Lesson 14: Separation of Variables

Definition: Separation of Variables is a technique to solve some differential equations. Separation of variable can be used when:

- All the y terms (including dy) can be moved to one side of the equation, and
- All the x terms (including dx) to the other side.

Example 1: Consider the differential equation $\frac{dy}{dx} = ky$ where the proportionally constant $k>0$. Find the general solution.

Rewrite:

$$\frac{dy}{dx} = ky$$
$$dx\left(\frac{dy}{dx}\right) = (ky)\,dx$$
$$dy = ky\,dx$$
$$\frac{dy}{y} = \frac{ky\,dx}{y}$$
$$\frac{dy}{y} = k\,dx$$

Now integrate:

$$\int \frac{dy}{y} = \int k\,dx$$
$$\ln|y| = kx + C$$
$$e^{\ln|y|} = e^{kx+C}$$
$$|y| = e^{kx}e^{C}$$
$$\pm y = e^{C}e^{kx}$$
$$y = \boxed{\pm e^{C}}e^{kx}$$

All of this is a constant... So call it all C.

$$y = Ce^{kx}$$

In the future, proportionality $\Rightarrow y' = ky \Rightarrow y = Ce^{kt}$

Example 2: Suppose that $y' = ky$, $y(0) = 5$ and $y'(0) = 10$. What is y as a function of t?

$$y' = ky \Rightarrow y = Ce^{kt}$$

When $y(0) = 5$

$$5 = Ce^{k(0)}$$
$$5 = C \qquad \Rightarrow y = 5e^{kt}$$

Find y': $y = 5ke^{kt}$

when $y'(0) = 10$,

$$10 = 5k\,e^{k(0)}$$
$$10 = 5k$$
$$2 = k \qquad \Rightarrow y = 5e^{2t}$$

Example 3: Solve the IVP:

$$\frac{dy}{dx} = 5x \quad \text{when } y = 10,\ x = 0$$

Solve like we did in Example 1.

Rewrite: $dy = 5x\,dx$

$$\int dy = \int 5x\,dx$$
$$y = \frac{5x^2}{2} + C$$

Plug $y = 10$, $x = 10$ to find C.

$$10 = \frac{5}{2}(0)^2 + C$$
$$10 = C \qquad \Rightarrow y = \frac{5x^2}{2} + 10$$

Example 4: Find the general solution for the following:

ⓐ $\frac{dy}{dx} = -\frac{x}{y}$

Rewrite: $y\,dy = -x\,dx$

$$\int y\,dy = \int -x\,dx$$
$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$
$$y^2 = -x^2 + \underbrace{2C}_{}$$

This is a constant...
So let it be C.

$$y^2 = -x^2 + C$$
$$y = \pm\sqrt{C - x^2}$$

(b) $\frac{dy}{dt} = y\sin(t)$

Rewrite: $\frac{dy}{y} = \sin(t)\,dt$

$$\int \frac{dy}{y} = \int \sin(t)\,dt$$

$$\ln|y| = -\cos(t) + C$$

$$\exp[\ln|y|] = \exp[-\cos(t) + C]$$

$$|y| = \exp[-\cos(t)] \cdot e^{C}$$

$$\pm y = e^{C} \cdot \exp[-\cos(t)]$$

$$y = \underbrace{\pm e^{C}}_{\text{All a constant}} \cdot \exp[-\cos(t)]$$

$$y = C\exp[-\cos(t)]$$

(c) $\frac{dy}{dt} = 7e^{-4t-y}$

Rewrite: $\frac{dy}{dt} = 7e^{-4t}e^{-y}$

$$e^{y}\,dy = 7e^{-4t}\,dt$$

$$\int e^{y}\,dy = \int 7e^{-4t}\,dt$$

Done with a u-sub.

$$e^{y} = 7 \cdot \frac{-1}{4}e^{-4t} + C$$

$$e^{y} = \frac{-7}{4}e^{-4t} + C$$

$$\ln(e^{y}) = \ln\left(\frac{-7}{4}e^{-4t} + C\right)$$

$$y = \ln\left(\frac{-7}{4}e^{-4t} + C\right)$$

(d) $\frac{dy}{dx} = 3x^2(5+y)$

Rewrite: $\frac{dy}{5+y} = 3x^2\,dx$

$$\int \frac{dy}{5+y} = \int 3x^2\,dx$$

$$\ln|5+y| = \frac{3x^3}{3} + C$$

$$\ln|5+y| = x^3 + C$$

$$\exp[\ln|5+y|] = \exp[x^3 + C]$$

$$|5+y| = \exp[x^3] \cdot e^C$$

$$\pm(5+y) = e^C \cdot \exp[x^3]$$

$$5+y = \underbrace{\pm e^C}_{\text{All a constant}} \cdot \exp[x^3]$$

$$5+y = C\exp[x^3]$$

$$y = C\exp[x^3] - 5$$

(e) $\frac{dy}{dx} = \frac{5x+1}{4y^2}$

Rewrite: $4y^2\,dy = (5x+1)\,dx$

$$\int 4y^2\,dy = \int (5x+1)\,dx$$

$$\frac{4y^3}{3} = \frac{5x^2}{2} + x + C$$

$$y^3 = \frac{3}{4}\left(\frac{5}{2}x^2 + x + C\right)$$

$$y^3 = \frac{15}{8}x^2 + \frac{3}{4}x + \underbrace{\frac{3}{4}C}_{\text{All a constant}}$$

$$y^3 = \frac{15}{8}x^2 + \frac{3}{4}x + C$$

$$y = \left(\frac{15}{8}x^2 + \frac{3}{4}x + C\right)^{1/3}$$