Lesson 6: The Natural Logarithmic Function: Integration

The function $f(t) = 1/t$ is continuous on $(0, \infty)$.

By FTC, $f(t) = 1/t$ has an antiderivative on the interval with endpoints x and 1 when $x > 0$.

Definition: $\ln(x) = \int_1^x \frac{1}{t}\, dt$

But in general, $\ln|x| = \int \frac{dt}{t}$

Example 1: Let $f(x) = \ln(x^5 + 7x + 12)$. Compute $f'(x)$.

$$f'(x) = \frac{1}{x^5 + 7x + 12} \cdot (x^5 + 7x + 12)'$$

$$= \frac{1}{x^5 + 7x + 12} \cdot (5x^4 + 7)$$

$$= \frac{5x^4 + 7}{x^5 + 7x + 12}$$

Example 2: Compute the following:

ⓐ $\int \frac{5x^4 + 7}{x^5 + 7x + 12}\, dx$ $\quad \frac{u = x^5 + 7x + 12}{du = (5x^4 + 7)dx}$ $\quad \int \frac{du}{u} = \ln|u| + C$

$$= \ln|x^5 + 7x + 12| + C$$

ⓑ $\int \frac{10t}{9t^2 + 7}\, dt$ $\quad \frac{u = 9t^2 + 7}{du = 18t\, dt}$ $\quad \int \frac{10t}{u} \cdot \frac{du}{18t} = \frac{5}{9} \int \frac{du}{u}$

$\frac{du}{18t} = dt$

$$= \frac{5}{9} \ln|u| + C = \frac{5}{9} \ln|9t^2 + 7| + C$$

ⓒ $\int \frac{1000}{10+3x}\,dx \quad \frac{u=10+3x}{du=3dx} \quad \int \frac{1000}{u}\cdot\frac{du}{3} = \frac{1000}{3}\int\frac{du}{u}$

$\frac{du}{3}=dx$

$= \frac{1000}{3}\ln|u| + C = \frac{1000}{3}\ln|10+3x| + C$

ⓓ $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx \quad \xrightarrow[du=\cos x\,dx]{u=\sin x} \int\frac{du}{u} = \ln|u| + C$

$= \ln|\sin x| + C$

ⓔ $\int \tan x\,dx = \int\frac{\sin x}{\cos x}\,dx \quad \frac{u=\cos x}{du=-\sin x\,dx} \quad \int\frac{-du}{u}$

$-du = \sin x\,dx$

$= -\ln|u| + C = -\ln|\cos x| + C$

ⓕ $\int \frac{8\sin x}{3+\cos x}\,dx \quad \frac{u=3+\cos x}{du=-\sin x\,dx} \quad \int\frac{8\cancel{\sin x}}{u}\cdot\frac{du}{-\cancel{\sin x}}$

$\frac{du}{-\sin x} = dx$

$= -8\int\frac{du}{u} = -8\ln|u| + C = -8\ln|3+\cos x| + C$

Example 3: Compute the following:

ⓐ $\int \frac{dx}{5x(\ln(3x))^2} \quad \frac{u=\ln(3x)}{du=\frac{1}{3x}\cdot 3dx} \quad \int\frac{x\,du}{5x\cdot u^2} = \int\frac{du}{5u^2}$

$du = \frac{1}{x}dx$

$x\,du = dx$

$= \frac{1}{5}\int u^{-2}\,du = \frac{1}{5}\frac{u^{-1}}{-1} + C = \frac{1}{5u} + C$

$= \frac{1}{5\ln(3x)} + C$

(b) $\int \frac{dx}{x\ln(x^{19})} = \int \frac{dx}{x \cdot 19\ln(x)} = \frac{1}{19}\int \frac{dx}{x\ln(x)}$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$\frac{1}{19}\int \frac{\cancel{x}\,du}{\cancel{x}\,u}$

$= \frac{1}{19}\ln|u| + C$

$= \frac{1}{19}\ln|(\ln(u))| + C$

(c) $\int \frac{4(\ln x)^3}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$\int \frac{4u^3}{x} \cdot x\,du = \int 4u^3\,du$

$= \frac{4u^4}{4} + C = u^4 + C = (\ln(x))^4 + C$

(d) $\int \frac{\ln\sqrt{x}}{x}\,dx = \int \frac{\ln(x^{1/2})}{x}\,dx = \int \frac{\frac{1}{2}\ln x}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}dx$

$x\,du = dx$

$= \frac{1}{2}\int \frac{u}{x} \cdot x\,du = \frac{1}{2}\int u\,du = \frac{1}{2}\frac{u^2}{2} + C$

$= \frac{1}{4}(\ln x)^2 + C$