MA 16020: Lesson 18
Volume By Revolution
Shell Method
Pt 2: Rotation around any non-Axis

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RECAP: When should I use Shell Method? How do I use Shell Method?

If you find **solving for x or y**, for either Disk or Washer Method, **is hard**
$\Rightarrow$ **Shell Method**

For rotation around x-axis:

$$V = 2\pi \int_c^d y \cdot (Right - Left)\, dy$$

For rotation around y-axis:

$$V = 2\pi \int_a^b x \cdot (Top - Bottom)\, dx$$

		Axis of Rotation	
		x-axis	y-axis
Method	Disk/Washer	dx	dy
Method	Shells	dy	dx

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What happens if we are revolving around non-Axes (like $x = a$ or $y = b$) ?

For the most part, everything stays the same except for the **RADIUS**.

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How did I find the radius, $r(x)$, last class?

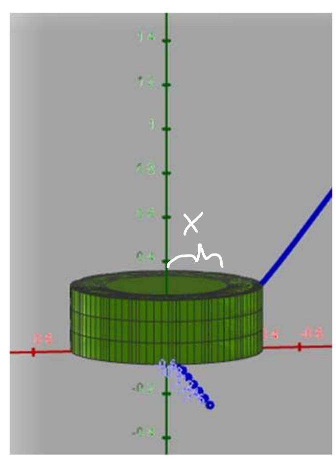
- **To find $r(x)$, we need to find the distance of the shell from the axis of rotation**
- The shell is x units away from the y-axis. So,

$$r(x) = x$$

- So, the formula:

$$V = 2\pi \int_a^b r(x) \cdot (Top - Bottom)\, dx$$

$$V = 2\pi \int_a^b x \cdot (Top - Bottom)\, dx$$



https://www.geogebra.org/m/jqfyndpu

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How did I find the radius, $r(x)$, last class?

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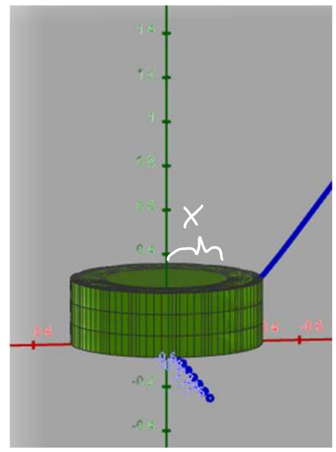

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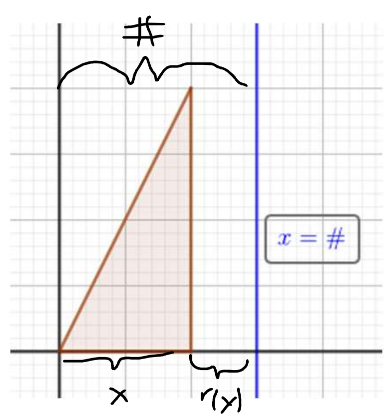
What is $r(x)$ when we are rotating around something other than y-axis?

- **Again, to find $r(x)$, we need to find the distance of the shell from the axis of rotation**
- From the picture we see, # is the distance from the y-axis and the line of rotation. So,
  $$x + r(x) = \# \quad \Rightarrow \quad r(x) = \# - x$$
- If the axis of rotation is on the right of your region,
  $$V = 2\pi \int_a^b r(x) \cdot (Top - Bottom)\, dx$$
  $$V = 2\pi \int_a^b (\# - x) \cdot (Top - Bottom)\, dx$$



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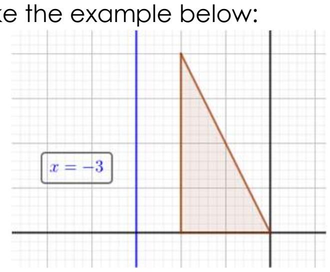
What if the axis of rotation is on the left of the region? Is it the same formula?

Take the example below:



- If $r(x) = \# - x$, then $r(x) = -3 - x$.
  - Choose a value in the region (i.e. triangle).
  - Is r(x) still positive? **No.**

What if $r(x) = x - \#$? **Yes.**

- **Almost…. If the axis of rotation $x = \#$ is on the left, then**

$$r(x) = x - \#$$

  - Why? The radius needs to always positive

- If the axis of rotation is on the left of your region,

$$V = 2\pi \int_a^b r(x) \cdot (Top - Bottom)\, dx$$

$$V = 2\pi \int_a^b (x - \#) \cdot (Top - Bottom)\, dx$$

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Overall, the Shell Method Formulas around any non-axis are…

- Rotating around x = #
  - If the axis of rotation is on the right of your region,

$$V = 2\pi \int_a^b (\# - x) \times (Top - Bottom)\ dx$$

  - If the axis of rotation is on the left of your region,

$$V = 2\pi \int_a^b (x - \#) \times (Top - Bottom)\ dx$$

- Rotating around y = #
  - If the axis of rotation is above of your region,

$$V = 2\pi \int_a^b (\# - y) \times (Right - Left)\ dy$$

  - If the axis of rotation is below of your region,

$$V = 2\pi \int_a^b (y - \#) \times (Right - Left)\ dy$$

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Example 1: Using the **Shell Method,** set up the integral that represents the volume of solid obtained by revolving the region defined by a triangle with vertices at $(0,0)$, $(2,0)$, and $(2,3)$ about the line indicated

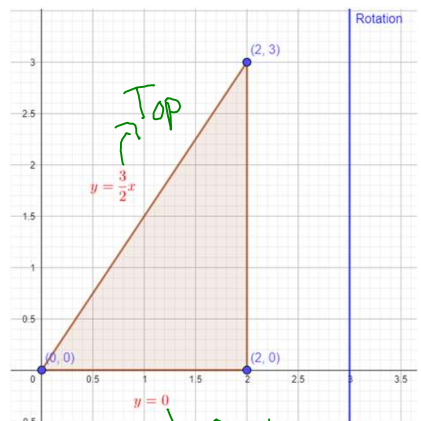
a) about $x = 3$ ⇒ dx problem and $x=3$ is to the right of our region

So $V = 2\pi \int (3-x)(\text{Top} - \text{Bottom})\,dx$

From the graph, the bounds are 0 to 2 and Top $= \frac{3}{2}x$ Bottom $= 0$

$$V = 2\pi \int_0^2 (3-x)\left(\frac{3}{2}x - 0\right) dx$$

Draw the region.



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Example 1: Using the **Shell Method,** set up the integral that represents the volume of solid obtained by revolving the region defined by a triangle with vertices at $(0,0)$, $(2,0)$, and $(2,3)$ about the line indicated

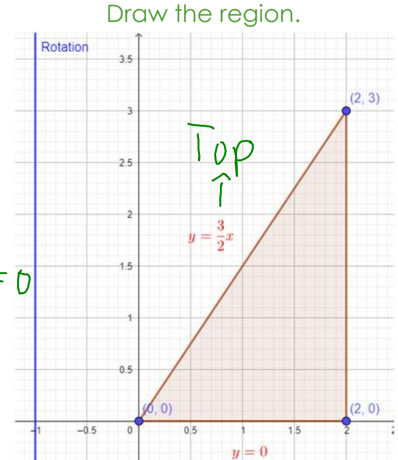
b) about $x = -1$ ⇒ dx problem and $x=-1$ is on the left of our region

So $V = 2\pi \int (x-(-1))(\text{Top} - \text{Bottom})\,dx$

From the graph, the bounds are 0 to 2 and Top $= \frac{3}{2}x$ and Bottom $= 0$

$$V = 2\pi \int_0^2 (x+1)\left(\frac{3}{2}x - 0\right) dx$$

Draw the region.



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Example 2: Using the **Shell Method,** set up the integral that represents the volume of solid obtained by revolving the region defined by

a triangle with vertices at (0,0), (5,0), and (5,6)

about the line indicated

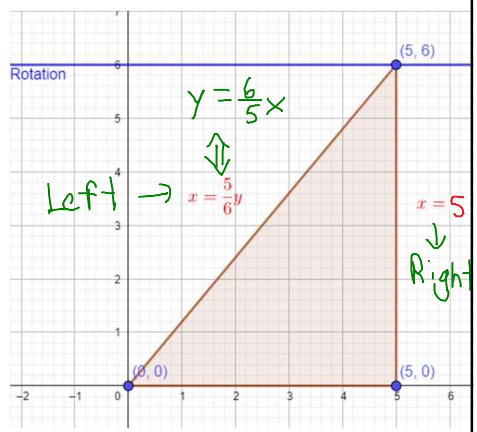
a) About $y = 6$ ⇒ dy problem and y=6 is above our region

So $V = 2\pi \int (6-y)(\text{Right}-\text{Left})\,dy$

From the graph, the bounds are 0 to 6 and Right = 5 and Left = $\frac{5}{6}y$

$$V = 2\pi \int_0^6 (6-y)\left(5-\frac{5}{6}y\right)dy$$

Draw the region.



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Example 2: Using the **Shell Method,** set up the integral that represents the volume of solid obtained by revolving the region defined by

a triangle with vertices at (0,0), (5,0), and (5,6)

about the line indicated

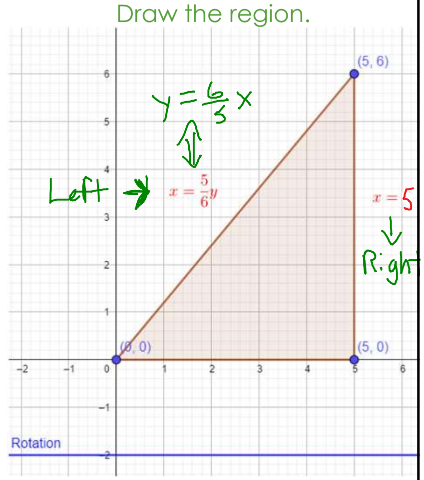
b) About $y = -2$ ⇒ dy problem and y=−2 is below our region

So $V = 2\pi \int (y-(-2))(\text{Right}-\text{Left})\,dy$

From the graph, the bounds are 0 to 6 and Right = 5 and Left = $\frac{5}{6}y$

$$V = 2\pi \int_0^6 (y+2)\left(5-\frac{5}{6}y\right)dy$$

Draw the region.



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Example 3: Consider the region bounded by:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad \text{and} \quad x = 4$$

Set up the integral that represents the volume of solid obtained by the rotating the region using the **Shell Method**

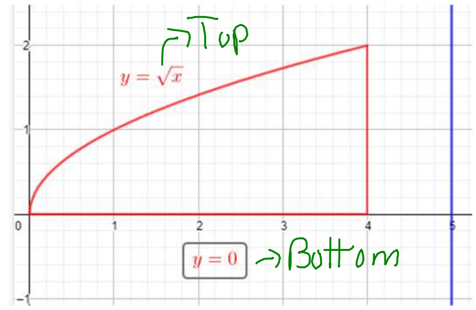
a) about $x = 5$ ⇒ dx problem and x=5 is to the right of our region

So $V = 2\pi \int (5-x)(\text{Top}-\text{Bottom})\,dx$

From the graph, the bounds are 0 to 4 and Top $=\sqrt{x}$ and Bottom $=0$

$$V = 2\pi \int_0^4 (5-x)(\sqrt{x}-0)\,dx$$

Draw the region.



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Example 3: Consider the region bounded by:

$$y = \sqrt{x}\,, \quad y = 0\,, \quad \text{and} \quad x = 4$$

Set up the integral that represents the volume of solid obtained by the rotating the region using the **Shell Method**

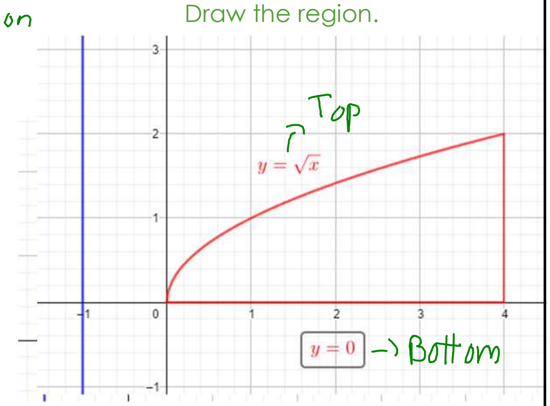
b) about $x = -1$ ⇒ dx problem and x=-1 is on the left of our region

So $V = 2\pi \int (x-(-1))(\text{Top}-\text{Bottom})\,dx$

From the graph, the bounds are 0 to 4 and Top $=\sqrt{x}$ and Bottom $=0$

$$V = 2\pi \int_0^4 (x+1)(\sqrt{x}-0)\,dx$$

Draw the region.



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Example 4: Consider the region bounded by:

$$x = y^2 + 1, \quad \text{and} \quad x = 2$$

Set up the integral that represents the volume of solid obtained by the rotating the region using the **Shell Method**

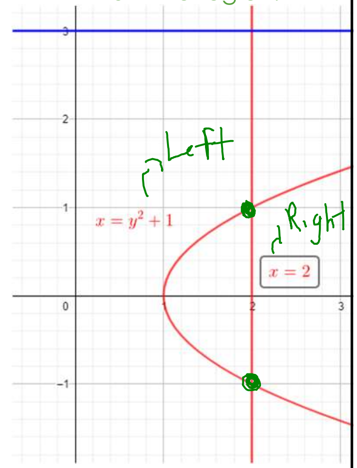
Draw the region.

a) about $y = 3$ ⇒ dy problem and y=3 is above our region

So $V = 2\pi \int (3-y)(\text{Right} - \text{Left})\,dy$

From the graph, the bounds are −1 to 1 and Right = 2 and Left = y^2+1

$$V = 2\pi \int_{-1}^{1} (3-y)\left[2-(y^2+1)\right]dy$$



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Example 4: Consider the region bounded by:

$$x = y^2 + 1, \quad \text{and} \quad x = 2$$

Set up the integral that represents the volume of solid obtained by the rotating the region using the **Shell Method**

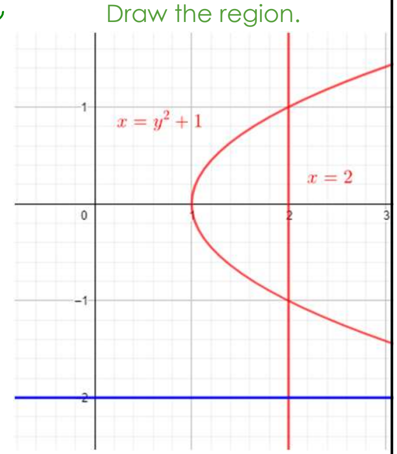
Draw the region.

b) about $y = -2$ ⇒ dy problem and y=−2 is below our region

So $V = 2\pi \int (y-(-2))(\text{Right} - \text{Left})\,dy$

From the graph, the bounds are −1 to 1 and Right = 2 and Left = y^2+1

$$V = 2\pi \int_{-1}^{1} (y+2)(2-(y^2+1))\,dy$$



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When do we apply Disk Method or Washer Method or Shell Method?

- When the region **"hugs"** the axis of rotation
  ⇒ **Disk Method**
- When there is a **"gap"** between the region and axis of rotation
  ⇒ **Washer Method**
- But if you find **solving for x or y**, in either method, **is hard**
  ⇒ **Shell Method**

		Axis of Rotation	
		x-axis or $y = \#$	y-axis or $x = \#$
Method	Disk/Washer	dx	dy
	Shells	dy	dx

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Formulas from Lessons 14 and 15 and 17 Rotation around x-axis or y-axis

For rotation around x-axis:

- Disk Method:

$$V = \pi \int_a^b [f(x)]^2\ dx$$

- Washer Method:

$$V = \pi \int_a^b (R^2 - r^2)\ dx$$

- Shell Method:

$$V = 2\pi \int_c^d y \cdot (Right - Left)\ dy$$

For rotation around y-axis:

- Disk Method:

$$V = \pi \int_c^d [g(y)]^2\ dy$$

- Washer Method:

$$V = \pi \int_c^d (R^2 - r^2)\ dy$$

- Shell Method:

$$V = 2\pi \int_a^b x \cdot (Top - Bottom)\ dx$$

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Formulas from Lesson 15 and 18
Rotation around any non-Axis Formulas

For rotation around the line $x = \#$:

- Disk Method:

$$V = \pi \int_a^b [f(x) - \#]^2 \, dx$$

- Washer Method:

$$V = \pi \int_a^b [(R - \#)^2 - (r - \#)^2] \, dx$$

- Shell Method:
  - If the axis of rotation is on the left of your region,

$$V = 2\pi \int_a^b (x - \#) \times (Top - Bottom) \, dx$$

  - If the axis of rotation is on the right of your region,

$$V = 2\pi \int_a^b (\# - x) \times (Top - Bottom) \, dx$$

Note: That these formulas work for the case of x-axis ($y = 0$) and y-axis ($x = 0$).

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Formulas from Lesson 15 and 18
Rotation around any non-Axis Formulas

For rotation around the line $y = \#$:

- Disk Method:

$$V = \pi \int_c^d [g(y) - \#]^2 \, dy$$

- Washer Method:

$$V = \pi \int_c^d [(R - \#)^2 - (r - \#)^2] \, dy$$

- Shell Method:
  - If the axis of rotation is below your region,

$$V = 2\pi \int_a^b (y - \#) \times (Right - Left) \, dy$$

  - If the axis of rotation is above your region,

$$V = 2\pi \int_a^b (\# - y) \times (Right - Left) \, dy$$

Note: That these formulas work for the case of x-axis ($y = 0$) and y-axis ($x = 0$).

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GeoGebra Link for Lesson 18

- https://www.geogebra.org/m/knnfvjub
- Note click on the play buttons on the left-most screen and the animation will play/pause.

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