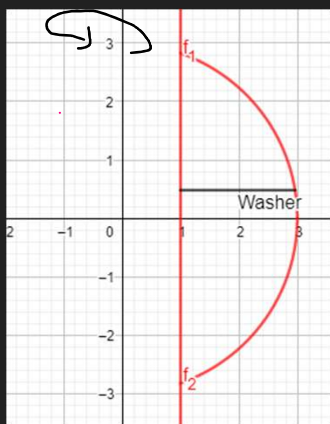
<u>Example 4:</u> Find the volume of the solid that results by revolving the region inside the circle $x^2 + y^2 = 9$ and to the right of the line $x = 1$ about the y-axis.

$$V = \pi\left(8y - \frac{y^3}{3}\right)\Bigg]_{-\sqrt{8}}^{\sqrt{8}}$$

$$= \frac{64\sqrt{2}}{3}\pi$$



https://www.geogebra.org/m/c2wzbrbf

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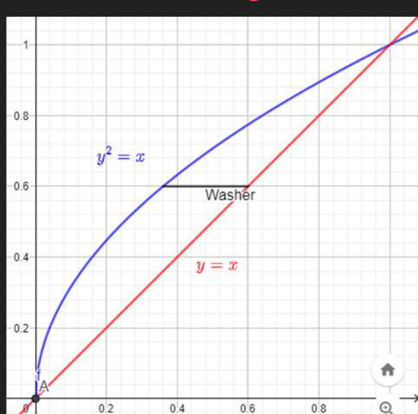
<u>Example 5:</u> Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

a) About the y-axis

First draw the region.



https://www.geogebra.org/m/qtt49cqx

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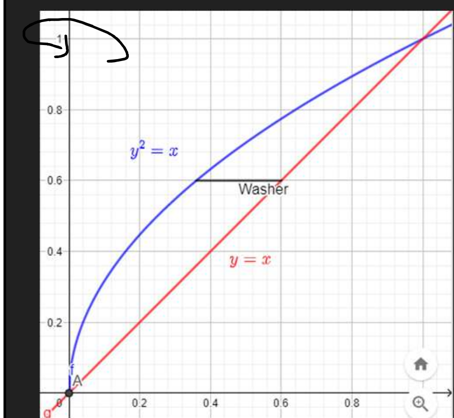
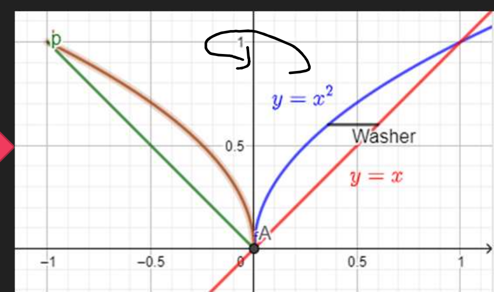
<u>Example 5:</u> Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

a) About the y-axis



Rotation about y-axis



https://www.geogebra.org/m/qtt49cqx

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<u>Example 5:</u> Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

a) About the y-axis

Using the graph, we have

$$V = \pi \int_0^1 (y)^2 - (y^2)^2 \, dy$$

$$= \pi \int_0^1 (y^2 - y^4) \, dy$$

$$= \pi \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \Big]_0^1 = \frac{2\pi}{15}$$

https://www.geogebra.org/m/qtt49cqx

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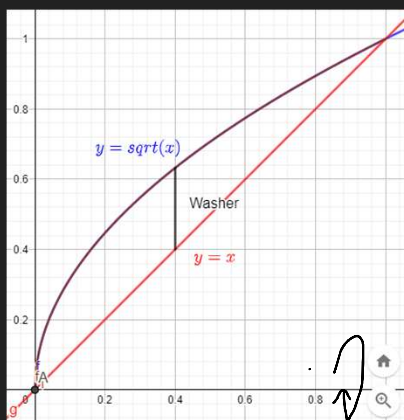
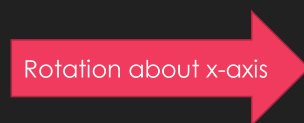
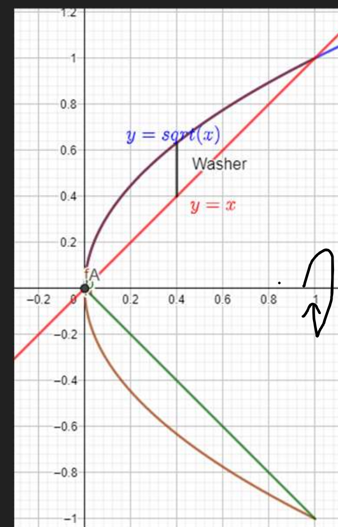
Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

b) About the x-axis







https://www.geogebra.org/m/esmxtp7s

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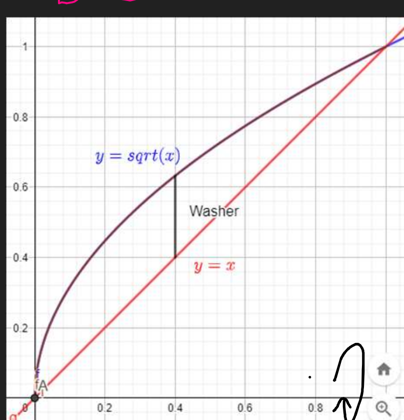
Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

$$y^2 = x, \quad \text{and} \quad x = y$$

b) About the x-axis

Note we are using the same graph



Except we rewrite the lines to be x=

Also we are now looking @Top & Bottom

https://www.geogebra.org/m/esmxtp7s

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Example 5: Find the volume of the solid obtained by revolving the region enclosed by the curves

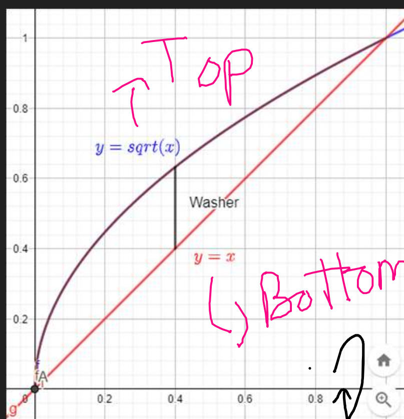
$$y^2 = x, \quad \text{and} \quad x = y$$

b) About the x-axis

$$V = \pi \int_0^1 (\sqrt{x})^2 - (x)^2 \, dx$$

$$= \pi \int_0^1 (x - x^2) \, dx$$

$$= \pi \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big]_0^1 = \frac{\pi}{6}$$



https://www.geogebra.org/m/esmxtp7s

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RECAP: Disk vs. Washer Method

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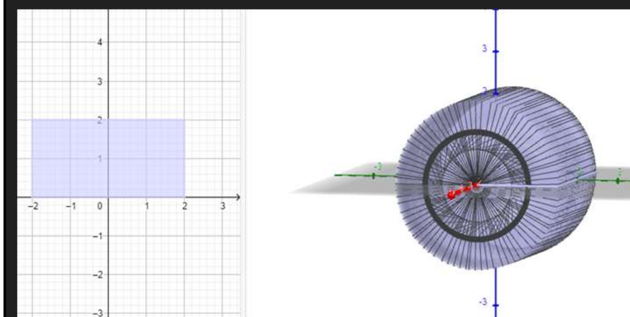
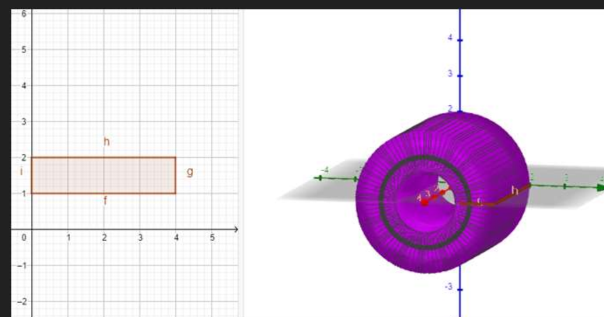
When do apply Disk Method or Washer Method?

Disk Method | **Washer Method**





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When do we apply Disk Method or Washer Method?

- When the region "hugs" the axis of rotation

  $\Rightarrow$ Disk Method

- When there is a "gap" between the region and axis of rotation

  $\Rightarrow$ Washer Method

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GeoGebra Link for Lesson 15

- https://www.geogebra.org/m/f73zjxfe
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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