MA 16020 LESSON 30: EXTREMA OF FUNCTIONS OF TWO VARIABLES (WORKSHEET)

How do we solve Optimization Problems via Extrema?

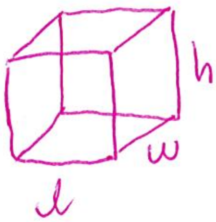
- Determine an **objective function** that we need to maximize or minimize.

- Determine if there are some constraints on the variables which yields **constraint equations.**
    - If there are constraint equations, rewrite the objective function as a function of only TWO variables.

- Then we can solve for the maximum or minimum via **the Second Derivative Test for Multivariable Functions** from Last Class.
    1. Find all the critical points.
       i.e. All (x_0, y_0) such that $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
    2. Compute f_{xx}, f_{xy}, f_{yy} and
       $$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$
       where D is known as the discriminant.
    3. For every given critical point (x_0, y_0), evaluate D and f_{xx} at (x_0, y_0)
    4. Apply the Second Derivative Test for Multivariable Functions.
       i. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0 \quad \Rightarrow \quad$ relative min
       ii. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0 \quad \Rightarrow \quad$ relative max
       iii. If $D(x_0, y_0) < 0 \quad \Rightarrow \quad$ saddle point
       iv. If $D(x_0, y_0) = 0 \quad \Rightarrow \quad$ Test is inconclusive

- Reread the question and be sure you have answered exactly what was asked.

Example 1: **We are tasked with constructing a rectangular box with a volume of 13 cubic feet. The material for the top costs 9 dollars per square foot, the material for the 4 sides cost 2 dollars per square foot, and the material for the bottom costs 1 dollars per square foot. To the nearest cent, what is the minimum cost for such a box?**



Constraint: $13 = V = lwh$ ①

Objective: $C = \$9lw + \$2(2wh + 2lh) + \$1lw$

$= 10lw + 4wh + 4lh$

Solve ① for l.

$l = \frac{13}{wh}$

Plug $l = \frac{13}{wh}$ into ②

$C = 10 \cdot \frac{13}{wh} \cdot w + 4wh + 4 \cdot \frac{13}{wh} \cdot h$

$= 130h^{-1} + 4wh + 52w^{-1}$

Now Second Derivative Test with C.

① Critical Point(s):

$C_w = 4h - 52w^{-2} = 0 \iff 4h - \frac{52}{w^2} = 0$ (I)

$C_h = -130h^{-2} + 4w = 0 \iff 4w - \frac{130}{h^2} = 0$ (II)

Solve (I) for h.

$4h = \frac{52}{w^2}$

$h = \frac{13}{w^2}$

Rewrite (II).

$4w = \frac{130}{h^2}$

$wh^2 = \frac{65}{2}$ (II)

Plug h into new (II).

$w\left(\frac{13}{w^2}\right)^2 = \frac{65}{2}$

$w \frac{169}{w^4} = \frac{65}{2}$

$\frac{169}{w^3} = \frac{65}{2}$

$65w^3 = 338$

$w^3 = \frac{26}{5}$

$w = \sqrt[3]{\frac{26}{5}}$

Plug $w = \sqrt[3]{\frac{26}{5}}$ into

$h = 13\left(\frac{26}{5}\right)^{-2/3}$

Critical Pt:

$(w,h) = \left(\left(\frac{26}{5}\right)^{1/3}, 13\left(\frac{26}{5}\right)^{-2/3}\right)$

② Compute C_{ww}, C_{wh}, C_{hh}, and D.

$C_w = 4h - 52w^{-2}$

$C_{ww} = 104w^{-3} = \frac{104}{w^3}$ $\quad C_{wh} = 4$

$C_h = -130h^{-2} + 4w$

$C_{hh} = 260h^{-3} = \frac{260}{h^3}$

$$D = C_{ww}C_{hh} - (C_{wh})^2 = \left(\frac{104}{w^3}\right)\left(\frac{260}{h^3}\right) - 4^2 = \frac{21424}{w^3h^3} - 16$$

③ Plug critical point (from ①) into C_{ww} and D.

$\underline{\left(\left(\frac{26}{5}\right)^{1/3}, 13\left(\frac{26}{5}\right)^{-2/3}\right)}$: $\quad C_{ww} = \frac{104}{26/5} > 0$

$$D = \frac{21424}{\frac{26}{5} \cdot 13^3\left(\frac{26}{5}\right)^{-2}} - 16 \approx 34.7 > 0$$

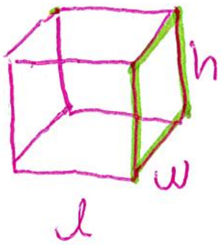
$\Rightarrow$ relative min

Note we want minimize cost.

$C = 130h^{-1} + 4wh + 52w^{-1}$

$$C\left(\left(\frac{26}{5}\right)^{1/3}, 13\left(\frac{26}{5}\right)^{-2/3}\right) \approx \$150.19$$

Example 2: **The post office will accept packages whose combined length and girth is at most 71 inches. (The girth is the perimeter/distance around the package perpendicular to the length; for a rectangular box, the length is the largest of the three dimensions.) What is the largest volume that can be sent in a rectangular box?**



Constraint: $71 = \text{Girth} + l$

$\text{Girth} = 2w + 2h$

$71 = l + 2w + 2h$ ①

Objective: $V = lwh$ ②

Solve ① for l.

$l = 71 - 2w - 2h$

Plug $l = 71 - 2w - 2h$ into ②

$V = (71 - 2w - 2h)wh$

$V = 71wh - 2w^2h - 2wh^2$

Now Second Derivative Test with V.

Note $h=0, w=0$ can't happen b/c then there is no package

① Critical Point(s):

$V_w = 71h - 4wh - 2h^2 = 0 \Rightarrow h(71 - 4w - 2h) = 0 \Rightarrow h = 0$ or $71 = 4w + 2h$

$V_h = 71w - 2w^2 - 4wh = 0 \Rightarrow w(71 - 2w - 4h) = 0 \Rightarrow w = 0$ or $71 = 2w + 4h$

Hence our equations are $\begin{cases} 71 = 4w + 2h & \text{I} \\ 71 = 2w + 4h & \text{II} \end{cases}$

Let's solve via Elimination Method.
Multiply II by 2 and subtract.

$$\begin{aligned} 71 &= \cancel{4w} + 2h \\ -(142 &= \cancel{4w} + 8h) \\ \hline -71 &= -6h \\ h &= \frac{71}{6} \end{aligned}$$

Solve I for w.

$71 = 4w + 2h$

$71 - 2h = 4w$

$\frac{71 - 2h}{4} = w$

Plug $h = 71/6$ into w.

$\frac{71 - 2(71/6)}{4} = w$

$w = \frac{71}{6}$

② Compute V_{ww}, V_{wh}, V_{hh}, and D.

$V_w = 71h - 4wh - 2h^2$

$V_{ww} = -4h$ $\qquad$ $V_{wh} = 71 - 4w - 4h$

$V_h = 71w - 2w^2 - 4wh$

$V_{hh} = -4w$

$$D = V_{ww}V_{hh} - (V_{wh})^2 = (-4h)(-4w) - (71 - 4w - 4h)^2$$
$$= 16hw - \left(5041 - 568w - 568h + 16w^2 + 32wh + 16h^2\right)$$
$$= -5041 + 568w + 568h - 16w^2 - 16wh - 16h^2$$

	71	$-4w$	$-4h$
71	5041	$-284w$	$-284h$
$-4w$	$-284w$	$16w^2$	$16wh$
$-4h$	$-284h$	$16wh$	$16h^2$

③ Plug critical point (from ①) into V_{ww} and D.

$\left(\frac{71}{6}, \frac{71}{6}\right)$: $\quad V_{ww}\left(\frac{71}{6}, \frac{71}{6}\right) = -4\left(\frac{71}{6}\right) = \frac{-142}{3}$

$\quad D\left(\frac{71}{6}, \frac{71}{6}\right) = \frac{5041}{3}$

④ Second Derivative Test

Using ③, we make the following conclusion:

$\left(\frac{71}{6}, \frac{71}{6}\right)$: $\left.\begin{matrix} V_{ww} < 0 \\ D > 0 \end{matrix}\right\} \Rightarrow$ relative max

Note we want the largest volume.

$V = 71wh - 2w^2h - 2wh^2$

$V\left(\frac{71}{6}, \frac{71}{6}\right) \approx 3313.99$

Example 3: A manufacturer is planning to sell a new product at the price of 350 dollars per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, consumers will buy approximately

$$\frac{110y}{y+4}+\frac{190x}{x+9}$$

Units of the product. If the manufactor costs for the product are 170 dollars per unit, how much should the manufactor spend on development and how much on promotion to generate max profit?

Profit = Revenue − Cost

$$\text{Revenue} = \$350\left(\frac{110y}{y+4}+\frac{190x}{x+9}\right)$$

- b/c \$ is spent to develop and promote.
- 1000 b/c x and y is measured in thousands

$$\text{Cost} = \$170\left(\frac{110y}{y+4}+\frac{190x}{x+9}\right)+\underbrace{1000x+1000y}$$

$$P = 350\left(\frac{110y}{y+4}+\frac{190x}{x+9}\right)-170\left(\frac{110y}{y+4}+\frac{190x}{x+9}\right)-1000x-1000y$$

$$= 180\left(\frac{110y}{y+4}+\frac{190x}{x+9}\right)-1000x-1000y$$

$$= \frac{19800y}{y+4}+\frac{34200x}{x+9}-1000x-1000y$$

Now Second Derivative Test with P.

① Critical Point(s)

$$P_x = 34200\cdot\frac{1(x+9)-x\cdot 1}{(x+9)^2}-1000=0 \Rightarrow \frac{307800}{(x+9)^2}-1000=0 \quad \text{(I)}$$

$$P_y = 19800\cdot\frac{1(y+4)-y\cdot 1}{(y+4)^2}-1000=0 \Rightarrow \frac{79200}{(y+4)^2}-1000=0 \quad \text{(II)}$$

Solve (I) for x.

$$\frac{307800}{(x+9)^2} = 1000$$

$$\frac{3078}{10} = (x+9)^2$$

$$\pm\sqrt{\frac{3078}{10}} = x+9$$

$$x = -9 \pm \sqrt{\frac{3078}{10}}$$

but x is positive, so

$$x = -9 + \sqrt{\frac{3078}{10}}$$

$$\approx 8.544 \text{ thousands}$$

Solve (II) for y.

$$\frac{79200}{(y+4)^2} = 1000$$

$$\frac{792}{10} = (y+4)^2$$

$$\pm\sqrt{\frac{792}{10}} = y+4$$

$$y = -4 \pm \sqrt{\frac{792}{10}}$$

but y is positive, so

$$y = -4 + \sqrt{\frac{792}{10}}$$

$$\approx 4.899 \text{ thousands}$$

(2) Compute P_{xx}, P_{xy}, P_{yy}, and D.

$$P_x = 307800(x+9)^{-2} - 1000$$

$$P_{xx} = 307800(-2)(x+9)^{-3} = -\frac{615600}{(x+9)^3}$$

$$P_{xy} = 0$$

$$P_y = 79200(y+4)^{-2} - 1000$$

$$P_{yy} = 79200(-2)(y+4)^{-3} = -\frac{158400}{(y+4)^3}$$

$$D = P_{xx}P_{yy} - (P_{xy})^2 = \left(-\frac{615600}{(x+9)^3}\right)\left(-\frac{158400}{(y+4)^3}\right) - 0^2$$

$$= \frac{97511040000}{(x+9)^3(y+4)^3}$$

③ Plug critical point (from ①) into P_{xx} and D.

$(8.544, 4.899)$: $P_{xx}(8.544, 4.899) = \dfrac{-615600}{\frac{3078}{10}} = -2000$

$$D(8.544, 4.899) = \frac{97511040000}{\frac{3078}{10} \cdot \frac{792}{10}} = 4000000$$

④ Second Derivative Test

Using ③, we make the following conclusion:

$(8.544, 4.899)$: $P_{xx} < 0 \quad D > 0 \Rightarrow$ relative max ✓

We want the amount spent on development and promotion. Note x and y are measured in thousands. So

Development: \$8,544

Promotion: \$4,899