

Lesson 35: Double Integrals

Recall from Lesson 2, the formula for average value:
For $f(x)$ defined on $[a, b]$, the average value of $f(x)$ on $[a, b]$ is

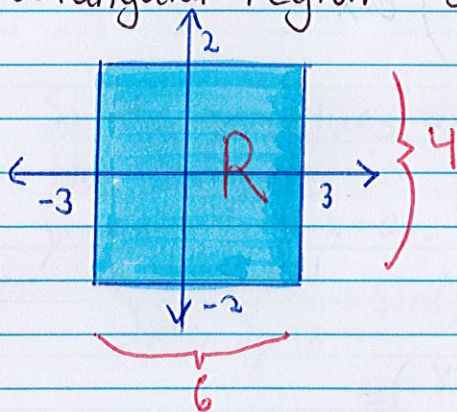
$$f_{AVE}(x) = \frac{1}{b-a} \int_a^b f(x) dx$$

The multivariable average value formula follows:

For $f(x, y)$ defined on a region, R , the average value of $f(x, y)$ over the region, R , is given by

$$f_{AVE}(x, y) = \frac{1}{A} \iint_R f(x, y) dA \quad \text{where } A \text{ is the area of } R.$$

Example 1: Find the average of $f(x, y) = 12 - x^2 - y^2$ in a rectangular region $-3 \leq x \leq 3$, $-2 \leq y \leq 2$.



First draw the region. With the drawing, find the area of R .

$$\text{Area} = 6 \times 4 = 24$$

of R

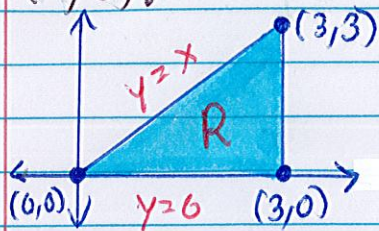
Note we are given the bounds for our integral. So

$$f_{AVE}(x, y) = \frac{1}{24} \int_{x=-3}^{x=3} \int_{y=-2}^{y=2} (12 - x^2 - y^2) dy dx$$

Now integrate.

$$\begin{aligned} f_{AVE}(x, y) &= \frac{1}{24} \int_{x=-3}^{x=3} \left(12y - x^2y - \frac{y^3}{3} \right) \Big|_{y=-2}^{y=2} dx \\ &= \frac{1}{24} \int_{x=-3}^{x=3} \left(12(2) - 2x^2 - \frac{2^3}{3} - \left(12(-2) - x^2(-2) - \frac{(-2)^3}{3} \right) \right) dx \\ &= \frac{1}{24} \int_{x=-3}^{x=3} \left(24 - 2x^2 - \frac{8}{3} + 24 - 2x^2 - \frac{8}{3} \right) dx \\ &= \frac{1}{24} \int_{x=-3}^{x=3} \left(\frac{128}{3} - 4x^2 \right) dx \\ &= \frac{1}{24} \left(\frac{128}{3}x - \frac{4x^3}{3} \right) \Big|_{x=-3}^{x=3} \\ &= \frac{1}{24} \left(\frac{128}{3}(3) - \frac{4(3)^3}{3} - \left(\frac{128}{3}(-3) - \frac{4(-3)^3}{3} \right) \right) = \frac{28}{3} \end{aligned}$$

Example 2: Find the average value of $f(x,y) = x^2 + 2xy + y^2$ in the triangular region with vertices $(0,0)$, $(3,0)$, and $(3,3)$.



First draw the region, with drawing, find the area of R,

$$\text{Area} = \frac{1}{2} (3)(3) = \frac{9}{2}$$

Next, we need to determine the bounds for our integral. The x -values are $0 \leq x \leq 3$. As for y , we see that y lies between the x -axis ($y=0$) and the line $y=x$. So

$$\begin{aligned} f_{\text{AVE}} &= \frac{1}{9/2} \int_{x=0}^{x=3} \int_{y=0}^{y=x} (x^2 + 2xy + y^2) dy dx \\ &= \frac{2}{9} \int_{x=0}^{x=3} \left(x^2 y + \frac{2xy^2}{2} + \frac{y^3}{3} \right) \Big|_{y=0}^{y=x} dx \\ &= \frac{2}{9} \int_{x=0}^{x=3} \left(x^3 + x \cdot x^2 + \frac{x^3}{3} \right) dx \\ &= \frac{2}{9} \int_{x=0}^{x=3} \frac{7}{3} x^3 dx \\ &= \frac{2}{9} \cdot \frac{7}{3} \cdot \frac{x^4}{4} \Big|_{x=0}^{x=3} \\ &= \frac{2}{9} \cdot \frac{7}{3} \cdot \frac{3^4}{4} = \frac{21}{2} \end{aligned}$$