Lesson 33: Double Integrals I

Let $z=f(x,y)$ be a function of two variables. Similar to taking partial derivatives with respect to x and y, we can take

- $\int_{x=a}^{x=b} f(x,y)\,dx$ – Integrate with respect to x and Treat y as a constant
- $\int_{y=c}^{y=d} f(x,y)\,dy$ – Integrate with respect to y and Treat x as a constant

Combining the above integrals, we obtain the following double integrals:

- $\int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy$
- $\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx = \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$

<u>Example 1</u>: Evaluate

ⓐ $\int_0^5\int_0^3 (x+y)\,dy\,dx = \int_{x=0}^{x=5}\left(\int_{y=0}^{y=3}(x+y)\,dy\right)dx$

$$= \int_{x=0}^{x=5}\left(\left(xy+\frac{y^2}{2}\right)\Big]_{y=0}^{y=3}\right)dx$$

$$= \int_{x=0}^{x=5}\left(3x+\frac{9}{2}-\left(0\cdot x-\frac{0^2}{2}\right)\right)dx$$

$$= \int_{x=0}^{x=5}\left(3x+\frac{9}{2}\right)dx$$

$$= \left(\frac{3x^2}{2}+\frac{9}{2}x\right)\Big]_{x=0}^{x=5}$$

$$= \frac{3(5)^2}{2}+\frac{9}{2}(5)-\left(\frac{3(0)^2}{2}+\frac{9}{2}(0)\right)$$

$$= 60$$

(b) $\displaystyle\int_1^2\int_0^1 x^2 y\,dy\,dx = \int_{x=1}^{x=2}\left(\int_{y=0}^{y=1} x^2 y\,dy\right)dx$

$$= \int_{x=1}^{x=2} x^2\cdot\left(\int_{y=0}^{y=1} y\,dy\right)dx$$

$$= \int_{x=1}^{x=2} x^2\left(\frac{y^2}{2}\right)\Bigg]_{y=0}^{y=1} dx$$

$$= \int_{x=1}^{x=2} x^2\left(\frac{1^2}{2}-\frac{0^2}{2}\right)dx$$

$$= \int_{x=1}^{x=2} \frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}\cdot\frac{x^3}{3}\Bigg]_{x=1}^{x=2}$$

$$= \frac{2^3}{6}-\frac{1^3}{6}$$

$$= \frac{7}{6}$$

(c) $\displaystyle\int_0^{\pi/2}\int_0^1 16y^4\cos(x)\,dy\,dx = \int_{x=0}^{x=\pi/2}\left(\int_{y=0}^{y=1} 16y^4\cos(x)\,dy\right)dx$

$$= \int_{x=0}^{x=\pi/2}\cos(x)\cdot\left(\int_{y=0}^{y=1}16y^4\,dy\right)dx$$

$$= \int_{x=0}^{x=\pi/2}\cos(x)\cdot\left(\frac{16y^5}{5}\right)\Bigg]_{y=0}^{y=1}dx$$

$$= \int_{x=0}^{x=\pi/2}\cos(x)\left(\frac{16(1)^5}{5}-\frac{16(0)^5}{5}\right)dx$$

$$= \int_{x=0}^{x=\pi/2}\frac{16}{5}\cos(x)\,dx$$

$$= \frac{16}{5}\sin(x)\Bigg]_{x=0}^{x=\pi/2}$$

$$= \frac{16}{5}\left(\sin\left(\frac{\pi}{2}\right)-\sin(0)\right)$$

$$= \frac{16}{5}(1-0)$$

$$= 16/5$$

(d) $\int_{\pi/6}^{\pi/2}\int_{-1}^{2} \cos(y)\,dx\,dy = \int_{y=\pi/6}^{y=\pi/2}\left(\int_{x=-1}^{x=2} \cos(y)\,dx\right)dy$

$= \int_{y=\pi/6}^{y=\pi/2} \cos(y)\cdot\left(\int_{x=-1}^{x=2} dx\right)dy$

$= \int_{y=\pi/6}^{y=\pi/2} \cos(y)\cdot x\Big]_{x=-1}^{x=2} dy$

$= \int_{y=\pi/6}^{y=\pi/2} \cos(y)\cdot(2-(-1))\,dy$

$= \int_{y=\pi/6}^{y=\pi/2} 3\cos(y)\,dy$

$= 3\sin y\Big]_{y=\pi/6}^{y=\pi/2}$

$= 3\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right]$

$= 3\left[1-\frac{1}{2}\right]$

$= \frac{3}{2}$

(e) $\int_0^5\int_0^3 x^2e^{2y}\,dx\,dy = \int_{y=0}^{y=5}\int_{x=0}^{x=3} x^2e^{2y}\,dx\,dy$

$= \int_{y=0}^{y=5} e^{2y}\cdot\left(\int_{x=0}^{x=3} x^2\,dx\right)dy$

$= \int_{y=0}^{y=5} e^{2y}\left(\frac{x^3}{3}\right)\Big]_{x=0}^{x=3} dy$

$= \int_{y=0}^{y=5} e^{2y}\left(\frac{3^3}{3}-\frac{0^3}{3}\right)dy$

$= \int_{y=0}^{y=5} 9e^{2y}\,dy$

$= \frac{9}{2}e^{2y}\Big]_0^5$

$= \frac{9}{2}\left(e^{10}-e^{0}\right) = \frac{9}{2}\left(e^{10}-1\right)$

Example 2: Evaluate

ⓐ $\int_0^7 \int_0^y 5xy \, dx \, dy = \int_{y=0}^{y=7} \left(\int_{x=0}^{x=y} 5xy \, dx \right) dy$

$= \int_{y=0}^{y=7} 5y \cdot \left(\int_{x=0}^{x=y} x \, dx \right) dy$

$= \int_{y=0}^{y=7} 5y \left(\frac{x^2}{2} \right) \Big]_{x=0}^{x=y} dy$

$= \int_{y=0}^{y=7} 5y \left(\frac{y^2}{2} - \frac{0^2}{2} \right) dy$

$= \int_{y=0}^{y=7} 5y \cdot \frac{y^2}{2} \, dy$

$= \frac{5}{2} \int_{y=0}^{y=7} y^3 \, dy$

$= \frac{5}{2} \cdot \frac{y^4}{4} \Big]_0^7$

$= \frac{5}{8} \left(7^4 - 0^4 \right) = \frac{12005}{8}$

ⓑ $\int_3^4 \int_3^x \frac{6x}{y^2} \, dy \, dx = \int_{x=3}^{x=4} \left(\int_{y=3}^{y=x} 6xy^{-2} \, dy \right) dx$

$= \int_{x=3}^{x=4} 6x \cdot \left(\int_{y=3}^{y=x} y^{-2} \, dy \right) dx$

$= \int_{x=3}^{x=4} 6x \cdot \left(-y^{-1} \right) \Big]_{y=3}^{y=x} dx$

$= \int_{x=3}^{x=4} 6x \left(\frac{-1}{y} \right) \Big]_{y=3}^{y=x} dx$

$= \int_{x=3}^{x=4} 6x \left(-\frac{1}{x} + \frac{1}{3} \right) dx$

Remember, we introduced

$$\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy \text{ and } \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx$$

A geometric interpertation of these double integrals is we are finding the volume below $z=f(x,y)$ above the region

$$R=\{(x,y) \mid a\le x\le b;\ c\le y\le d\}$$

We can denote the integrals above by just one

$$\iint_R f(x,y)\,dA$$

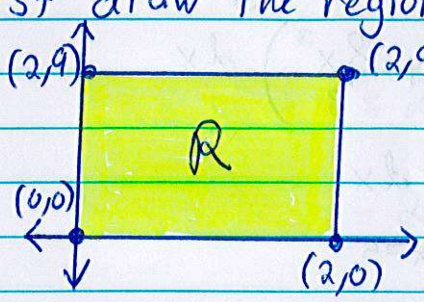
where R is the **domain of integration.**

To solve the integrals of the form $\iint_R f(x,y)\,dA$, we start by drawing the region. We do this to determine the bounds of our integrals.

Example 3: Evaluate the integral $\iint_R 10x^3y\,dA$ where R is the rectangle with vertices $(0,0)$, $(2,0)$, $(0,9)$, and $(2,9)$.

First draw the region, R. We can see that $0\le x\le 2$ and $0\le y\le 9$. So

$$R=\{(x,y) \mid 0\le x\le 2,\ 0\le y\le 9\}$$



Hence

$$\iint_R 10x^3y\,dA = \int_{x=0}^{x=2}\left(\int_{y=0}^{y=9} 10x^3\cdot y\,dy\right)dx$$

$$=\int_{x=0}^{x=2}\left(10x^3\cdot \frac{y^2}{2}\Big]_{y=0}^{y=9}\right)dx$$

$$=\int_{x=0}^{x=2}\left(10x^3\left(\frac{9^2}{2}-\frac{0^2}{2}\right)\right)dx$$

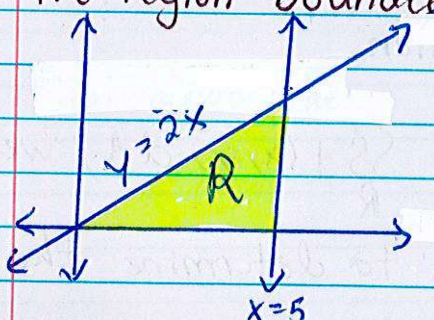
$$= \int_{x=0}^{x=2} \left(10x^3 \cdot \frac{81}{2}\right) dx$$

$$= \int_{x=0}^{x=2} 405x^3\,dx$$

$$= \frac{405}{4} x^4 \Big]_{x=0}^{x=2}$$

$$= \frac{405}{4}(2^4 - 0^4) = 1620$$

Example **4**: Evaluate the integral $\iint_R (x^2+y^2)\,dA$ where R is the region bounded by the lines $y=2x$, $x=5$, and the x-axis.



First draw the region, R. We can see that

$$R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 2x\}$$

Hence $\iint_R (x^2+y^2)\,dA = \int_{x=0}^{x=5}\left(\int_{y=0}^{y=2x} (x^2+y^2)\,dy\right) dx$

$$= \int_{x=0}^{x=5} \left(\left(x^2 \cdot y + \frac{y^3}{3}\right)\Big]_{y=0}^{y=2x}\right) dx$$

$$= \int_{x=0}^{x=5} \left(x^2(2x) + \frac{(2x)^3}{3} - \left(x^2 \cdot 0 + \frac{0^3}{3}\right)\right) dx$$

$$= \int_{x=0}^{x=5} \left(2x^3 + \frac{8x^3}{3}\right) dx$$

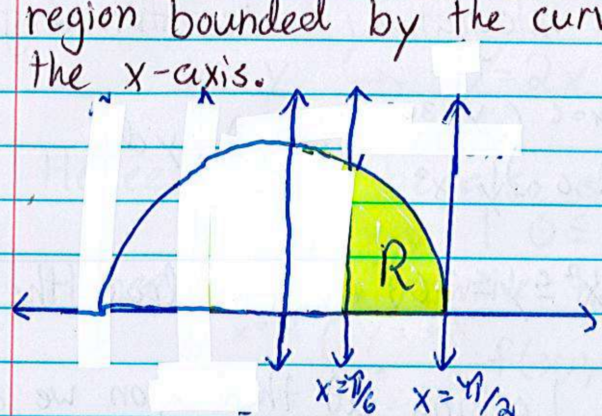
$$= \int_{x=0}^{x=5} \frac{14}{3} x^3\,dx$$

$$= \frac{14}{3} \cdot \frac{x^4}{4}\Big]_{x=0}^{x=5}$$

$$= \frac{7}{6} x^4 \Big]_{x=0}^{x=5}$$

$$= \frac{7}{6}(5^4 - 0^4) = \frac{4375}{6}$$

__Example 5:__ Evaluate the integral $\iint_R 6\sin^2(x)\,dA$ where R is the region bounded by the curves $y=\cos(x)$, $x=\pi/6$, $x=\pi/2$ and the x-axis.



First draw the region, R. We can see that

$$R=\left\{(x,y)\;\middle|\;\frac{\pi}{6}\le x\le\frac{\pi}{2},\ 0\le y\le\cos x\right\}$$

Hence

$$\iint_R 6\sin^2(x)\,dA=\int_{x=\pi/6}^{x=\pi/2}\left(\int_{y=0}^{y=\cos(x)}6(\sin x)^2\,dy\right)dx$$

$$=\int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2\cdot y\Big]_{y=0}^{y=\cos x}\right)dx$$

$$=\int_{x=\pi/6}^{x=\pi/2}\left(6(\sin x)^2\cdot(\cos x-0)\right)dx$$

$$=\int_{x=\pi/6}^{x=\pi/2}6(\sin x)^2\cdot\cos x\,dx$$

$$\frac{u=\sin x}{du=\cos x\,dx}\quad\int 6u^2\,du$$

$$=\frac{6u^3}{3}=2u^3$$

$$=2(\sin x)^3\Big]_{x=\pi/6}^{x=\pi/2}$$

$$=2\left(\left(\sin\left(\frac{\pi}{2}\right)\right)^3-\left(\sin\left(\frac{\pi}{6}\right)\right)^3\right)$$

$$=2\left((1)^3-\left(\frac{1}{2}\right)^3\right)=2\left(1-\frac{1}{8}\right)=\frac{2\cdot 7}{8}=\frac{7}{4}$$

Note that Examples **3, 4, and 5** could have been done with dxdy as the order of integration.

So a good question to ask is when to use dxdy or dydx? We will answer that next time.