MA 16020: Lesson 16
Volume By Revolution
Rotation around any non-Axis

By Alexandra Cuadra

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RECAP of Formulas from Lessons 14 and 15

For rotation around x-axis:

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

- Washer Method:

$$V = \pi \int_a^b [R^2 - r^2] \, dx$$

For rotation around y-axis:

- Disk Method:

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

- Washer Method:

$$V = \pi \int_c^d [R^2 - r^2] \, dy$$

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RECAP: When do we apply Disk Method or Washer Method?

- When the region "hugs" the axis of rotation
  ⇒ Disk Method

- When there is a "gap" between the region and axis of rotation
  ⇒ Washer Method

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Today's Lecture

- In the previous two lessons, we looked at rotations around the x-axis or y-axis.

- Today we are going to rotate about **ANY** arbitrary axis.
  - Don't worry. We are going to limit ourselves to any vertical or horizontal line parallel to the x-axis or y-axis

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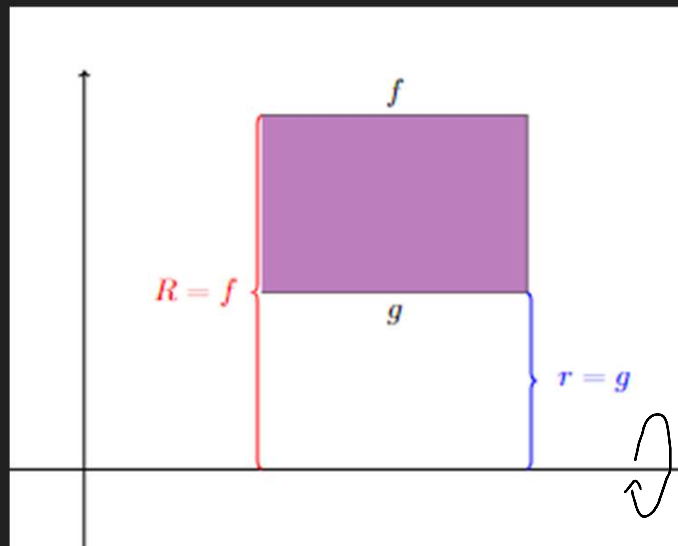
Let's Backtrack a Bit…

Remember when we first described Washers, we talked about **farthest** and **closest**.

Consider the case of x-axis rotation.

In terms of distance,

- R is the length that is **FARTHEST** from x-axis
  - i.e. $\mathrm{R} = f$
- r is the length that is **CLOSEST** to x-axis
  - i.e. $r = g$



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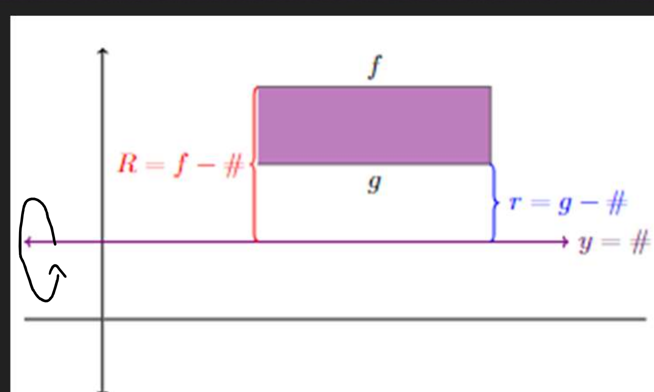
When rotating around the line $y = \#$ …

- Since f is the **FARTHEST**,
  - Distance b/w f and $y = \#$ is $R = f - \#$
- Since g is the **CLOSEST**,
  - Distance b/w g and $y = \#$ is $r = g - \#$
- Washer Method for around $y = \#$:

$$V = \pi \int_a^b [(R - \#)^2 - (r - \#)^2]\, dx$$



Note this formula is also true for the x-axis case, because the x-axis is simply the line $y = 0$

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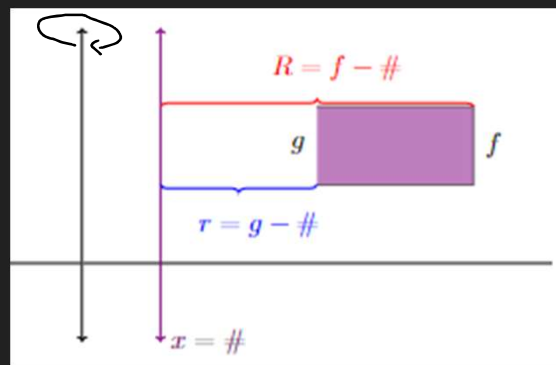
GOOD NEWS EVERYBODY:
When rotating around the line $x = \#$ …

- The same formulas, for R and r, from the case of $y = \#$ applies.
- Washer Method for around $x = \#$:

$$V = \pi \int_a^b [(R - \#)^2 - (r - \#)^2]\, dy$$



Note this formula is also true for the y-axis case, because the y-axis is simply the line $x = 0$

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Note that though we did all these calculations for the Washer Problems; this idea also applies for the Disk Problems.

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Rotation around any non-Axis Formulas

For rotation around the line $y = \#$:

- Disk Method:

$$V = \pi \int_a^b [f(x) - \#]^2 \ dx$$

- Washer Method:

$$V = \pi \int_a^b [(R - \#)^2 - (r - \#)^2] \, dx$$

For rotation around the line $x = \#$:

- Disk Method:

$$V = \pi \int_c^d [g(y) - \#]^2 \ dy$$

- Washer Method:

$$V = \pi \int_c^d [(R - \#)^2 - (r - \#)^2] \, dy$$

Note: That these formulas work for the case of x-axis ($y = 0$) and y-axis ($x = 0$).

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Note that

- **If you replace # with 0, and**
- **Remember that**
  - **x-axis => $y = 0$**
  - **y-axis => $x = 0$**

you get the formulas from Lessons 14 and 15 which are…

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Rotation around any Axis Formulas

For rotation around x-axis:

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2 \; dx$$

- Washer Method:

$$V = \pi \int_a^b [R^2 - r^2] \, dx$$

For rotation around y-axis:

- Disk Method:

$$V = \pi \int_c^d [g(y)]^2 \; dy$$

- Washer Method:

$$V = \pi \int_c^d [R^2 - r^2] \, dy$$

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AGAIN: When do we apply Disk Method or Washer Method?

- When the region "hugs" the axis of rotation

  ⇒ Disk Method

- When there is a "gap" between the region and axis of rotation

  ⇒ Washer Method

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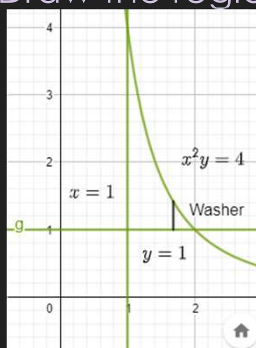
<u>Example 1:</u> Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

A) the x-axis

Draw the region.



https://www.geogebra.org/m/wrj2euhf

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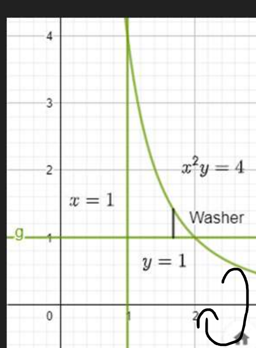
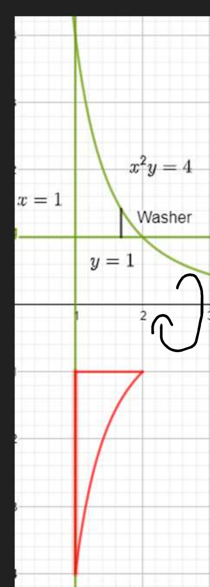
<u>Example 1:</u> Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

A) the x-axis

WASHER PROBLEM



Rotation about x-axis



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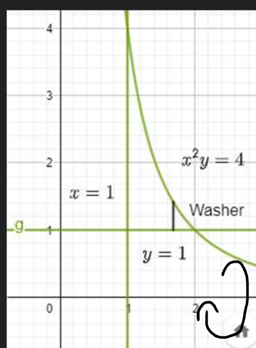
14

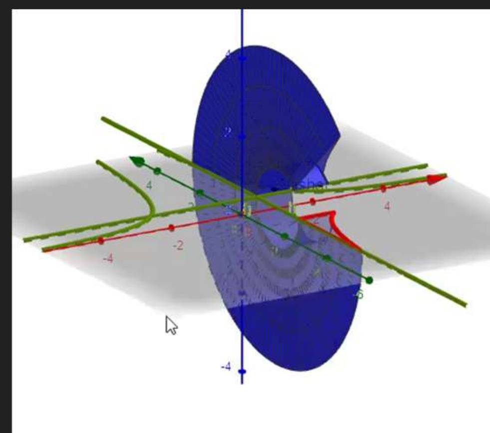
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

A) the x-axis



WASHER PROBLEM

Furthermore, 3-D



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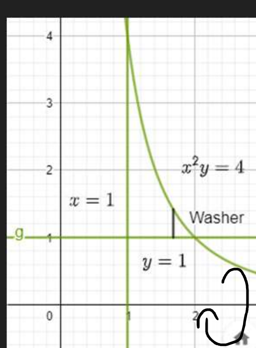
15

Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

A) the x-axis $\Rightarrow$ dx problem



Far $\Rightarrow x^2y = 4 \Rightarrow y = \frac{4}{x^2}$

close $\Rightarrow y = 1$

$$V = \pi \int \left(\frac{4}{x^2}\right)^2 - 1^2 \, dx$$

https://www.geogebra.org/m/wrj2euhf

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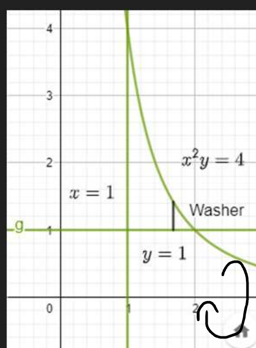
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

A) the x-axis

Now the bounds we are given the smallest on $x=1$ next find the other by putting $y=1$ into $x^2y=4$



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17

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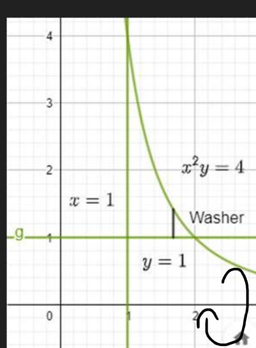
A) the x-axis

So $x^2=4$, $x=\pm 2$ But we are looking for $x>1$, so $x=2$ Hence

$$V=\pi\int_1^2 \left(\frac{4}{x^2}\right)^2 - 1^2 \, dx$$



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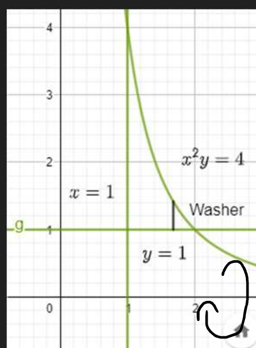
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

A) the x-axis

$$V = \pi \int_1^2 \frac{16}{x^4} - 1 \, dx$$

$$= \pi \int_1^2 16x^{-4} - 1 \, dx$$

$$= \pi \left(\frac{16x^{-3}}{-3} - x \right) \Big]_1^2 = \frac{11\pi}{3}$$



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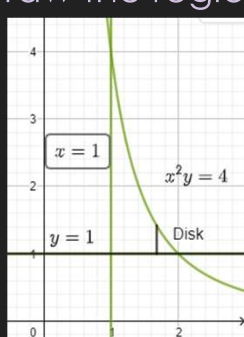
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

B) the line $y = 1$

Draw the region.



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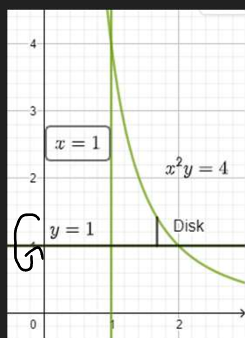
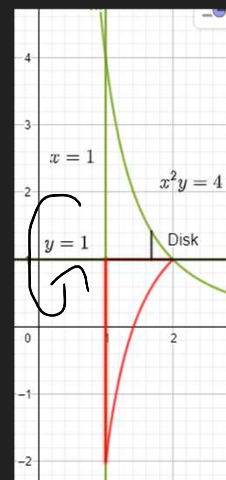
B) the line $y = 1$

DISK PROBLEM

Rotation about $y = 1$





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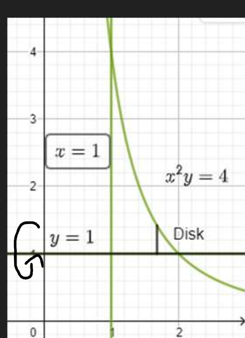
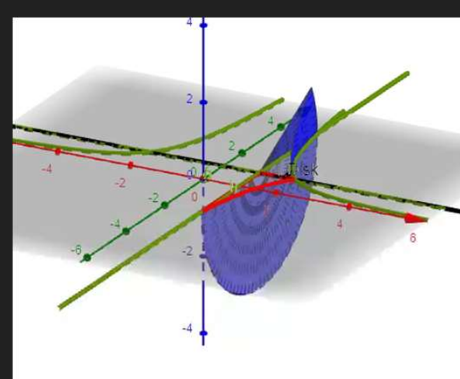
B) the line $y = 1$

DISK PROBLEM

Furthermore, 3-D





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Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

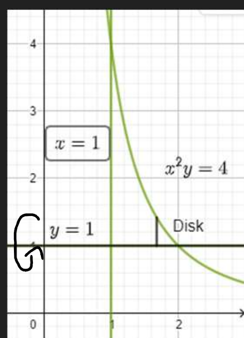
B) the line $y = 1$

dx problem

But now it is a disk problem so

$$V = \pi \int_1^2 \left(\frac{4}{x^2} - 1\right)^2 dx$$

Note our bounds are the same as (a)



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Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

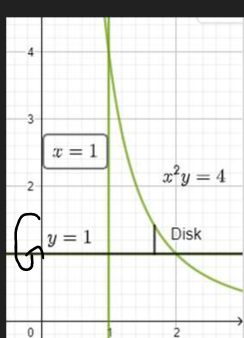
B) the line $y = 1$

$$V = \pi \int_1^2 \left(\frac{16}{x^4} - \frac{8}{x^2} + 1\right) dx$$

$$= \pi \int_1^2 \left(16x^{-4} - 8x^{-2} + 1\right) dx$$

$$= \pi \left(\frac{16x^{-3}}{-3} - \frac{8x^{-1}}{-1} + x\right)\Big]_1^2$$

$$= 5\pi/3$$



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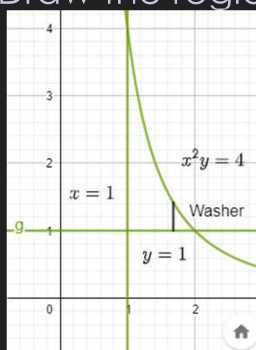
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

C) the y-axis

Draw the region.



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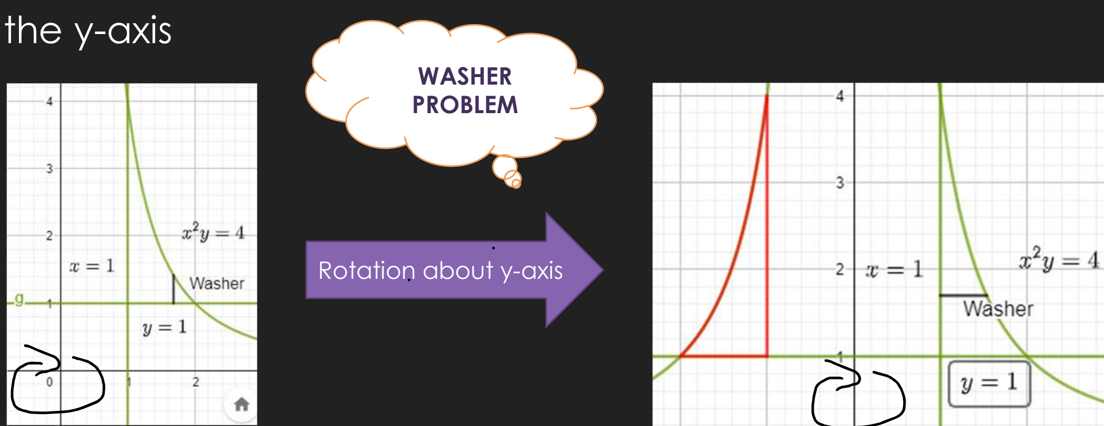
C) the y-axis



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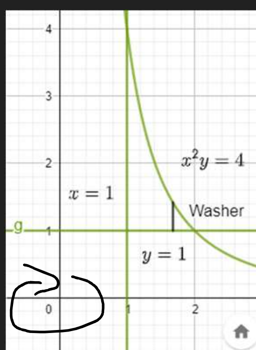
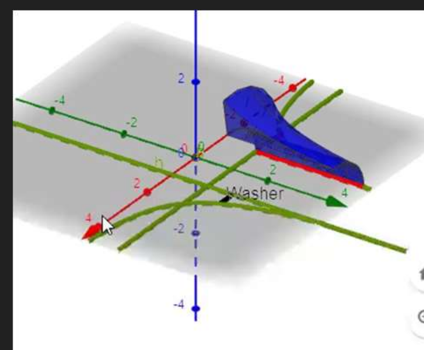
C) the y-axis



WASHER PROBLEM

Furthermore, 3-D



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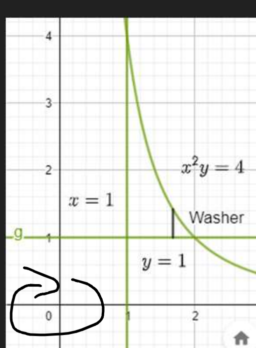
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

C) the y-axis $\Rightarrow$ dy problem

Far $\Rightarrow x^2y = 4 \Rightarrow x = \sqrt{\frac{4}{y}}$

Close $\Rightarrow x = 1$

$V = \pi \int \left(\sqrt{\frac{4}{y}}\right)^2 - 1^2 \, dy$



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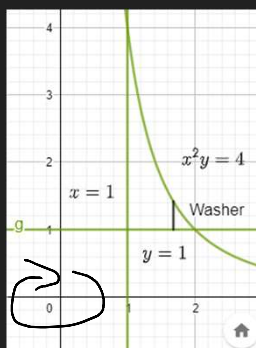
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

C) the y-axis



Now the bounds we are given the smallest one $y=1$

Next find the other by putting $x=1$ into $x^2y=4$

So $y=4$

https://www.geogebra.org/m/wzbm2xbt

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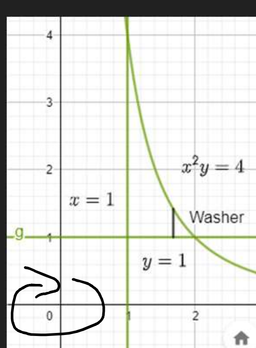
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

C) the y-axis



$$V = \pi \int_1^4 \left(\sqrt{\frac{4}{y}}\right)^2 - 1^2 \, dy$$

$$= \pi \int_1^4 \left(\frac{4}{y} - 1\right) dy$$

$$= \pi \left(4\ln|y| - y\right)\Big]_1^4$$

$$\approx 7.9959$$

https://www.geogebra.org/m/wzbm2xbt

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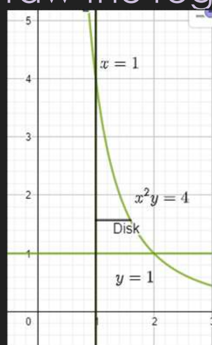
<u>Example 1:</u> Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

D) the line $x = 1$

Draw the region.



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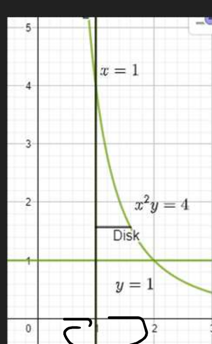
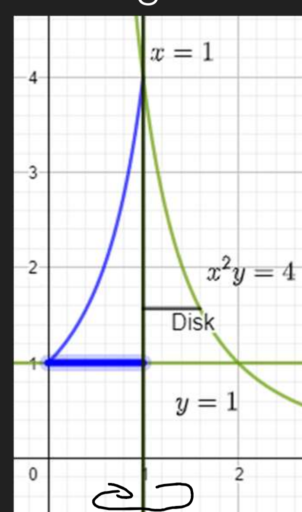
D) the line $x = 1$



Rotation about $x = 1$



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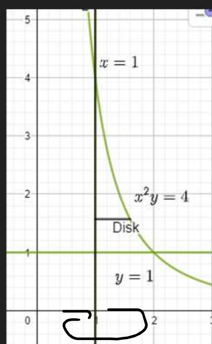
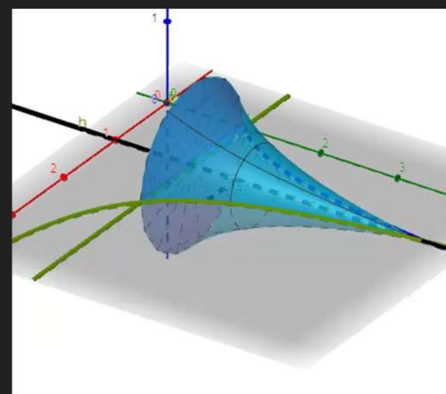
D) the line $x = 1$

DISK PROBLEM



Furthermore, 3-D



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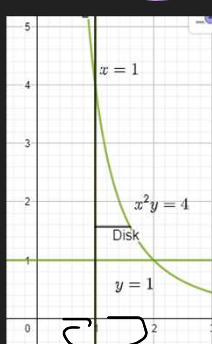
Example 1: Let R be the region of the xy-plane bounded by the curves $x^2y = 4$ below by the line $y = 1$, on the left by the line $x = 1$. Find the volume of the solid obtained by rotating R around

D) the line $x = 1$ → dy problem

But now it is a disk problem so

$$V = \pi \int_1^4 \left(\sqrt{\frac{4}{y}} - 1\right)^2 dy$$

Note our bounds are the same as (C)



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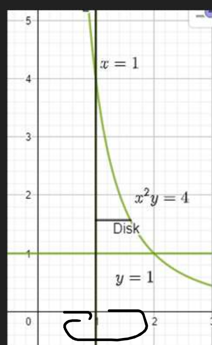
D) the line $x = 1$



$$V = \pi \int_1^4 \left(\frac{4}{y} - 2\sqrt{\frac{4}{y}} + 1 \right) dy$$

$$= \pi \int_1^4 \left(\frac{4}{y} - \frac{2 \cdot 2}{y^{1/2}} + 1 \right) dy$$

$$= \pi \int_1^4 \left(\frac{4}{y} - 4y^{-1/2} + 1 \right) dy$$

$$= \pi \left(4\ln|y| - 4 \cdot \frac{2}{1} y^{1/2} + y \right) \Big]_1^4$$

$$\approx 1.7127$$

https://www.geogebra.org/m/cppyhnqk

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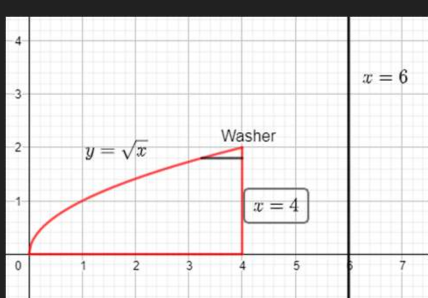
Example 2: Find the volume of the solid generated by revolving the given region about the line $x = 6$:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad x = 4$$

Draw the region.



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Example 2: Find the volume of the solid generated by revolving the given region about the line $x = 6$:

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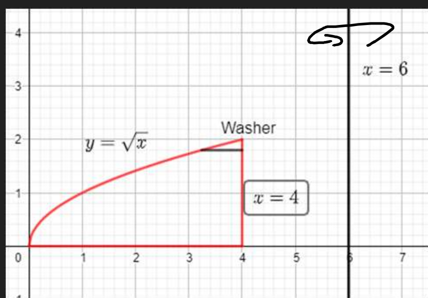
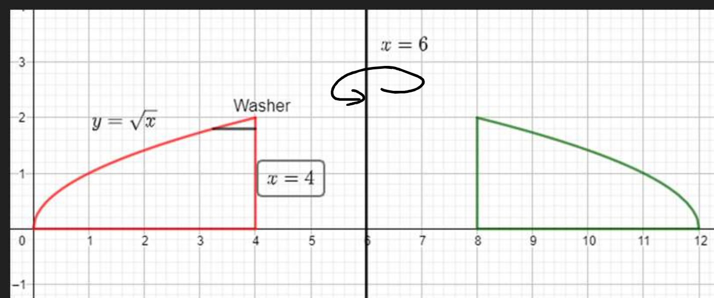
WASHER PROBLEM





Rotation about $x = 6$

https://www.geogebra.org/m/eyabfyya

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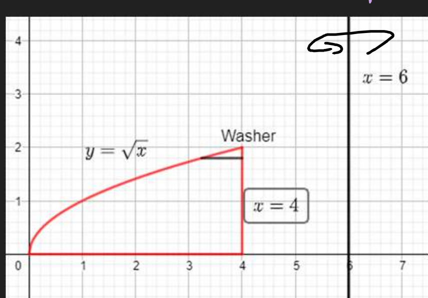
Example 2: Find the volume of the solid generated by revolving the given region about the line $x = 6$:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad x = 4$$

$x=6 \Rightarrow dy$ problem



Far $\Rightarrow y=\sqrt{x} \Rightarrow x=y^2$

Close $\Rightarrow x=4$

BUT we are going around $x=6$

Far $\Rightarrow x= y^2-6$ Close $\Rightarrow x=4-6$

https://www.geogebra.org/m/eyabfyya

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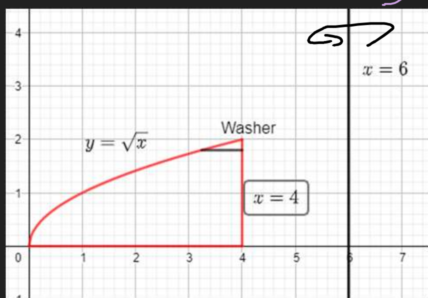
Example 2: Find the volume of the solid generated by revolving the given region about the line $x = 6$:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad x = 4$$

So $V = \pi \int (y^2 - 6)^2 - (4-6)^2 \, dy$



To find the bounds set Far = Close

$y^2 = 4 \Rightarrow y = 2$

B/c I want greater than $y = 0$ b/c that a bound

https://www.geogebra.org/m/eyabfyya

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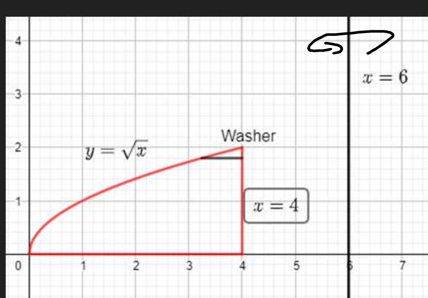
Example 2: Find the volume of the solid generated by revolving the given region about the line $x = 6$:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad x = 4$$



$V = \pi \int_0^2 (y^2 - 6)^2 - (4-6)^2 \, dy$

$= \pi \int_0^2 (y^4 - 12y^2 + 36 - (-2)^2) \, dy$

$= \pi \int_0^2 (y^4 - 12y^2 + 36 - 4) \, dy$

$= \pi \int_0^2 (y^4 - 12y^2 + 32) \, dy$

https://www.geogebra.org/m/eyabfyya

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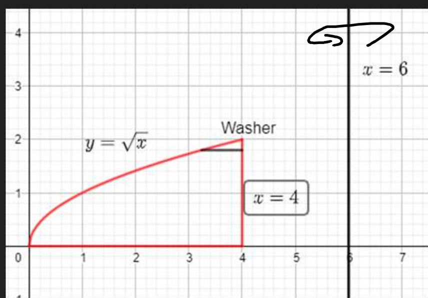
<u>Example 2:</u> Find the volume of the solid generated by revolving the given region about the line $x = 6$:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad x = 4$$

$$V = \pi\left(\frac{y^5}{5} - \frac{12y^3}{3} + 32y\right)\Big]_0^2$$

$$= \frac{192}{5}\pi$$



https://www.geogebra.org/m/eyabfyya

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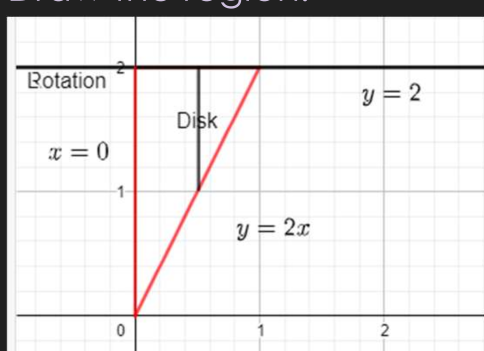
<u>Example 3:</u> Find the volume of the solid generated by revolving the given region about the line $y = 2$:

$$y = 2x\,, \quad x = 0, \quad y = 2$$

Draw the region.



https://www.geogebra.org/m/z6tjgnn9

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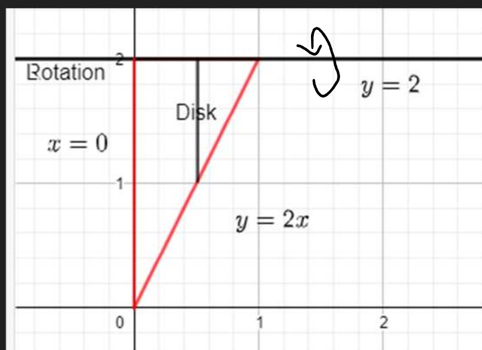
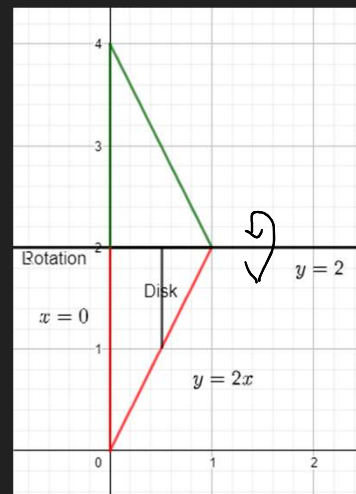
Example 3: Find the volume of the solid generated by revolving the given region about the line $y = 2$:

$$y = 2x\ , \qquad x = 0, \qquad y = 2$$



DISK PROBLEM

Rotation about $y = 2$



https://www.geogebra.org/m/z6tjgnn9

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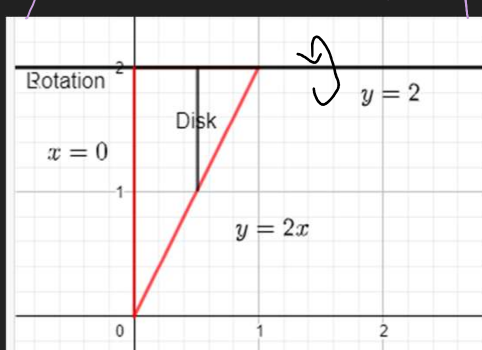
Example 3: Find the volume of the solid generated by revolving the given region about the line $y = 2$:

$$y = 2x\ , \qquad x = 0, \qquad y = 2$$

$y=2 \Rightarrow dx$-problem



$$V = \pi\int_0^1 (2x-2)^2\,dx$$

$$= \pi\int_0^1 (4x^2-8x+4)\,dx$$

$$= \pi\left(\frac{4x^3}{3} - \frac{8x^2}{2} + 4x\right)\Big]_0^1$$

$$= 4\pi/3$$

https://www.geogebra.org/m/z6tjgnn9

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GeoGebra Link for Lesson 16

- https://www.geogebra.org/m/y4pqm3mr
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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