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ON CERTAIN L-FUNCTIONS

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Introduction. To generalize our previous study of certain Langlands L-functions, in this paper we develop a general theory for certain local coefficients. It turns out that the holomorphy of these local coefficients determines the irreducibility of certain induced representations, and furthermore they can be used to normalize the intertwining operators. These local coefficients take on a global significance, as we find them appearing in the functional equations satisfied by these L-functions. In particular, we establish the functional equations satisfied by the L-functions attached to the cusp forms on $PGL_2(\mathbf{A})$, and five and six dimensional irreducible representations of $SL_2(\mathbf{C})$, its L-group. Finally we prove certain non-vanishing theorems for several of these L-functions at the line $\mathrm{Re}(s) = 1$. As it is the case with the classical L-functions, some interesting consequences are expected (see the remarks before Theorem 5.2).

As was shown in [15, 23], the theory of Eisenstein series plays an important role in establishing the analytic continuation and functional equation of certain Langlands L-functions. This general philosophy is due to R. P. Langlands [15]. The local L-functions and root numbers are defined in general only for almost all places. But another function related to them can be defined for all places. In Section 3, we develop a general theory for these functions, which we will call the "local coefficients". In fact, let G be a connected reductive algebraic group defined over a local field F (we assume that G is quasi-split if $F = \mathbf{R}$), and let A_0 be a maximal split torus of G over F. Fix a set of simple roots Δ of A_0. Given $\theta \subset \Delta$, let $P_\theta = M_\theta N_\theta$ be the standard parabolic subgroup of G corresponding to θ (cf. Section 2). Let A_θ be the maximal split torus in the center of M_θ, and denote by $\mathfrak{a}_\theta$ its real Lie algebra. Finally let χ be a non-degenerate character of $N_\phi = U$, and fix an element w in the Weyl group of (G, A_0) such that $w(\theta) \subset \Delta$.

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Then given an irreducible admissible non-degenerate (also fine, cf. 1.3, when $F = \mathbf{R}$) representation π of M_θ and a linear functional $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$, we define a complex number $C_\chi(\nu, \pi, \theta, w)$. This is what we call a local coefficient. It is these factors whose holomorphy determines the irreducibility of the induced representation $I(\nu, \pi, \theta)$ (cf. Section 2). In fact Theorem 3.3.1 states that if F is non-archimedean, π is supercuspidal, and ν_0 is so that $\pi \otimes q^{\langle \nu_0, H_\theta(\ )\rangle}$ is unramified, then $I(\nu_0, \pi, \theta)$ is irreducible if and only if both $C_\chi(\nu, \pi, \theta, w_\theta)$ and $C_\chi(w_\theta(\nu), w_\theta(\pi)w_\theta(\theta), w_\theta{}^{-1})$ are holomorphic at $\nu = \nu_0$.

Now, as we mentioned above, these local coefficients appear in the functional equations satisfied by the previously mentioned L-functions. More precisely, let a_i and r_i, $1 \le i \le m$, be as in Section 2.3. Fix a cuspidal representation π of $M_{\theta,\mathbf{A}}$. Assume that G splits over $\mathbf{Q}$. Write $\pi = \otimes_v \pi_v$. Let χ be a character of $U_{\mathbf{A}}$ with $\chi = \otimes_v \chi_v$. Finally denote by S the finite set of places such that $v \notin S$ implies that both π_v and χ_v are unramified. Given i, $1 \le i \le m$, let

$$L_S(s, \pi, r_i) = \prod_{v \notin S} L(s, \pi_v, r_i).$$

Then Theorem 4.1 establishes the functional equation

$$\prod_{i=1}^{m} L_S(a_i s, \pi, r_i) = \prod_{v \in S} C_{\chi_v}(-2s\rho_\theta, \pi_v, \theta, w_\theta) \cdot \prod_{i=1}^{m} L_S(1 - a_i s, \pi, \tilde{r}_i).$$

Finally comparing this equation with the functional equations conjectured by Langlands [16], we conclude that each $C_{\chi_v}(-2s\rho_\theta, \pi_v, \theta, w_\theta)$ must be of the form

$$\prod_{i=1}^{m} \epsilon(a_i s, \pi_v, r_i, \chi_v) \frac{L(1 - a_i s, \pi_v, \tilde{r}_i)}{L(a_i s, \pi_v, r_i)}, \tag{1}$$

where the factors involved are the conjecturally defined Langlands root numbers and L-functions. Consequently, the irreducibility of $I(-2s\rho_\theta, \pi, \theta)$ is explicitly related to the poles of the L-functions $L(1 - a_i s, \pi, \tilde{r}_i)$, $1 \le i \le m$.

When $G = GL_{n+m}$ and $M_\theta = GL_n \times GL_m$, the L-functions are $L(s, \pi_n \times \pi_m)$ defined by H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika (cf. [10] and Section 2.3 here).

The most important tool used in the definition of these local coefficients is the theory of intertwining integrals. In the case of real groups they are relatively well studied (cf. [7, 14, 22]). When F is non-archimedean the only general theory is due to Harish-Chandra [8] (cf. A. Silberger's notes [28]). We have adopted his approach, and we have extensively used Silberger's notes on this subject. This is done in Sections 2-2.4, and in fact Sections 2.2 and 2.4 are entirely due to Harish-Chandra (cf. [28]). Section 2.1 provides the factorization of the intertwining operators. Finally in Section 2.3 we have specified the class of L-functions in which we are interested.

The theory of local coefficients is developed in Section 3. A normalization of intertwining operators using the local coefficients is provided in Section 3.1. Section 3.2 is devoted to the computation of these local coefficients. In fact Proposition 3.2.1 reduces their computation to the rank one cases, and Theorem 3.2.2 computes them for real groups. A particularly interesting result of this section is Theorem 3.2.1 which provides the factorization of Jacquet-Shalika local coefficients (granting their equality with the corresponding local coefficients). It is interesting to know that their factorization is equivalent to that of intertwining operators. Finally we should mention that the computation of the local coefficients is effectively equivalent to the computation of Plancherel measures, and therefore in general by no means an easy question.

Functional equations are proved in Section 4. They appear as Theorems 4.1, 4.2, and 4.3. As an application we have used Deligne's idea (cf. [4]) to prove the functional equations satisfied by $L(s, \pi, \mathrm{Sym}^4(\rho_2))$ and $L(s, \pi, \mathrm{Sym}^5(\rho_2))$, where π is a non-monomial cuspidal representation of $PGL_2(\mathbf{A})$, and $\mathrm{Sym}^4(\rho_2)$ and $\mathrm{Sym}^5(\rho_2)$ are respectively five and six dimensional irreducible representations of $SL_2(\mathbf{C})$. The latter case seems to be new; the former is originally due to P. Deligne [4].

With regard to the proofs in Section 4, we have mostly referred to [23]. In fact the results provided in [24] make it now possible to carry the proofs to the general case.

Aside from the functional equations, the most interesting number theoretic result of this paper is Theorem 5.1. It provides a non-vanishing theorem for a product of L-functions along the line $\mathrm{Re}(s) = 1$. The case of $L(s, \pi_n \times \pi_m)$ which has already been announced [25] (Theorem 5.2 here) has a beautiful application to the classification of automorphic forms on GL_n (due to H. Jacquet and J. A. Shalika). The case $m = 1$ is originally due to H. Jacquet and J. A. Shalika [12]. A similar result (with some reservations at $s = 1$) is proved here for $L(s, \pi, \mathrm{Sym}^3(\rho_2))$ and $L(s, \pi, \mathrm{Sym}^4(\rho_2))$ (Theorem 5.3). Finally, as we have remarked, one can generalize Theorem 5.1 to conclude that for every quasi-split group, there is a product of L-functions which does not vanish at $s = 1$, and therefore these non-vanishing theorems are only special cases of a more general theory. Whether this can be used to prove some density theorems remains to be seen.

I would like to thank Joseph Shalika for many helpful discussions and communications. My thanks are also due to the referee for his suggestions, especially on Theorem 5.3. In fact the proof of $L_S(1, \pi, \mathrm{Sym}^3(\rho_2)) \neq 0$ is due to him.

1. Preliminary notations. There are two different sets of notations that we shall be using throughout this paper.

1.1. In Sections 2-2.4, we let F be a non-archimedean local field, and we let G be a connected reductive algebraic group defined over F. Since most of the time we shall be dealing with the representations of the group of F-rational points of G, we allow ourselves to use the same notation for this latter group. We hope that no confusion will arise.

We fix a maximal split torus A_0 of G over F, and we let P_0 be a fixed minimal parabolic subgroup of G which has A_0 as its split component. Let U be the unipotent radical of P_0.

Let ψ be the set of roots of G with respect to A_0. A root $\alpha \in \psi$ is *reduced* if $\frac{1}{2}\cdot\alpha \notin \psi$. We fix a set of simple roots Δ in ψ (corresponding to P_0), and we let ψ^+ and ψ^- be the sets of positive and negative roots with respect to Δ, respectively. Given α, let U^α be the corresponding root group. We denote by W the Weyl group of G with respect to A_0. Finally we let K be an A_0-good maximal compact subgroup of G (cf. Section 0.6 of [28]).

It is only in Section 2.3 that the group G is more restricted. The restriction is explained there.

In Section 3 the group G is either as above or a quasi-split real group.

1.2. Throughout Sections 4 and 5, the group G is a reductive algebraic group which splits over $\mathbf{Q}$. Then A_0 and P_0 are respectively, a Cartan and a Borel subgroup of G. Occasionally we may denote them by T and B, respectively. As in [15] we also define a group ${}^\circ G$. More precisely if $P = MN$ is a parabolic subgroup of G with $M \supset T$, we let ${}^\circ G = M/A$, where A is the center of M.

Every other symbol has been explained during the text of the paper.

1.3. Fine representations. (cf. [24]). Suppose $F = \mathbf{R}$ and let π be an irreducible admissible Fréchet representation of G. Then by Casselman's subrepresentation theorem there exists a principal series representation I of G such that $\delta: \pi \hookrightarrow I$ infinitesimally. We say π is *fine* if every embedding δ is bicontinuous with respect to the Schwartz topologies of π_∞ and I_∞. When π is unitary it would be enough that δ be continuous. In fact the continuity of the other map follows from the continuity of the intertwining operators between the principal series representations of real groups which is a result of W. Casselman.

It is a result of W. Casselman and N. Wallach that every unitary representation of $GL_n(\mathbf{R})$ or any complex group is fine.

2. Intertwining operators and L-functions. In this section we shall assume that G is as in Section 1.1, i.e. it is a connected reductive algebraic group defined over a p-adic field F. By abuse of notation we shall also use the same notation for its subgroup of F-rational points. This should not cause any major confusion, as most of the time we shall be dealing with the representations and therefore G would stand for the group of F-rational points.

Given a subset θ of Δ, let Σ_θ be the subset of roots in the linear span of θ. Let $\Sigma_\theta{}^+ = \psi^+ \cap \Sigma_\theta$ and $\Sigma_\theta{}^- = \psi^- \cap \Sigma_\theta$. Also let $A_\theta = \bigcap_{\alpha \in \theta} \operatorname{Ker} \theta$. Denote by M_θ the centralizer of A_θ in G and set

$$N_\theta = \prod_{\alpha \in \psi^+ - \Sigma_\theta{}^+} U^\alpha$$

Finally, let $P_\theta = M_\theta N_\theta$. Clearly $P_\theta \supset P_0$. Then P_θ is called the standard parabolic subgroup corresponding to θ. Furthermore $A_\theta \subset A_0 \subset M_\theta$. Observe that $P_0 = P_\phi$, $G = P_\Delta$, and $A_0 = A_\phi$.

Let $\Sigma(\theta)$ be the set of all the roots of (P_θ, A_θ), and let $\mathfrak{a}_\theta$ be the real Lie algebra of A_θ. We say $\alpha \in \Sigma(\theta)$ is positive if there exists $\beta \in \Psi^+ - \Sigma_\theta{}^+$ such that $\beta \mid \mathfrak{a}_\theta = \alpha$, and we denote by $\Sigma^+(\theta)$ and $\Sigma^-(\theta)$ the corresponding subsets of positive and negative roots in $\Sigma(\theta)$, respectively. From now on, we shall identify every root α of (P_θ, A_θ) with all those roots $\beta \in \Psi^+ - \Sigma_\theta{}^+$ for which $\beta \mid \mathfrak{a}_\theta = \alpha$.

Given a simple root $\alpha \in \Sigma(\theta)$, let

$$A_\alpha = \bigcap_{\alpha' \in \Delta - \{\alpha\}} \operatorname{Ker} \alpha'.$$

Then A_α is a subgroup of A_θ.

Finally, set

$$\rho_\theta = \frac{1}{2} \sum_{\alpha \in \Psi^+ - \Sigma_\theta^+} \alpha$$

Now, let $(\mathfrak{a}_\theta)^*_{\mathbf{C}}$ be the complex dual of $\mathfrak{a}_\theta$. More precisely, $(\mathfrak{a}_\theta)^*_{\mathbf{C}}$ is the complexification of

$$\mathfrak{a}_\theta^* = X(A_\theta) \otimes \mathbf{R} = X(M_\theta) \otimes \mathbf{R},$$

where $X(A_\theta)$ and $X(M_\theta)$ respectively denote the group of all rational characters of A_θ and M_θ which are defined over F (cf. [8]).

Let H_θ be the homomorphism H_{M_θ} from M_θ into $\mathrm{Hom}(X(M_\theta), \mathbf{Z})$ defined in Section 7 of [8], and let q be the number of elements in the residue class field of F.

Now, let $(\pi, H(\pi))$ be an irreducible unitary representation of M_θ. Let $H(\pi)_K$ be the subspace of K finite vectors. Fix $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$ and set

$$I(\nu, \pi, \theta) = \operatorname*{Ind}_{P_\theta \uparrow G} (\pi \otimes q^{\langle \nu, H_\theta(\ )\rangle})$$

Then the space $V(\nu, \pi, \theta)$ of $I(\nu, \pi, \theta)$ consists of all the smooth functions f from G into $H(\pi)_K$ satisfying

$$f(gnm) = \pi(m^{-1}) q^{\langle -\nu - \rho_\theta, H_\theta(m)\rangle} f(g) \; (m \in M_\theta, n \in N_\theta).$$

The representation $I(\nu, \pi, \theta)$ acts by left inverse translations.

Let us now establish a duality between $V(\nu, \pi, \theta)$ and $V(-\bar{\nu}, \pi, \theta)$.

Given $f \in V(\nu, \pi, \theta)$ and $f' \in V(-\bar{\nu}, \pi, \theta)$, we have

$$(f(gnm), f'(gnm)) = q^{\langle -2\rho_\theta, H_\theta(m)\rangle} (f(g), f'(g)),$$

where (,) is the inner product on $H(\pi)$. Therefore

$$(f(g), f'(g)) \in V(\rho_\theta, 1, \theta).$$

Now, given $h \in V(\rho_\theta, 1, \theta)$, choose $\phi \in C_c^\infty(G)$ such that

$$h(g) = \int_{M_\theta \times N_\theta} \phi(gmn) q^{\langle 2\rho_\theta, H(m)\rangle} \, dmdn.$$

Then there exists (cf. [22]) a relatively bounded linear form μ such that

$$\mu(h) = \int_G \phi(g)dg,$$

where dg is a fixed Haar measure on G. In this section, we shall use

$$\mu(h) = \oint_G h(g)d\mu(g)$$

to denote the above linear form. Then

$$\langle f, f' \rangle = \oint_G (f(g), f'(g))d\mu(g)$$

defines a duality between $V(\nu, \pi, \theta)$ and $V(-\bar{\nu}, \pi, \theta)$.

Now, let θ and θ' be two subsets of Δ and let

$$W(\theta, \theta') = \{w \in W | w(\theta) = \theta'\}.$$

We say θ and θ' are associate if $W(\theta, \theta')$ is not empty. If $w(\theta) = \theta'$, we shall say that θ and θ' are associate by w. As in [2], let $\{\theta\}$ be the set of subsets associate to θ.

We shall now start our study of intertwining integrals.

Fix two associated subsets θ and θ' of Δ, and let w be in $W(\theta, \theta')$. Let N_θ^- be the unipotent group generated by U^α, $\alpha \in \psi^- - \Sigma_\theta^-$, and set

$$N_w = U \cap wN_\theta^- w^{-1}.$$

Given $f \in V(\nu, \pi, \theta)$, define

$$A(\nu, \pi, w) f(g) = \int_{N_w} f(gnw)dn. \tag{2.1}$$

The convergence of this integral will be studied in a moment, but for a moment suppose that ν is so that the integral converges absolutely.

For $m \in M_{\theta'}$, define a representation of $M_{\theta'}$ by

$$w(\pi)(m) = \pi(w^{-1}mw).$$

Then $A(\nu, \pi, w)f \in V(w(\nu), w(\pi), \theta')$.

Now, let $|| \ ||$ be the Hilbert space norm on $H(\pi)$. Given $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$, define $f_\nu \in V(\nu, 1, \theta)$ by

$$f_\nu(kmn) = q^{\langle -\nu-\rho_\theta, H_\theta(m)\rangle}$$

for $k \in K$, $m \in M_\theta$, and $n \in N_\theta$. Now, given $f \in V(\nu, \pi, \theta)$ and $g = kmn$ in G, we have

$$||f(kmn)|| = |f_\nu(kmn)| \cdot ||f(k)||.$$

Set

$$\nu_K(f) = \sup_{k \in K} ||f(k)||.$$

Then

$$||f(g)|| \leq \nu_K(f)|f_\nu(g)|.$$

Consequently if ν is so that

$$\int_{N_w} |f_\nu(nw)|dn < +\infty, \tag{2.2}$$

then (2.1) would converge absolutely.

Now, fix a W-invariant inner product (,) on $\mathfrak{a}_\phi$ and restrict it to $\mathfrak{a}_\theta$. Given $\alpha \in \Sigma(\theta)$, let $\mathfrak{a}_\alpha$ be the orthogonal complement of Ker(α) in $\mathfrak{a}_\theta$ with respect to (,). Let $H_\alpha \in \mathfrak{a}_\alpha$ be a generator for the lattice $H_\theta(A_\theta) \cap \mathfrak{a}_\alpha$ such that $\langle \alpha, H_\alpha \rangle > 0$. Then H_α is uniquely defined. It is well known that (2.2) holds if

$$\langle \mathrm{Re}(\nu), H_\alpha \rangle \ll 0 \tag{2.3}$$

for every reduced root $\alpha \in \Sigma^+(\theta)$ for which $w(\alpha) \in \psi^-$.

Finally let ${}^{\circ}A_\theta = \mathrm{Ker}(H_\theta) \cap A_\theta$, and denote by h_α the unique coset of ${}^{\circ}A_\theta$ in A_θ which, under H_θ, is mapped to H_α. Now, let χ be a quasi-character of A_θ. Then we define χ at h_α to be $\chi(h_\alpha) = \chi(a)$ with $a \in h_\alpha$ if $\chi \mid {}^{\circ}A_\theta \equiv 1$; and $\chi(h_\alpha) = 0$, otherwise. Then clearly $\chi(h_\alpha)$ is well-defined.

2.1. Factorization of intertwining operators. Suppose ν satisfies (2.3) and put

$$\overline{N_w} = w^{-1} N_w w.$$

Then by a suitable normalization of Haar measure on $\overline{N_w}$, (2.1) can be written as

$$A(\nu, \pi, w) f(g) = \int_{\overline{N_w}} f(gw\bar{n}) d\bar{n}. \tag{2.1.1}$$

Observe that

$$\overline{N_w} = N_\theta^- \cap w^{-1} N_{\theta'} w.$$

Now, let $\bar{\mathfrak{n}}_w$ be the Lie algebra of $\overline{N_w}$. Then

$$\bar{\mathfrak{n}}_w = \oplus\ \mathfrak{g}_{-\alpha},$$

where the sum runs over all $\alpha \in \psi^+ - \Sigma_\theta^+$ with $w(\alpha) \in \psi^-$. Here $w(\alpha)$ (as well as $w(\nu)$) is defined by:

$$\langle w(\alpha), H \rangle = \langle \alpha, \mathrm{Ad}(w^{-1}) H \rangle \qquad \forall H \in \mathfrak{a}_\phi.$$

In this section we shall use the same notation as in [2] and for this reason let us recall some notation from [2].

Given $w \in W$, let

$$\Sigma_w = \{\alpha \in \Psi^+ \mid w(\alpha) \in \Psi^-\}.$$

Also, given $\theta \subseteq \Delta$ and $\alpha \in \Delta - \theta$, let $\Omega = \theta \cup \{\alpha\}$.

Define $\bar{\theta}$ by

$$\bar{\theta} = w_{l,\Omega} w_{l,\theta}(\theta) \subseteq \Omega.$$

Here $w_{l,\Omega}$ and $w_{l,\theta}$ denote the longest elements in the Weyl groups of M_Ω and M_θ, respectively. $\bar{\theta}$ is called the conjugate of θ in Ω.

We need the following lemma.

LEMMA 2.1.1. *Suppose $\theta, \theta' \subseteq \Delta$ are associate. Take $w \in W(\theta, \theta')$. Suppose β_1 and β_2 in $\psi^+ - \Sigma_\theta^+$ have the same restriction to $\mathfrak{a}_\theta$,* i.e. *they represent the same root of (P_θ, A_θ). Then $w(\beta_1)$ and $w(\beta_2)$ will have the same restrictions to $\mathfrak{a}_{\theta'}$. In particular $w(\beta_1)$ and $w(\beta_2)$ represent the same root of $(P_{\theta'}, A_{\theta'})$.*

Proof. Let $\theta = \{\alpha_1, \ldots, \alpha_m\}$. We may assume

$$\beta_1 = a_0\alpha_0 + \sum_{i=1}^{m} a_i\alpha_i$$

and

$$\beta_2 = a_0\alpha_0 + \sum_{i=1}^{m} b_i\alpha_i,$$

where $\alpha_0 \in \Delta - \theta$ and $a_0, a_1, \ldots, a_m, b_1, \ldots, b_m$ are nonnegative integers. Clearly $\alpha_i | \mathfrak{a}_\theta = 0$ for $i = 1, \ldots, m$. Now

$$w(\beta_1) = a_0w(\alpha_0) + \sum_{i=1}^{m} a_iw(\alpha_i)$$

and

$$w(\beta_2) = a_0w(\alpha_0) + \sum_{i=1}^{m} b_iw(\alpha_i),$$

and $w(\alpha_i) | \mathfrak{a}_{\theta'} = 0$ for $i = 1, \ldots, m$ which proves the lemma.

Let θ, θ', and w be as in Lemma 2.1.1. Given $\alpha \in \Psi - \Sigma_\theta$, let $[\alpha]$ be the subset of all roots in $\Psi - \Sigma_\theta$ which represent the same root of $\mathfrak{a}_\theta$. Then by Lemma 2.1.1, $[w(\alpha)]$ will represent a root of $\mathfrak{a}_{\theta'}$. Let $S \subseteq \Psi$; we say $[\alpha] \in S$ if and only if $\beta \in S$ for all $\beta \in [\alpha]$.

The following lemma is crucial to this section. It is basically due to Langlands (Lemma 2.13 of [17]). Here we give an algebraic proof.

LEMMA 2.1.2. *Suppose $\theta, \theta' \subset \Delta$ are associate. Take $w \in W(\theta, \theta')$. Then, there exists a family of subsets $\theta_1, \theta_2, \ldots, \theta_n \subseteq \Delta$ such that*

a) *$\theta_1 = \theta$ and $\theta_n = \theta'$;*
b) *fix $1 \le i \le n - 1$; then there exists a root $\alpha_i \in \Delta - \theta_i$ such that θ_{i+1} is the conjugate of θ_i in $\Omega_i = \theta_i \cup \{\alpha_i\}$;*
c) *set $w_i = w_{l,\Omega_i} w_{l,\theta_i}$ in $W(\theta_i, \theta_{i+1})$ for $1 \le i \le n - 1$, then $w = w_{n-1} \cdots w_1$;*
d) *if one sets $w_1' = w$ and $w_{i+1}' = w_i' w_i^{-1}$ for $1 \le i \le n - 1$, then $w_n' = 1$ and*

$$\overline{\mathfrak{n}}_{w_i'} = \overline{\mathfrak{n}}_{w_i} \oplus Ad(w_i^{-1}) \overline{\mathfrak{n}}_{w_{i+1}'}. \tag{2.1.2}$$

Proof of Lemma 2.1.2. Suppose first that there is no $\alpha \in \Delta - \theta$ such that $w(\alpha) \in \Psi^-$. Then since $w(\theta) = \theta'$, we conclude that w sends every positive root to a positive root and therefore $w = 1$, i.e. $\theta = \theta'$. Consequently we may assume that there exists a simple root $\alpha_1 \in \Delta - \theta_1$ such that $w(\alpha_1) \in \Psi^-$. Let θ_2 be the conjugate of θ_1 in $\Omega_1 = \theta_1 \cup \{\alpha_1\}$, and set $w_1 = w_{l,\Omega_1} w_{l,\theta_1}$. For the sake of induction let us let $w_1' = w$. Then $w_1^{-1} \in W(\theta_2, \theta_1)$ and $w_2' = w_1' \cdot w_1^{-1}$ belongs to $W(\theta_2, \theta')$.

Let

$$S_2 = \{[\beta] \,|\, \beta \in \Psi^+ - \Sigma_{\theta_2}^+, w_2'(\beta) \in \Psi^-\}.$$

and

$$S_1 = \{[\alpha] \,|\, \alpha \in \Psi^+ - \Sigma_{\theta_1}^+, w_1'(\alpha) \in \Psi^-\}.$$

Given $\beta \in [\beta] \in S_2$, let $\alpha = w_1^{-1}(\beta)$. Then clearly $w_1'(\alpha) = w_2'(\beta)$ and thus $w_1'(\alpha) \in \Psi^-$. Also $\alpha \in \Psi^+ - \Sigma_{\theta_1}^+$. In fact for $\beta \notin \Sigma_{\Omega_1}^+ - \Sigma_{\theta_2}^+$, it follows from

$$\Sigma_{w_1^{-1}} = \Sigma_{\Omega_1}^+ - \Sigma_{\theta_2}^+$$

(Lemma 1.1.7 of [2]) that $w_1^{-1}(\beta)$ is positive and clearly in $\Psi^+ - \Sigma_{\theta_1}^+$. Now suppose $\beta \in \Sigma_{\Omega_1}^+ - \Sigma_{\theta_2}^+$. Then

$$\beta = -a_0 w_1(\alpha_1) + \sum_{i=1}^{m} b_i \beta_i,$$

where $a_0, b_1, \ldots, b_m$ are non-negative integers, $a_0 \neq 0$, and $\theta_2 = \{\beta_1, \ldots, \beta_m\}$. But then

$$w_2'(\beta) = -a_0 w_1'(\alpha_1) + \sum_{i=1}^{m} b_i w_2'(\beta_i)$$

is in Ψ^+ and thus $\beta \notin S_2$. Hence $[\alpha] = [w_1^{-1}(\beta)]$ is in S_1.

Now, we claim that there is no $[\beta] \in S_2$ such that $[\alpha] = [w_1^{-1}(\beta)]$ for every $\alpha \in \Sigma_{\Omega_1}{}^+ - \Sigma_{\theta_1}{}^+$. Suppose false, then there exists $\beta \in [\beta] \in S_2$ such that $w_1^{-1}(\beta) = \alpha$. But, then $\beta = w_1(\alpha)$ is in Ψ^-, a contradiction.

Finally, take $[\alpha] \in S_1$, $\alpha \notin \Sigma_{\Omega_1}{}^+ - \Sigma_{\theta_1}{}^+$. Let $\beta = w_1(\alpha)$. Then $\beta \in \Psi^+ - \Sigma_{\theta_2}{}^+$ since $\Sigma_{w_1} = \Sigma_{\Omega_1}{}^+ - \Sigma_{\theta_1}{}^+$. Also

$$w_2'(\beta) = w_1'(\alpha) \in \Psi^-,$$

and hence $[\beta] \in S_2$.

Therefore, the correspondence

$$[\beta] \mapsto [w_1^{-1}(\beta)]$$

defines a proper bijection from S_2 onto a certain subset of S_1 and consequently $\mathrm{Card}(S_2) < \mathrm{Card}(S_1)$.

The relation

$$\bar{\mathfrak{n}}_{w_1'} = \bar{\mathfrak{n}}_{w_1} \oplus \mathrm{Ad}(w_1^{-1})\bar{\mathfrak{n}}_{w_2'}$$

is now a simple consequence of this correspondence. In fact

$$\bar{\mathfrak{n}}_{w_1} = \oplus\, \mathfrak{g}_{-\alpha} \qquad \alpha \in \Sigma_{w_1}$$

and

$$\bar{\mathfrak{n}}_{w_2'} = \oplus\, \mathfrak{g}_{-\beta},$$

where the sum runs over $\bigcup_{[\beta]\in S_2} [\beta]$. But then

$$\mathrm{Ad}(w_1^{-1})\bar{\mathfrak{n}}_{w_2'} = \bigoplus_{\beta} \mathfrak{g}_{-w_1^{-1}(\beta)} \qquad \forall \beta \in \bigcup_{[\beta]\in S_2} [\beta]$$

$$= \bigoplus_{\alpha} \mathfrak{g}_{-\alpha} \qquad \forall \alpha \in \bigcup_{[\alpha]\in S_1} [\alpha] - \Sigma_{w_1}.$$

Now the lemma follows by induction on $\mathrm{Card}(S_i)$, $1 \le i \le n-1$, and finally there exists a positive integer n such that there is no $\alpha \in \Delta - \theta_n$

with $w_n'(\alpha) \in \Psi^-$, where $w_n' = w'_{n-1}w_{n-1}^{-1}$. Then $w_n' = 1$ and $w = w_{n-1} \cdots w_1$.

Let $\Sigma_r^+(\theta)$ be the set of all the reduced roots in $\Sigma^+(\theta)$. Let

$$\Sigma_r(\theta, \theta', w) = \{[\beta] \in \Sigma_r^+(\theta) \mid \beta \in \psi^+ - \Sigma_\theta^+, w(\beta) \in \psi^-\}.$$

Then the following corollary is a simple consequence of the proof of Lemma 2.1.2 (observe the similarity with Proposition 1.2 of [22]).

COROLLARY. *Suppose $\theta, \theta' \subset \Delta$ are associate and fix $w \in W(\theta, \theta')$. Write $w = w_{n-1} \cdots w_1$ as in Lemma 2.1.2, and let $\alpha_1, \ldots, \alpha_n \in \Delta$ be the corresponding simple roots. Realize each α_i as an element $[\alpha_i] \in \Sigma_r^+(\theta_i)$, $1 \leq i \leq n-1$. Then $[\beta_i] = w_1^{-1} \cdots w_{i-1}^{-1}([\alpha_i])$, $1 \leq i \leq n-1$, are all distinct elements of $\Sigma_r(\theta, \theta', w)$. Furthermore, given $[\beta] \in \Sigma_r(\theta, \theta', w)$, there exists an i, $1 \leq i \leq n-1$, such that $[\beta] = w_1^{-1} \cdots w_{i-1}^{-1}([\alpha_i])$.*

PROPOSITION 2.1.1. *For each i, $1 \leq i \leq n-1$, $\overline{N}_{w_i'}$ is semi-direct product of $\overline{N}_{w_i}$ and $w_i^{-1}\overline{N}_{w_{i'+1}}w_i$ as a P-adic Lie group.*

Proof. We first show that

$$[\overline{\mathfrak{n}}_{w_i}, \operatorname{Ad}(w_i^{-1})\overline{\mathfrak{n}}_{w_{i'+1}}] \subset \operatorname{Ad}(w_i^{-1})\overline{\mathfrak{n}}_{w_{i'+1}}.$$

Otherwise, take $\alpha, \beta \in \Psi^+$ such that $\mathfrak{g}_{-\alpha} \subset \overline{\mathfrak{n}}_{w_i}$ and $\mathfrak{g}_{-\beta} \subset \overline{\mathfrak{n}}_{w_{i'+1}}$. Since

$$\overline{\mathfrak{n}}_{w_i'} = \overline{\mathfrak{n}}_{w_i} \oplus \operatorname{Ad}(w_i^{-1})\overline{\mathfrak{n}}_{w_{i'+1}}$$

it is clear that if $\gamma = \alpha + w_i^{-1}(\beta)$ is a root, we must have

$$\mathfrak{g}_{-\gamma} \subset \overline{\mathfrak{n}}_{w_i}.$$

Then $w_i^{-1}(\beta) = \gamma - \alpha \in \Sigma_{\Omega_i}$ and therefore $\beta \in \Sigma_{\Omega_i}$. But this is a contradiction. Consequently, $\overline{N}_{w_i'}$ is semi-direct product of $\overline{N}_{w_i}$ and $w_i^{-1}\overline{N}_{w_{i'+1}}w_i$ as abstract groups. The fact that it is a semi-direct product as Lie groups follows from (2.1.2).

Now, for each i, $1 \leq i \leq n-1$, by Proposition 2.1.1 we can normalize Haar measures on $\overline{N}_{w_i'}$, $\overline{N}_{w_i}$, and $\overline{N}_{w_{i'+1}}$ so that

$$\int_{\overline{N}_{w_i'}} f_i(\overline{n}_i)d\overline{n}_i' = \int_{\overline{N}_{w_{i'+1}} \times \overline{N}_{w_i}} f_i(w_i^{-1}\overline{n}'_{i+1}w_i\overline{n}_i)d\overline{n}'_{i+1}d\overline{n}_i \tag{2.1.3}$$

for $f_i \in I(\nu_i, \pi_i, \theta_i)$, where $\nu_i = w_{i-1}(\nu_{i-1})$, $\pi_i = w_{i-1}(\pi_{i-1})$, $\nu_1 = \nu$, and $\pi_1 = \pi$. The following theorem is important in the study of intertwining operators, as well as factorization of local coefficients which will be done later.

THEOREM 2.1.1. *Fix $\theta, \theta' \subseteq \Delta$ and choose $w \in W(\theta, \theta')$. Let $\theta_1, \ldots, \theta_n \subseteq \Delta$, and $w_i \in W(\theta_i, \theta_{i+1})$ be as in Lemma 2.1.2. Take $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$ satisfying* (2.3). *Then each ν_i satisfies* (2.3) *with respect to A_{θ_i} and*

$$A(\nu, \pi, w) = A(\nu_{n-1}, \pi_{n-1}, w_{n-1}) \cdot \cdots \cdot A(\nu_1, \pi_1, w_1), \quad (2.1.4)$$

where $\nu_i = w_{i-1}(\nu_{i-1})$, $\pi_i = w_{i-1}(\pi_{i-1})$, $2 \leq i \leq n-1$, $\nu_1 = \nu$, and $\pi_1 = \pi$.

Proof. For i, $2 \leq i \leq n-1$, we need to check

$$\langle \mathrm{Re}(\nu_i), H_\alpha \rangle \ll 0$$

for every reduced root $\alpha \in \Sigma^+(\theta_i)$ for which $w_i(\alpha) \in \psi^-$. We do this by induction. Suppose

$$\langle \mathrm{Re}(\nu_{i-1}), H_\alpha \rangle \ll 0$$

for every reduced root $\alpha \in \Sigma^+(\theta_{i-1})$ for which $w_{i-1}(\alpha) \in \Psi^-$. By (2.1.2), every reduced root $\alpha \in \Sigma^+(\theta_i)$ satisfying $w_i(\alpha) \in \Psi^-$ is of the form $w_{i-1}(\beta)$ for some $\beta \in \Sigma^+(\theta_{i-1})$ satisfying $w'_{i-1}(\beta) \in \Psi^-$. But

$$\begin{aligned} \langle \mathrm{Re}(\nu_i), H_\alpha \rangle &= \langle \mathrm{Re}(w_{i-1}(\nu_{i-1})), H_\alpha \rangle \\ &= \langle \mathrm{Re}(\nu_{i-1}), \mathrm{Ad}(w_{i-1}^{-1}) H_\alpha \rangle \\ &= \langle \mathrm{Re}(\nu_{i-1}), H_\beta \rangle, \end{aligned}$$

since $w_{i-1}^{-1}(\alpha) = \beta$. But this last term is sufficiently small by the induction hypothesis. This proves the first statement of the theorem. Now (2.1.4) is an inductive application of (2.1.3).

2.2. Analytic continuation of intertwining operators. In this section, we shall study the analytic properties of intertwining operators. It is based entirely on Harish-Chandra's results on p-adic groups (cf. [8, 28]) and follows the same techniques as in the case of real groups [7].

We shall first explain some of the notation which we need from [8, 28].

Let τ be a smooth and unitary double representation of K on a Hilbert space V. Given an admissible representation π of G, let $\mathcal{A}(\pi)$ denote the complex space generated by all the matrix coefficients of π. Now let $\mathcal{A}(G) = \cup_\pi \mathcal{A}(\pi)$, where the union runs over all the admissible representations of G. Denote by $C(G, \tau)$ the space of all the functions $f: G \to V$ such that

$$f(k_1 x k_2) = \tau(k_1) f(x) \tau(k_2),$$

where k_1 and k_2 are in K and $x \in G$. Let

$$\mathcal{A}(G, \tau) = (\mathcal{A}(G) \otimes V) \cap C(G, \tau)$$

and

$$\mathcal{A}(\pi, \tau) = (\mathcal{A}(\pi) \otimes V) \cap C(G, \tau).$$

Now, let P be a parabolic subgroup of G and fix a split component A of P. Write $P = MN$ for its Levi decomposition and denote by P^- the parabolic subgroup of G opposed to P.

Denote by $\mathcal{P}(A)$ the set of all the parabolic subgroups of G which have A as their split component. Then $P, P^- \in \mathcal{P}(A)$. Given P_1 and P_2 in $\mathcal{P}(A)$, let $V(P_1 | P_2)$ be the subspace of all $v \in V$ such that

$$\tau(n_1) v \tau(n_2) = v \qquad (n_i \in N_i \cap K, i = 1, 2),$$

where N_1 and N_2 denote the unipotent radicals of P_1 and P_2, respectively. Let τ_M denote the restriction of τ on $K_M = K \cap M$. Then $V(P_1 | P_2)$ is stable under τ_M. Let $\tau_{P_1 | P_2}$ denote the restriction of τ_M on $V(P_1 | P_2)$.

Now, let (π, U) be an admissible representation of M. Replacing G by M, we let

$$L(\pi, P) = \mathcal{A}(\pi, \tau_{P|P})$$

and

$$\mathcal{L}(\pi, P) = \mathcal{A}(\pi, \tau_{P|P^-}).$$

Given $\psi \in L(\pi, P)$, we extend ψ to G by

$$\psi(kmn) = \tau(k)\psi(m) \qquad (k \in K,\ m \in M,\ n \in N);$$

we let H_P be the extension of H_M to G defined by

$$H_P(kmn) = H_M(m) \qquad (k \in K,\ m \in M,\ n \in N).$$

Now, given $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ (complex dual of the real Lie algebra of A), we define the corresponding Eisenstein integral by

$$E(P; \psi: \nu: x) = \int_K \psi(xk)\tau(k^{-1})q^{\langle \nu-\rho, H_P(xk)\rangle}\, dk \qquad (x \in G),$$

where ρ is the half of the sum of the positive roots of P. We use $W(A)$ to denote the Weyl group of A. Now, given $P_1, P_2 \in \mathcal{P}(A)$ and $w \in W(A)$, we let

$$c_{P_2|P_1}(w, \pi, \nu): L(\pi, P_1) \to \mathcal{L}(w(\pi), P_2)$$

be the Harish-Chandra's c-function introduced in [8].

The mapping $T \to \kappa_T$. As before, let (π, U) be an admissible representation of M. Denote by $(\tilde{\pi}, \tilde{U})$ its contragredient. Put $\pi_o = \pi | K_M$, and $\tilde{\pi}_o = \tilde{\pi} | K_M$, and set $\Pi_o = \mathrm{Ind}_{K_M|K}\, \pi_o$ and $\tilde{\Pi}_o = \mathrm{Ind}_{K_M|K}\, \tilde{\pi}_o$. Then

$$\mathcal{H} = \{h \in C^\infty(K:U) \,|\, h(km) = \pi_o(m^{-1})h(k),\ k \in K,\ m \in K_M\}$$

is the space of Π_o. We define $\tilde{\mathcal{H}}$ for $\tilde{\Pi}_o$ the same way. Then by restriction on K, every space $V(\nu, \pi, \theta)$, $\theta \subset \Delta$, may be realized as a subspace of $\mathcal{H}$ (cf. Section 5.2.1 of [28]). Let F_θ be the projection of $\mathcal{H}$ onto $V(\nu, \pi, \theta)$ explained in Section 5.2.1 of [28].

Let $\mathfrak{I} = \mathrm{End}^o(\mathcal{H})$ be the space of smooth endomorphisms of $\mathcal{H}$. More precisely, the space of those $T \in \mathrm{End}(\mathcal{H})$ for which the functions $x \to \Pi_o(x)T$ and $x \to T\Pi_o(x^{-1})$, $\forall x \in K$, are smooth (cf. Section 1.11 of [28]).

Now, to each $T \in \mathfrak{I}$ we associate a smooth function

$$\kappa_T: K \times K \to \mathrm{End}^o(U)$$

as follows.

As in Section 1.11 of [28] we first realize $\mathrm{End}^{\circ}(\mathcal{H})$ with $\mathcal{H} \otimes \tilde{\mathcal{H}}$ by means of the isomorphism i: $\mathcal{H} \otimes \tilde{\mathcal{H}} \cong \mathrm{End}^{\circ}(\mathcal{H})$ which is defined by $i(v \otimes \tilde{v})(u) = \tilde{v}(u)v$ for v, $u \in \mathcal{H}$ and $\tilde{v} \in \tilde{\mathcal{H}}$. Now, given $T \in \mathfrak{I}$, write

$$T = \sum_{i=1}^{r} h_i \otimes \tilde{h}_i \in \mathcal{H} \otimes \tilde{\mathcal{H}}.$$

Then

$$\mathrm{Th} = \sum_{i=1}^{r} (h_i \otimes \tilde{h}_i)(h)$$

$$= \sum_{i=1}^{r} \tilde{h}_i(h) h_i$$

for $h \in \mathcal{H}$. Now, take $k_2 \in K$, then

$$(\mathrm{Th})(k_2) = \sum_{i=1}^{r} k_i(k_2) \cdot \tilde{h}_i(h).$$

Now, if we realize $\tilde{\mathcal{H}}$ by the space of $\mathrm{Ind}_{K_M \uparrow K}\, \tilde{\pi}_0$ through

$$\tilde{h}(h) = \int_K \langle \tilde{h}(k), h(k) \rangle dk,$$

$\tilde{h} \in \mathrm{Ind}_{K_M \uparrow K}\, \tilde{\pi}_0$ and $h \in \mathcal{H}$ (cf. Prop. 3.1.3 of [2]), we can write

$$(\mathrm{Th})(k_2) = \sum_{i=1}^{r} h_i(k_2) \int_K \langle \tilde{h}_i(k_1), h(k_1) \rangle dk_1.$$

Here $\langle\ ,\ \rangle$ denotes the pairing on $\tilde{U} \times U$.

Set

$$\kappa_T(k_2 : k_1) = \sum_{i=1}^{r} h_i(k_2) \otimes \tilde{h}_i(k_1{}^{-1}) \in U \otimes \tilde{U}.$$

But, then clearly $\kappa_T(k_2 : k_1) \in \mathrm{End}^{\circ}(U)$ through the isomorphism $U \otimes \tilde{U} \cong \mathrm{End}^{\circ}(U)$. Furthermore

$$\int_K \kappa_T(k_2: k_1)h(k_1^{-1})dk_1 = \int_K \sum_{i=1}^{r} h_i(k_2)\langle \tilde{h}_i(k_1^{-1}), h(k_1^{-1})\rangle dk_1$$

$$= \sum_{i=1}^{r} h_i(k_2) \int_K \langle \tilde{h}_i(k_1), h(k_1)\rangle dk_1$$

$$= (\mathrm{Th})(k_2),$$

and for $m_1, m_2 \in K_M$, $k_1, k_2 \in K$,

$$\kappa_T(k_2 m_2^{-1}: m_1^{-1}k_1) = \sum_{i=1}^{r} h_i(k_2 m_2^{-1}) \otimes \tilde{h}_i(k_1^{-1}m_1)$$

$$= \sum_{i=1}^{r} \pi_0(m_2)h_i(k_2) \otimes \tilde{\pi}_0(m_1^{-1})\tilde{h}_i(k_1^{-1})$$

$$= \pi_0(m_2)\kappa_T(k_2: k_1)\pi_0(m_1),$$

where, by abuse of notation, π_0 also denotes the representation of $K \times K$ on $\mathrm{End}^{\circ}(U)$ given by

$$\pi_0(k_2: k_1)T = \pi_0(k_2)T\pi_0(k_1^{-1})$$

for $k_1, k_2 \in K$ and $T \in \mathrm{End}^{\circ}(U)$ (cf. Section 1.11 of [28]).

Therefore κ_T satisfies

1) $\kappa_T(k_2 m_2^{-1}: m_1^{-1}k_1) = \pi_0(m_2)\kappa_T(k_2: k_1)\pi_0(m_1)$

for $m_1, m_2 \in K_M$, $k_1, k_2 \in K$, and

2) $(\mathrm{Th})(k_2) = \int_K \kappa_T(k_2: k_1)h(k_1^{-1})dk_1,$

where $h \in \mathcal{H}$ and $k_1, k_2 \in K$.

Moreover, the correspondence $T \mapsto \kappa_T$ is a bijection from $\mathfrak{J}$ onto the space of smooth functions from $K \times K$ onto $\mathrm{End}^{\circ}(U)$ satisfying condition 1 above.

The mapping $T \mapsto \psi_T$. We again refer to Section 5.2.1 of [28].

Let $V^{\circ} = C^{\infty}(K \times K)$, and define a double representation ${}^{\circ}\tau$ of K on V° by

$$^{\circ}\tau(l_1)v^{\circ}\tau(l_2)\colon (k_1; k_2) \mapsto v(l_1^{-1}k_1; k_2l_2^{-1}),$$

where $v \in V^{\circ}$ and $k_1, k_2, l_1, l_2 \in K$. Let $^{\circ}\tau_M = {}^{\circ}\tau | K_M$.

Now, given $T \in \mathfrak{Z}$, we define $\psi_T \in C(M, V^{\circ})$ by

$$\begin{aligned}\psi_T(m)(k_1; k_2) &= \psi_T(k_1\colon m\colon k_2)\\ &= \operatorname{tr}(\pi(m)\kappa_T(k_2\colon k_1)),\end{aligned}$$

where $m \in M$ and $k_1, k_2 \in K$. Notice that the trace is defined since π is admissible (or equally, since $\kappa_T(k_2\colon k_1) \in \operatorname{End}^{\circ}(U)$).

Lemma 2.2.1. *Choose $T \in \mathfrak{Z}$; then $\psi_T \in \mathfrak{A}(\pi, {}^{\circ}\tau_M)$.*

Proof. Taking $m_1, m_2 \in K_M$, we have

$$\begin{aligned}\psi_T(k_1\colon m_1mm_2\colon k_2) &= \operatorname{tr}(\pi(m_1mm_2)\kappa_T(k_2\colon k_1))\\ &= \operatorname{tr}(\pi(m)\kappa_T(k_2m_2^{-1}\colon k_1)\pi(m_1))\\ &= \operatorname{tr}(\pi(m)\kappa_T(k_2m_2^{-1}\colon m_1^{-1}k_1))\\ &= \psi_T(m_1^{-1}k_1\colon m\colon k_2m_2^{-1})\\ &= {}^{\circ}\tau_M(m_1)\psi_T(k_1\colon m\colon k_2){}^{\circ}\tau_m(m_2).\end{aligned}$$

and therefore $\psi_T \in C(M, {}^{\circ}\tau_M)$.

Now, since $\kappa_T \in V^{\circ} \otimes \operatorname{End}^{\circ}(U)$, take $v_i \in V^{\circ}$ and $T_i \in \operatorname{End}^{\circ}(U)$ such that

$$\kappa_T = \sum_i v_i \otimes T_i.$$

But, then

$$\psi_T(k_1\colon m\colon k_2) = \sum_i v_i(k_2\colon k_1)\operatorname{tr}(\pi(m)T_i).$$

Now, if we realize $U \otimes \tilde{U}$ by $\operatorname{End}^{\circ}(U)$, then

$$\begin{aligned}\operatorname{tr}(\pi(m)T_i) &= \sum_j \operatorname{tr}(\pi(m)v_{ij} \otimes \tilde{v}_{ij})\\ &= \sum_j \langle \tilde{v}_{ij}, \pi(m)v_{ij}\rangle\end{aligned}$$

for $T_i = \Sigma_j v_{ij} \otimes \tilde{v}_{ij}$ with $v_{ij} \in U$ and $\tilde{v}_{ij} \in \tilde{U}$. Therefore

$$\psi_T(k_1 : m : k_2) = \sum_{i,j} v_i(k_2 : k_1)\langle \tilde{v}_{ij}, \pi(m)v_{ij}\rangle$$

which shows that $\psi_T \in \mathcal{A}(\pi) \otimes V^\circ$. This completes the lemma.

The following lemma, is Lemma 5.2.1.3 of [28].

LEMMA 2.2.2. *The linear map $T \to \psi_T$ is a surjection of $\mathfrak{Z}$ on $\mathcal{A}(\pi, {}^\circ\tau_M)$. The mapping $T \to \psi_T$ is a bijection if and only if π is irreducible.*

Relation with intertwining operators. We shall now go back to the intertwining operators introduced in Sections 2 and 2.1. By Theorem 2.1.1, it is clear that we only need to know the analytic continuation for the maximal parabolic subgroups. Therefore, for the moment, let us assume that P is maximal. Then $\mathcal{P}(A) = \{P, P^-\}$ and $W(A)$ consists of at most two elements. Furthermore there is only one reduced root $\alpha \in \Sigma^+(\theta)$.

Using the notation in Section 2. Suppose $P = P_\theta$ for some $\theta \subset \Delta$, and take $f \in I(\nu, \pi, \theta)$. Now, take ν so that (2.3) holds for the root α. Observe that $w(\alpha) \in \psi^-$, $(w = w_l w_{l,\theta})$. Then

$$A(\nu, \pi, w)f(g) = \int_{N^-} f(gwn^-)dn^-,$$

where $P^- = MN^-$ is the parabolic subgroup opposed to P. Let $h_f(g) = f(gw)$. Then $h_f \in I(w(\nu), w(\pi), -w(\theta))$, $-w(\theta) \subset -\Delta \subset \psi^-$. Furthermore, if we let $\bar{\theta} = w(\theta)$, and define

$$J_{P_{\bar{\theta}}|P_{\bar{\theta}}^-}(\nu, \pi): I(w(\nu), w(\pi), -w(\theta)) \to I(w(\nu), w(\pi), w(\theta))$$

by

$$J_{P_{\bar{\theta}}|P_{\bar{\theta}}^-}(\nu, \pi)h_f(g) = \int_{N_{\bar{\theta}}} h_f(g\bar{n})d\bar{n},$$

then

$$A(\nu, \pi, w)f = J_{P_{\bar{\theta}}|P_{\bar{\theta}}^-}(\nu, \pi)h_f,$$

for all ν satisfying (2.3).

Consequently, we need to study the operator

$$J_{P^-|P}(\nu, \pi)h = \int_{N^-} h(gn^-)dn^- \ (h \in I(\nu, \pi, \theta)) \tag{2.2.1}$$

for every maximal subgroup $P = P_\theta$. First of all observe that the right hand side of (2.2.1) converges whenever

$$\langle \mathrm{Re}(\nu), H_\alpha \rangle \ll 0.$$

Now, let us realize $V(\nu, \pi, \theta)$ and $V(\nu, \pi, -\theta)$ as subspaces of

$$\mathcal{H} = \mathcal{H}(\pi) = \{h \in C^\infty(K: U) | h(km) = \pi(m^{-1})h(k);\ m \in K_M\}.$$

Let F_θ and $F_{-\theta}$ be the corresponding projections. Fix a scalar product on $\mathcal{H}$ as explained in Section 5.2.1 of [28]. Extend $J_{P^-|P}(\nu, \pi)$ to $\mathcal{H}$ by putting it equal to zero on the complement of $V(\nu, \pi, \theta)$. Then $J_{P^-|P}(\nu, \pi) \in \mathrm{End}(\mathcal{H})$. Given $g \in G$, write $g = k(g)\mu(g)n(g)$, where $k(g) \in K$, $\mu(g) \in M$, and $n(g) \in N$. The following lemma is trivial.

LEMMA 2.2.3. *Suppose ν satisfies* (2.3) *with respect to α, and take $h \in F_\theta(\mathcal{H})$; then*

$$J_{P^-|P}(\nu, \pi)h(k) = \int_{N^-} q^{\langle -\nu-\rho_\theta, H_\theta(\mu(n^-))\rangle}\ \pi(\mu(n^-)^{-1})h(kk(n^-))dn^-.$$

Now, let $\mathcal{E}(K)$ be the set of classes of irreducible unitary representations of K. For $\delta \in \mathcal{E}(K)$, let ξ_δ denote the character and $d(\delta)$ the degree of δ. Write $\alpha_\delta = d(\delta)\overline{\xi}_\delta$. For a finite subset F of $\mathcal{E}(K)$, let

$$\alpha_F = \sum_{\delta \in F} \alpha_\delta,$$

and

$$E_F = \int_K \alpha_F(k)\pi(k)dk.$$

Finally, set $\mathcal{H}_F = E_F\mathcal{H}$, where $\mathcal{H} = \mathcal{H}(\pi)$.

Now, given $T \in \mathfrak{Z} = \mathrm{End}^\circ(\mathcal{H}(\pi))$, choose $F \subset \mathcal{E}(k)$ such that $T \in \mathrm{End}(\mathcal{H}_F)$. Set

$$j_{P^-|P}(\nu, \pi) = E_F J_{P^-|P}(\nu, \pi)E_F$$

and

$$\gamma(G/P) = \int_{N^-} q^{\langle -2\rho_\theta, H_\theta(\mu(n^-))\rangle} dn^-.$$

The following lemma is crucial. (In the case of real groups this is Lemma 11.1 of [7].)

LEMMA 2.2.4. *Take ν so that* (2.3) *holds. Then*

$$c_{P|P}(1, \pi, \nu)\psi_T = \gamma(G/P)^{-1}\psi_{j_{P^-|P(\nu,\pi)}TF_\theta}$$

Proof. Let $j = j_{P^-|P}(\nu, \pi)$. Then

$$(jTF_\theta h)(k_2) = \int_K \kappa_{jTF_\theta}(k_2 : k_1)h(k_1{}^{-1})dk_1 \tag{2.2.2}$$

with $h \in \mathcal{K}$ and $k_2 \in K$. Also by the assumption on ν,

$$(jTF_\theta h)(k_2) = \int_{N^-} q^{\langle -\nu-\rho_\theta, H_\theta(\mu(n^-))\rangle}\pi(\mu(n^-)^{-1})(TF_\theta h)(k_2 k(n^-))dn^- \tag{2.2.3}$$

Now the right hand side of (2.2.3) is equal to

$$\int_{K\times N^-} q^{\langle -\nu-\rho_\theta, H_\theta(\mu(n^-))\rangle}\, \pi(\mu(n^-)^{-1})\kappa_{TF_\theta}(k_2 k(n^-) : k_1)h(k_1{}^{-1})dk_1 dn^-. \tag{2.2.4}$$

Using the injectivity of the map $T \to \kappa_T$ and comparing (2.2.2) and (2.2.4), we conclude

$$\kappa_{jTF_\theta}(k_2 : k_1) = \int_{N^-} q^{\langle -\nu-\rho_\theta, H_\theta(\mu(n^-))\rangle}\, \pi(\mu(n^-)^{-1})\kappa_{TF_\theta}(k_2 k(n^-) : k_1)dn^-.$$

Consequently

$$\psi_{jTF_\theta}(k_1 : m : k_2) = \int_{N^-} q^{\langle -\nu-\rho_\theta, H_\theta(\mu(n^-))\rangle}\, \psi_{TF_\theta}(k_1 : m\mu(n^{-1})^{-1} : k_2 k(n^-))dn^-.$$

But now by Theorem 5.3.5.4 of [28]

$$(c_{P|P}(1;\ \pi;\ \nu)\psi_T)(k_1\colon m\colon k_2) = \gamma(G/P)^{-1}$$

$$\times \int_{N^-} q^{\langle -\nu-\rho_\theta, H_\theta(\mu(n^-))\rangle}\ \psi_{TF_\theta}$$

$$\psi_{TF_\theta}\,(k_1\colon m\mu(n^-)^{-1}\colon k_2k(n^-))dn^-$$

which completes the lemma.

Now, let $\chi_{\pi,\nu}$ be the central character of the representation $\pi_\nu = \pi \otimes q^{\langle \nu, H_\theta(\ )\rangle}$, i.e. a quasi-character of the maximal split torus A in the center Z_M of M which satisfies

$$\pi(a)q^{\langle \nu, H_\theta(a)\rangle} = \chi_{\pi,\nu}(a)$$

for every $a \in A$. Observe that the irreducibility of π is now a part of our assumption.

LEMMA 2.2.5. *Suppose π is irreducible and supercuspidal. Also assume that ν is so chosen that $w(\chi_{\pi,\nu}) \neq \chi_{\pi,\nu}$ for $1 \neq w \in W(A)$. Then, as a function of ν, $J_{P^-|P}(\nu,\ \pi)$ is holomorphic. Furthermore, the possible poles when $w(\chi_{\pi,\nu}) = \chi_{\pi,\nu}$ for $w \neq 1$ are all simple.*

Proof. From Theorem 5.4.2.1 of [28], it is clear that the lemma holds for $c_{P|P}(1,\ \pi,\ \nu)$. Now, suppose $J_{P^-|P}(\nu,\ \pi)h$ has a pole for some $h \in \mathcal{H}$, and for some $\nu = \nu_0$ with $w(\chi_{\pi,\nu_0}) \neq \chi_{\pi,\nu_0}$, $w \neq 1$. Then $J_{P^-|P}(\nu_0,\ \pi)E_F$, and consequently $j_{P^-|P}(\nu_0,\ \pi)$ must have poles for some finite subset $F \subset \mathcal{E}(K)$. Now, taking T equal to the identity of $\mathcal{H}_F$, since the mapping $T \to \psi_T$ is a bijection, we conclude that $\psi_{j_{P^-|P}}(\nu_0,\ \pi)$ must have a pole. But this is a contradiction by Lemma 2.2.4. Observe that the same kind of argument establishes the meromorphicity of $J_{P^-|P}$ on $(\mathfrak{a}_\theta)^*_{\mathbf{C}}$ at the first place.

We say $P = P_\theta$ is *self conjugate* if $P_\theta = P_{w(\theta)}$, $w = w_l w_{l,\theta}$. In this case $W(A) = \{1,\ w\}$. Otherwise $W(A)$ is trivial.

COROLLARY. *Suppose P is not self conjugate; then $J_{P^-|P}(\nu,\ \pi)$ is entire.*

Now, let us consider the general case, i.e. when P is not necessarily maximal. Again suppose $P = P_\theta$, $P' = P_{\theta'}$ for some θ, $\theta' \subset \Delta$. Fix $w \in$

$W(\theta, \theta')$. Take $f \in I(\nu, \pi, \theta)$ and suppose ν satisfies (2.3) for every reduced root $\alpha \in \Sigma^+(\theta)$ satisfying $w(\alpha) \in \psi^-$. Write $w = w_{n-1} \cdots w_1$ as in Lemma 2.1.2. Then

$$A(\nu, \pi, w) = A(\nu_{n-1}, \pi_{n-1}, w_{n-1}) \cdots A(\nu_1, \pi_1, w_1)$$

with the notation of Theorem 2.1.1.

Let $\alpha_i \in \Delta - \theta_i$ be so that θ_{i+1} is the conjugate of θ_i in $\Omega_i = \theta_i \cup \{\alpha_i\}$, $1 \leq i \leq n-1$. Then $w_i = w_{l,\Omega_i} w_{l,\theta_i}$. Now, let A_i' be $A_{\Omega_i} = \bigcap_{\alpha \in \Omega_i} \mathrm{Ker}\, \alpha$, and let M_i' be the centralizer of A_i' in G.

Let $A_i = A_{\theta_i} = \bigcap_{\alpha \in \theta_i} \mathrm{Ker}\, \alpha$, then $A_i' \subset A_i$ and furthermore the centralizers of A_i in M_i' and G are the same. Let M_i be this centralizer, and let

$$N_i = \prod_{\alpha \in \Sigma_{\Omega_i}{}^+ - \Sigma_{\theta_i}{}^+} U^\alpha$$

Then $P_i = M_i N_i$ is a maximal parabolic subgroup of M_i'. Now, Ω_i may be considered as a basis for the system of roots of M_i' with respect to $\mathfrak{a}_0$. Denote by $I'(\nu_i, \pi_i, \theta_i)$, the corresponding induced representation of M_i' through the realization $\theta_i \subset \Omega_i$.

Given $f \in I(\nu_i, \pi_i, \theta_i)$, define f' by restriction on M_i'. Then $f' \in I'(\nu_i, \pi_i, \theta_i)$. Now, let $A'(\nu_i, \pi_i, w_i)$ be the restriction of $A(\nu_i, \pi_i, w_i)$ on $I'(\nu_i, \pi_i, \theta_i)$, i.e.

$$A'(\nu_i, \pi_i, w_i) f' = A(\nu_i, \pi_i, w_i) f.$$

But, now using the notation introduced before

$$A'(\nu_i, \pi_i, w_i) f' = J_{P_{\bar{\theta}_i} | P_{\bar{\theta}_i^-}}(\nu_i, \pi_i) h_{f'},$$

where $J_{P_{\bar{\theta}_i} | P_{\bar{\theta}_i^-}}(\nu_i, \pi_i)$ from

$$I'(w_i(\nu_i), w_i(\pi_i), -w_i(\theta_i)) \to I'(w_i(\nu_i), w_i(\pi_i), w_i(\theta_i))$$

is defined initially by (2.2.1). Let us now fix an inner product (which we explained before) on $\mathfrak{a}_\theta$.

LEMMA 2.2.6. *Suppose α_i is so that $w_i(\alpha_i) = -\alpha_i$. Then the reflection $w_{[\beta_i]}$ with respect to the hyperplane $[\beta_i] = 0$ (cf. corollary of Lemma*

2.1.2) is an element of $W(A_\theta)$. Conversely, given $[\beta]$ with $w_{[\beta]} \in W(A_\theta)$, choose α_i as before (in some decomposition of w), then $w_i \in W(A_{\theta_i})$.

Proof. The root $[\beta_i] \in \Sigma_r\,(\theta, \theta', w)$ is defined by $[\beta_i] = w_1{}^{-1} \cdots w_{i-1}^{-1}([\alpha_i])$. Let $\tilde{w}_{i-1} = w_{i-1} \cdots w_1$; then $\tilde{w}_i{}^{-1} w_i \tilde{w}_i([\beta_i]) = [-\beta_i]$, which is the reflection $w_{[\beta_i]}$ with respect to $[\beta_i] = 0$. But, then

$$\tilde{w}_i{}^{-1} w_i \tilde{w}_i \in W(\theta_1, \theta_i) \cdot W(\theta_i, \theta_i) \cdot W(\theta_i, \theta_1)$$

or $\tilde{w}_i{}^{-1} w_i \tilde{w}_i \in W(\theta_1, \theta_1)$, i.e. $w_{[\beta_i]} \in W(A_\theta)$. The converse can be proved the same way.

Now, let $\Sigma_r{}^{\circ}\,(\theta, \theta', w)$ be the set of those $[\beta] \in \Sigma_r\,(\theta, \theta', w)$ for which the reflection $w_{[\beta]}$ is in $W(A_\theta)$. Then by Lemma 2.2.6, this set provides all the roots $[\beta_i]$ for which P_i is self conjugate in M_i'. Let $\mathfrak{a} = \mathfrak{a}_\theta$. We now have

THEOREM 2.2.1. *Let π be an irreducible unitary supercuspidal representation of M. Then*

$$\prod_{\alpha \in \Sigma_r{}^{\circ}(\theta, \theta', w)} (1 - \chi^2_{\pi,\nu}(h_\alpha)) A(\nu, \pi, w)$$

is holomorphic on $\mathfrak{a}^*_{\mathbf{C}}$.

Proof. Suppose $\alpha = [\beta_i]$ for some i in some decomposition of w according to Lemma 2.1.2. Then $\chi^2_{\pi_i,\nu_i}(w_i{}^{-1} h_{\alpha_i} w_i) = \chi^2_{\pi,\nu}(h_{-[\beta_i]})$. Now, the theorem follows from the remarks we just made, Theorem 2.1.1, Lemma 2.2.5, its corollary and Lemma 2.2.6.

Remark 1. The unitary assumption is not really necessary. In fact Theorem 2.2.1 is true for any irreducible (admissible) supercuspidal representation.

Remark 2. This is another version of Theorem 22 of [8].

Now, suppose π is any irreducible admissible representation of M. By Jacquet's quotient theorem (cf. [2]), there exists a parabolic subgroup $P_* \subset M$, $P_* = M_* N_*$, $M_* \supset A_0$, and an irreducible supercuspidal representation σ of M_* such that

$$0 \to \pi \to \operatorname*{Ind}_{P_* \uparrow M} \sigma.$$

or

$$0 \to \pi \otimes q^{\langle \nu, H_\theta(\ )\rangle} \to \operatorname*{Ind}_{P_* \uparrow M} \sigma \otimes q^{\langle \nu, H_\theta(\ )\rangle}.$$

Now, if we restrict $q^{\langle \nu, H_\theta(\ )\rangle}$ to M_*, we have

$$0 \to \pi \otimes q^{\langle \nu, H_\theta(\ )\rangle} \to \operatorname*{Ind}_{P_* \uparrow M} (\sigma \otimes q^{\langle \nu, H_\theta(\ )\rangle}).$$

Then by inducing in stages

$$\operatorname*{Ind}_{P \uparrow G} (\pi \otimes q^{\langle \nu, H_\theta(\ )\rangle}) \hookrightarrow \operatorname*{Ind}_{P_* \uparrow G} (\sigma \otimes q^{\langle \nu, H_\theta(\ )\rangle}).$$

Now choose $\theta_* \subset \Delta$ such that $P_* = P_{\theta_*}$. Furthermore, let $\tilde{\nu} \in (\mathfrak{a}_{\theta_*})^*_{\mathbf{C}}$ be such that

$$\langle \tilde{\nu}, H_{\theta_*}(a)\rangle = \langle \nu, H_\theta(a)\rangle \qquad (\forall a \in A_{\theta_*}).$$

Observe that the choice of $\tilde{\nu}$ is unique. The imbedding $\pi \to \mathrm{Ind}_{P_* \uparrow M}\, \sigma$ is not unique, but the following lemma asserts that a certain product is independent of the choices of P_* and σ.

LEMMA 2.2.7. *The product*

$$\prod_{\alpha \in \Sigma_r{}^{\circ}(\theta_*, w\theta_*, w)} (1 - \chi^2_{\sigma,\tilde{\nu}}(h_\alpha))$$

is independent of the choices of P_* and σ.

Proof. Suppose π also has an imbedding into $\mathrm{Ind}_{P_*{}' \uparrow M}\, \sigma'$ for some $P_*{}' = P_{\theta_*{}'} \subset M$, $\theta_*{}' \subset \Delta$, $P_*{}' = M_*{}'N_*{}'$, $M_*{}' \supset A_0$, and an irreducible supercuspidal representation σ' of $M_*{}'$. By Theorem 6.3.4 of [2], θ_* and $\theta_*{}'$ are associate and there exists $\tilde{w} \in W(\theta_*, \theta_*{}')$ such that $\sigma' \cong w(\sigma)$.

Now, take $\alpha \in \Sigma_r{}^{\circ}(\theta_*, w\theta_*, w)$, then $w_\alpha \in W(\theta_*, \theta_*)$, and $w_{\tilde{w}(\alpha)} = \tilde{w} w_\alpha \tilde{w}^{-1} \in W(\theta_*{}', \theta_*{}')$. Consequently $\tilde{w}(\alpha) \in \Sigma_r{}^{\circ}(\theta_*{}' w\theta_*{}', w)$. Therefore changing θ_* to $\theta_*{}'$, the product will range over $\tilde{w}\Sigma_r{}^{\circ}(\theta_*, w\theta_*, w)$. But, now $\chi^2_{\sigma,\tilde{\nu}}(h_\alpha) = \chi^2_{\tilde{w}(\sigma),\tilde{\nu}'}(h_{\tilde{w}(\alpha)})$ for $\alpha \in \Sigma_r{}^{\circ}(\theta_*, w\theta_*, w)$, where $\tilde{\nu}'$ is the restriction of ν to $M_*{}'$. This proves the lemma.

Finally, since the matrix coefficients of π are among those of $\mathrm{Ind}_{P_* \uparrow M}\, \sigma$, we have the following consequences of Theorem 2.2.1 and Lemma 2.2.7.

THEOREM 2.2.2. *Let π be an irreducible admissible representation of M. Fix a parabolic subgroup $P_* = M_*N_*$ of M, $M_* \supset A_0$, and an irreducible supercuspidal representation σ of M_* such that $\pi \subset \mathrm{Ind}_{P_* \uparrow M}\, \sigma$. Then*

$$\prod_{\alpha \in \Sigma_r^{\mathrm{o}}(\theta_*, w\theta_*, w)} (1 - \chi^2_{\sigma,\tilde{\nu}}(h_\alpha)) A(\nu, \pi, w)$$

is holomorphic on $\mathfrak{a}^*_{\mathbf{C}}$.

COROLLARY. *The theorem holds for any irreducible unitary representation π of M.*

2.3. Preliminaries on L-functions. Let $\overline{F}$ be an algebraic closure of F, and denote by Γ_F the Galois group of $\overline{F}/F$. Then the L-group ${}^LG$ of G is the semi-direct product of a complex algebraic group ${}^LG^{\mathrm{o}}$ and Γ_F (cf. [1, 16]). Furthermore for every parabolic subgroup P of G, ${}^LP$ is a *relevant parabolic* subgroup of ${}^LG$ (cf. [1]), and ${}^LM$ is canonically isomorphic to the Levi factors of ${}^LP$. Write ${}^LP = {}^LM \cdot {}^LN$ with ${}^LN$ the unipotent radical of ${}^LP$. Then ${}^LN$ is the unipotent radical of ${}^LP \cap {}^LG^{\mathrm{o}}$. Let ${}^L\mathfrak{n}$ be the Lie algebra of ${}^LN$. The Galois group Γ_F acts on the root spaces of ${}^L\mathfrak{n}$. We shall now assume that G is such that every root $\alpha \in \psi^+ - \Sigma_\theta{}^+$ is reduced.

Now, let π be an irreducible admissible representation of $M = M_\theta$, $\theta \subset \Delta$. Let $W_\theta = W_{l,\Delta} \cdot w_{l,\theta}$. For a complex number s, set $\nu = -2s\rho_\theta$. Then $\nu \in \mathfrak{a}^*_{\mathbf{C}} = (\mathfrak{a}_\theta)^*_{\mathbf{C}}$.

Let $\{a_1, \ldots, a_m\}$ be distinct values of $\langle 2\rho_\theta, H_\alpha \rangle$, where α ranges over all the roots $\alpha \in \psi^+ - \Sigma_\theta{}^+$ (here $H_\alpha \in \mathfrak{a}_\phi$). The numbers a_i, $1 \leq i \leq m$, are all positive integers. For every root α, let α^ν be its dual root. Now fix i, $1 \leq i \leq m$, and let r_i be the restriction of the adjoint representation of ${}^LM$ on the subspace of ${}^L\mathfrak{n}$ which is generated by all the roots α^ν for which $\langle 2\rho_\theta, H_\alpha \rangle = a_i$. More precisely, this is the direct sum of the Lie algebras of the centralizers of $\mathrm{Ker}(\alpha^\nu)$, $\langle 2\rho_\theta, H_\alpha \rangle = a_i$, in ${}^LG$. Finally, let $\tilde{r}_i$ denote the contragredient representation of r_i.

Now for every complex number s, every irreducible admissible representation π of M, and every *representation* r of ${}^LM$ (cf. [1]), let $L(s, \pi, r)$ be the conjectural Langlands' L-function [16].

When everything is unramified, these L-functions are well-defined. More precisely, suppose G is unramified, i.e. G is a quasi-split group over F which splits over an unramified Galois extension of F. Also assume that π is a class one representation, i.e. it has a vector fixed by $K \cap M$. In this case, there exists an element ${}^{\circ}\mu$ of $(\mathfrak{a}_\phi)^*_{\mathbf{C}}$ such that π can be realized (by left inverse translation) on a constituent of the space of smooth, complex valued functions f on M satisfying

$$f(mtu) = q^{\langle -{}^{\circ}\mu - {}^{\circ}\rho_\phi, H_\phi(t)\rangle} f(m),$$

$m \in M$, $t \in M_0$ (centralizer of A_0), and $u \in M \cap U$. Here ${}^{\circ}\rho_\phi = \frac{1}{2}\Sigma_{\alpha \in \Sigma_\theta^+} \alpha$. Observe that ${}^{\circ}\mu$ can be represented by an element ${}^{\circ}t^\nu \in {}^LT^{\circ}$ ($(\mathfrak{a}_\phi)^*_{\mathbf{C}}$ is the Lie algebra of ${}^LT^{\circ}$).

Now, let r be a *representation* of ${}^LM$. Take a finite Galois extension F'/F such that M splits over F' and r is a representation of ${}^LM^{\circ} \times \Gamma_{F'/F}$. Let us assume that F'/F is unramified. Choose a Frobenious element σ_{Fr} of $\Gamma_{F'/F}$. Then the Langlands L-function is defined to be

$$L(s, \pi, r) = \det(I - r({}^{\circ}t^\nu, \sigma_{Fr})q^{-s})^{-1}$$

By the discussion in (7.5) of [1], ${}^{\circ}t^\nu$ may in fact be chosen to lie in $({}^LT^{\circ})^{\Gamma_F}$.

For the representations r_i, $1 \leq i \leq m$, F'/F may be chosen to be an extension over which G splits.

Now it follows from [19] that for every i, $1 \leq i \leq m$,

$$L(s, \pi, r_i) = \prod_{\substack{\alpha \in \psi^+ - \Sigma_\theta^+ \\ \langle 2\rho_\theta, H_\alpha\rangle = a_i}} (1 - q^{n_\alpha \langle {}^{\circ}\mu, H_\alpha\rangle} q^{-n_\alpha s})^{-1},$$

where n_α denotes the multiplicity of α.

We shall now study some of the examples considered in [15] more closely.

1. Relation with the Hecke theory of Jacquet-Shalika. By abuse of notation, let G be the group of F-rational points of GL_{n+m}, i.e. $G = GL_{n+m}(F)$, where m and n are two fixed positive integers. Then A_0 and P_0 may be chosen to be the subgroups of diagonal and the upper triangular elements of G, respectively. Let $P \supset P_0$ be the standard parabolic subgroup of G whose Levi factors are isomorphic to $GL_n(F) \times GL_m(F)$. Write $P = MN$. Let π_1 and π_2 be two irreducible admissible non-degenerate (cf. [10]) representations of $GL_n(F)$ and $GL_m(F)$, respectively. Then $\pi = \pi_1 \otimes \tilde{\pi}_2$ is an irreducible admissible non-degenerate representation of $M = GL_n(F) \times GL_m(F)$. The adjoint representation r of ${}^LM$ on ${}^L\mathfrak{n}$ is now irreducible and in fact is isomorphic to the representation $\rho_n \otimes$

$\tilde{\rho}_m$ of $GL_n(\mathbf{C}) \times GL_m(\mathbf{C})$ where ρ_n and ρ_m are the standard representations of $GL_n(\mathbf{C})$ and $GL_m(\mathbf{C})$, respectively. Observe that for the purpose of L-functions, the L-group of $GL_n(F) \times GL_m(F)$ may be chosen to be $GL_n(\mathbf{C}) \times GL_m(\mathbf{C})$.

Now, let us first consider the unramified case; more precisely, suppose the representations π_i, $i = 1, 2$, are of class one. Then $m = 1$, $a_1 = m + n$, and

$$L(s, \pi, r) = L(s, \pi_1 \times \pi_2)$$

where the L-function in the right hand side is defined by Jacquet-Shalika [10, 13] for any pair of irreducible admissible non-degenerate representation $\pi = (\pi_1, \pi_2)$ of $GL_n(F) \times GL_m(F)$.

On the other extreme, let us suppose that π_1 and π_2 are supercuspidal. Assume that $m \neq n$. Then $L(s, \pi_1 \times \pi_2) = 1$. Next suppose $m = n$. As in [10], let $W(\pi_1)$ and $W(\pi_2)$ denote the Whittaker models of π_1 and π_2, respectively. For a Schwartz-Bruhat function Φ on F^m set

$$f_\Phi(g) = \int_F \Phi((0, \ldots, 0, t)g)\omega(t)|t|^{ms}d^*t,$$

where $(0, \ldots, 0, t) \in F^m$. Here ω denotes the central character of $\pi_1 \otimes \tilde{\pi}_2$. Given $W_1 \in W(\pi_1)$, $W_2 \in W(\pi_2)$ and Φ as above, set (cf. [10]):

$$\psi(s, W_1, W_2, \Phi) = \int_{Z_m(F)N_m(f)\backslash GL_m(F)} W_1(g)W_2(\epsilon_m g)f_\Phi(g)|\det g|^s d\dot{g},$$

where $Z_m(F)$ is the center of $GL_m(F)$, $N_m(F)$ is its subgroup of upper triangulars with ones along the diagonals and $\epsilon_m = \mathrm{diag}(-1, 1, -1, \ldots) \in GL_m(F)$. Then according to [10] $L(s, \pi_1 \otimes \pi_2)$ is the G.C.D. of these integrals. Now, suppose π_1 and π_2 are supercuspidal. Consider the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

of $GL_{m-1}(F)$ into $GL_m(F)$. Then the restriction of the functions in $W(\pi_1)$ and $W(\pi_2)$ to the subgroup $GL_{m-1}(F)$ have all compact supports modulo $Z_m(F)N_m(F)$, and consequently the poles of $\psi(s, W_1, W_2, \Phi)$ depend only upon the poles of $f_\Phi(g)$. A simple computation shows that they are in fact the poles of $L(\omega, ms)$, the Hecke L-function attached to ω and ms. Consequently

$$L(s, \pi_1 \times \pi_2) = L(\omega, ms),$$

if $\pi_1 \cong \tilde{\pi}_2 \otimes |\det|^s$ for some s, and $L(s, \pi_1 \times \pi_2) = 1$ otherwise (the residue is proportional to the formal degree of π_1 in the first case and zero otherwise).

2. Let us now assume that G is as in Section 1.2, i.e. it is the adjoint group of a split semi-simple Lie algebra over Q. Let ${}^L({}^\circ G)$ be the L-group of ${}^\circ G$; then we have a natural inclusion ${}^L({}^\circ G) \hookrightarrow {}^L M$, and therefore the representations r_i, $1 \leq i \leq m$, of ${}^L M$ may be considered as the representations of ${}^L({}^\circ G)$. Thus every statement made in this section concerning the L-functions attached to the representations of M is still true for the representations of ${}^\circ G$.

Example. Let G be a simple algebraic group of type G_2 which splits over Q. Let $\theta = \{\beta\}$, where β is the longer of the two simple roots. Set $P = P_\theta$ and write $P = MN$. Then ${}^\circ G$ is isomorphic to $PSL_2(F)$ and ${}^L({}^\circ G) = SL_2(\mathbf{C})$. Furthermore $m = 2$, $a_1 = 10$, $a_2 = 5$, r_1 is the one-dimensional trivial representation of $SL_2(\mathbf{C})$, and r_2 is a four dimensional irreducible representation of $SL_2(\mathbf{C})$ (cf. [23]).

2.4. Adjoint of the intertwining operators and the relation with the Plancherel measure. We shall now show that the adjoint of an intertwining integral is again an intertwining integral.

First let π be an irreducible unitary supercuspidal representation of $M = M_\theta$. With the notation as in Section 2.2, let us for a moment assume that P is maximal. For $w \in W(A)$, $P_1, P_2 \in \mathcal{P}(A)$, and $\tau = {}^\circ\tau$, let $c_{P_2|P_1}(w, \pi, \nu)$ be as in Section 2.2. There is a pre-Hilbert structure (,) on $L(\pi, P_1)$, as well as on $\mathcal{L}(w(\pi), P_2)$ both coming from one on $\mathcal{A}(\pi, {}^\circ\tau_M)$ (cf. [8]). Let $c_{P_2|P_1}(w\colon \pi; \nu)^*$ be the adjoint of $c_{P_2|P_1}(w\colon \pi\colon \nu)$. As before take a finite subset $F \subset \mathcal{E}(K)$ and let

$$j_{P^-|P}(\nu, \pi) = E_F J_{P^-|P}(\nu, \pi) E_F.$$

Then we have

LEMMA 2.4.1. *Fix $T \in \mathfrak{Z}F_{-\theta}$ and take ν so that* (2.3) *holds. Then*

$$c_{P|P}(1, \pi, \nu)^* \psi_T = \gamma(G/P)^{-1} \psi_{j_{P^-|P}(\nu, \pi)^* T}$$

Proof. Given S and T in $\mathfrak{Z}$, let $d(\pi)$ be the formal degree of π defined in Section 5.2.4 of [28]. Then from Lemma 5.2.4.1 of [28] it follows that

$$(\psi_T, \psi_S) = d(\pi)^{-1} \operatorname{tr} T^* S.$$

Now if we let $j = j_{P^-|P}(\nu, \pi)$ and fix $T' \in \mathfrak{Z}$, arbitrarily, we conclude

$$\begin{aligned} (\psi_T, \psi_{jT'F_\theta}) &= d(\pi)^{-1} \operatorname{tr} (j^* T)^* T' F_\theta \\ &= (\psi_{j^* T}, \psi_{T'F_\theta}). \end{aligned} \tag{2.4.1}$$

But then by Lemma 2.2.4

$$\begin{aligned} (c_{P|P}(1, \pi, \nu)^* \psi_T, \psi_{T'F_\theta}) &= (\psi_T, c_{P|P}(1, \pi, \nu) \psi_{T'F_\theta}) \\ &= (\psi_T, \gamma(G/P)^{-1} \psi_{jT'F_\theta}), \end{aligned}$$

and consequently from (2.4.1), we get

$$(c_{P|P}(1, \pi, \nu)^* \psi_T, \psi_{T'F_\theta}) = (\gamma(G/P)^{-1} \psi_{j^* T}, \psi_{T'F_\theta}).$$

But now by Lemma 2.2.2 every element of $L(\pi, P)$ is of the form $\psi_{T'F_\theta}$ for some $T' \in \mathfrak{Z}$. This proves the lemma.

Now let $\mu(\pi\colon \nu)$ be the Plancherel measure attached to π and ν. Then by Theorem 5.3.5.2 of [28].

$$\mu(\pi, \nu) c_{P|P}(1, \pi, \bar{\nu})^* c_{P|P}(1, \pi, \nu)$$

acts like identity on $L(\pi, P)$ and consequently if we assume $\operatorname{Re}(\nu) = 0$, Lemmas 2.2.2, 2.2.4, and 2.4.1 would imply

$$\gamma(G/P)^{-2} \mu(\pi\colon \nu) J_{P^-|P}(\bar{\nu}, \pi)^* J_{P^-|P}(\nu, \pi) = 1.$$

On the other hand from Theorem 5.3.5.4 of [28], and Lemmas 2.2.4 and 2.4.1 of the present paper we conclude

$$J_{P^-|P}(\bar{\nu}, \pi)^* = J_{P|P^-}(\nu, \pi). \tag{2.4.2}$$

Now, going back to the discussion before Lemma 2.2.3, given $f \in V(\nu, \pi, \theta)$, let $h_f \in V(w(\nu), w(\pi), -w(\theta))$, $-w(\theta) \subset -\Delta$, be defined by $h_f(g) = f(gw)$, $w = w_l w_{l,\theta}$. Let T be the mapping $f \mapsto h_f$, and let $\bar{\theta} = w(\theta)$. Then

$$A(\nu, \pi, w) = J_{P_{\bar{\theta}} | P_{\bar{\theta}}^-}(\nu, \pi) \cdot T.$$

Taking adjoints and making use of (2.4.2), we get

$$A(\nu, \pi, w)^* = T^* \cdot J_{P_{\bar{\theta}}^- | P_{\bar{\theta}}}(\bar{\nu}, \pi),$$

where

$$A(\nu, \pi, w)^*: V(-w(\bar{\nu}), w(\pi), w(\theta)) \to V(-\bar{\nu}, \pi, \theta)$$

by the duality explained at the beginning of Section 2. It is easy to see that

$$(T^*f)(g) = f(gw^{-1})$$

with $f \in V(-w(\bar{\nu}), w(\pi), -w(\theta))$. On the other hand if ν satisfies (2.3), so does $-w(\bar{\nu})$, and consequently for such ν:

$$A(\nu, \pi, w)^* f(g) = \int_{N_{\bar{\theta}}^-} f(gw^{-1}n^-)dn^-$$

$$= \int_{N_{w^{-1}}} f(gnw^{-1})dn.$$

Thus

$$A(\nu, \pi, w)^* = A(-w(\bar{\nu}), w(\pi), w^{-1})$$

Now suppose P is of arbitrary rank. Then from Theorem 2.1.1 and the discussion before Lemma 2.2.6, we get the same formula for $A(\nu, \pi, w)^*$ in this case. Furthermore, using the same results we conclude, first for $\mathrm{Re}(\nu) = 0$, and then by analytic continuation for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, that

$$\gamma_w{}^{-2}(G/P)\mu_w(\pi: \nu)A(w(\nu), w(\pi), w^{-1})A(\nu, \pi, w) = 1,$$

where

$$\gamma_w{}^{-2}(G/P)\mu_w(\pi: \nu) = \prod_{i=1}^{n-1} \gamma^{-2}(M_i'/P_i)\mu(\pi_i: \nu_i).$$

We have observed

PROPOSITION 2.4.1. *Let π be an irreducible unitary supercuspidal representation of M. Suppose ν satisfies* (2.3) *for every reduced root $\alpha \in \Sigma^{+}(\theta)$. Then so does $-w(\bar{\nu})$, and furthermore the adjoint of the operator $A(\nu, \pi, w)$ is the operator:*

$$A(-w(\bar{\nu}), w(\pi), w^{-1}): V(-w(\bar{\nu}), w(\pi), w(\theta)) \to V(-\bar{\nu}, \pi, \theta).$$

Moreover

$$\gamma_w^{-2}(G/P)\mu_w(\pi: \nu)A(w(\nu); w(\pi), w^{-1})A(\nu, \pi, w) = 1.$$

Remark. The fact that, away from the poles, $A(w(\nu), w(\pi), w^{-1})A(\nu, \pi, w)$ is a scalar is almost clear since this operator commutes with $I(\nu, \pi, \theta)$. In fact this can be proved using the finite dimensional spaces $\mathcal{H}_F$, $F \subset \mathcal{E}(K)$, as in [7].

Now suppose π is an irreducible admissible unitary representation of M. As in Section 2.2, choose a parabolic subgroup $P_* \subset M$, $P_* = M_*N_*$, and an irreducible supercuspidal representation σ of M_* such that $\pi \hookrightarrow \mathrm{Ind}_{P_*\uparrow M}\,\sigma$. Suppose M_* is generated by $\theta_* \subset \theta$. Choose $\tilde{\nu} \in (\mathfrak{a}_{\theta_*})^*_{\mathbf{C}}$ such that $\langle \tilde{\nu}, H_{\theta_*}(a)\rangle = \langle \nu, H_\theta(a)\rangle$ for $\forall a \in A_{\theta_*}$. Let σ_0 be an irreducible unitary supercuspidal representation of M_* such that $\sigma = \sigma_0 \otimes q^{\langle \nu_0, H_{\theta_*}(\ )\rangle}$, where $\nu_0 \in \mathfrak{a}_{\theta_*}^*$. Let i be the inclusion

$$I(\nu, \pi, \theta) \hookrightarrow I(\tilde{\nu} + \nu_0, \sigma_0, \theta_*).$$

Denote by r the restriction map going the other way. The imbedding $\pi \to \mathrm{Ind}_{P_*\uparrow M}\,\sigma$ induces an imbedding $w(\pi) \hookrightarrow \mathrm{Ind}_{P_{w(\theta_*)}\uparrow M_{w(\theta)}}\,w(\sigma)$ in the obvious manner. Let i' be the induced inclusion

$$I(w(\nu), w(\pi), w(\theta)) \to I(w(\tilde{\nu}) + w(\nu_0), w(\sigma_0), w(\theta_*)).$$

Denote by r' the corresponding restriction. Then

$$A(\nu, \pi, w) = r' \cdot A(\tilde{\nu} + \nu_0, \sigma_0, w)\cdot i$$

and

$$A(w(\nu), w(\pi), w^{-1}) = r\cdot A(w(\tilde{\nu}) + w(\nu_0), w(\sigma_0), w^{-1})\cdot i'.$$

Then taking adjoints and using Proposition 2.4.1, we conclude

$$A(\nu, \pi, w)^* = i^* \cdot A(-w(\bar{\tilde{\nu}} + \bar{\nu}_0), w(\sigma_0), w^{-1}) \cdot r'^*$$

Therefore

$$A(\nu, \pi, w)^* = A(-w(\bar{\nu}), w(\pi), w^{-1}).$$

In fact i^* is the map

$$I(-\bar{\tilde{\nu}} - \bar{\nu}_0, \sigma_0, \theta_*) \xrightarrow{i^*} I(-\bar{\nu}, \pi, \theta) \to 0$$

and r'^* is simply the inclusion

$$0 \to I(-w(\bar{\nu}), w(\pi), w(\theta)) \xrightarrow{r'^*} I(-w(\bar{\tilde{\nu}} + \bar{\nu}_0), w(\sigma_0), w(\theta_*)).$$

Furthermore using $i' \cdot r' = 1$ and $r \cdot i = 1$, we have

$$\gamma_w^{-2}(G/P_*)\mu_w(\sigma_0, \tilde{\nu} + \nu_0)A(w(\nu), w(\pi), w^{-1})A(\nu, \pi, w) = 1,$$

and therefore we can state the following more general result.

PROPOSITION 2.4.2. *Let π be an irreducible admissible unitary representation of M. Fix a parabolic subgroup $P_* = M_*N_*$ of M, $M_* = M_{\theta_*} \supset A_0$, an irreducible unitary supercuspidal representation σ_0 of M_*, and $\nu_0 \in \mathfrak{a}_{\theta_*}{}^*$ such that $\pi \hookrightarrow \mathrm{Ind}_{P_*\uparrow M}(\sigma_0 \otimes q^{\langle \nu_0, H_{\theta_*}(\ )\rangle})$. Suppose $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$ is so that $\tilde{\nu} + \nu_0$ satisfies (2.3) for every reduced root $\alpha \in \Sigma^+(\theta_*)$. Then so does $-w(\bar{\tilde{\nu}} + \bar{\nu}_0)$, and furthermore the adjoint of the operator*

$$A(\nu, \pi, w)\colon V(\nu, \pi, \theta) \to V(w(\nu), w(\pi), w(\theta))$$

is the operator

$$A(-w(\bar{\nu}), w(\pi), w^{-1})\colon V(-w(\bar{\nu}), w(\pi), w(\theta)) \to V(-\bar{\nu}, \pi, \theta).$$

Moreover

$$\gamma_w^{-2}(G/P_*)\mu_w(\sigma_0, \tilde{\nu} + \nu_0)A(w(\nu), w(\pi), w^{-1})A(\nu, \pi, w) = 1.$$

Let $\langle\ ,\ \rangle$ be the pairing explained at the beginning of the Section 2. Observe that it is the same as the scalar product on $\mathcal{H}$ (cf. Section 5.2.1 of [28]). We then have

COROLLARY. *Let $f \in V(\nu, \pi, \theta)$ and $f' \in V(-w(\bar{\nu}), w(\pi), w(\theta))$; then*

$$\langle A(\nu, \pi, w)f, f' \rangle = \langle f, A(-w(\bar{\nu}), w(\pi), w^{-1})f' \rangle$$

3. Whittaker functionals and local coefficients. As it was shown in [23], Whittaker functionals for induced representations play an important role in the definition of local coefficients appearing in the functional equations satisfied by the L-function considered by Langlands in [15]. The purpose of this section is to study the properties of these local coefficients. We still resume the assumption of Section 2, i.e. G is any connected reductive algebraic group defined over any local field, except when the field is archimedean in which case we assume G is quasi-split and π is fine (cf. [24, 26]).

For every $\alpha \in \psi$, let N_α be the subgroup of U whose Lie algebra is $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. The subgroup $\Pi_{\alpha\in\psi^+-\Delta} N_\alpha$ is normal in U and the quotient is isomorphic to $\Pi_{\alpha\in\Delta} (N_\alpha/N_{2\alpha})$. For each $\alpha \in \Delta$, let χ_α be a smooth complex character of $N_\alpha/N_{2\alpha}$; then $\chi = \Pi_{\alpha\in\Delta} \chi_\alpha$, is a character of $\Pi_{\alpha\in\Delta} (N_\alpha/N_{2\alpha})$ and therefore one of U as well (cf. [3]). A character of U is said to be non-degenerate if it is of this form with no χ_α trivial. Throughout this section we shall fix a non-degenerate character χ of U.

Let (π, V) be an admissible representation of G. We say π is *non-degenerate* if there exists a linear functional λ on V (continuous with respect to the Schwartz topology of V_∞ if F is archimedean) such that

$$\lambda(\pi(u)v) = \overline{\chi(u)}\lambda(v) \qquad (u \in U,\ v \in V_\infty).$$

Such functionals are called *Whittaker functionals.*

Suppose now that π is irreducible. Denote by V_χ^* the complex vector space of Whittaker functionals on V. Let $\mathrm{Dim}_{\mathbf{C}} V_\chi^*$ be its dimension. We say that π satisfies *the multiplicity one* if $\mathrm{Dim}_{\mathbf{C}} V_\chi^* \leq 1$. Clearly if π is non-degenerate and satisfies the multiplicity one, then $\mathrm{Dim}_{\mathbf{C}} V_\chi^* = 1$. It is a result of J. A. Shalika [27] that every irreducible admissible representation of a quasi-split group satisfies the multiplicity one.

Suppose now that (π, V) is a non-degenerate irreducible admissible representation of G which satisfies the multiplicity one. Fix a Whittaker functional $\lambda \in V_\chi{}^*$. For every $v \in V_\infty$, define a *Whittaker function* W_v by $W_v(g) = \lambda(\pi(g^{-1})v)$. Then

$$W_v(gu) = \chi(u) W_v(g).$$

Let $W(\pi)$ be the vector space of all Whittaker functions on which G acts by the left inverse translations. The space $W(\pi)$ is unique and is called the *Whittaker model* for π.

Now let $\theta \subset \Delta$, and choose $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$. Let $M = M_\theta$ and let π be an irreducible admissible non-degenerate representation of M which satisfies the multiplicity one. This assumption will be kept fixed throughout this section. Let $W(\pi)$ be the Whittaker model for π. Then given $f \in I(\nu, \pi, \theta)$, we may assume that $f(g) \in W(\pi)$. Now at every $m \in M$, let $(f(g), m)$ denote the value of $f(g)$ at m.

As before let $w_\theta = w_{l,\Delta} \cdot w_{l,\theta}$ and let $M' = M_{w_\theta(\theta)}$. Write $P' = M'N'$, where N' is the corresponding unipotent radical. The following result is due to W. Casselman and J. A. Shalika (Proposition 2.1 of [3]). In their paper it is only proved for the case where P_θ is minimal and π is the trivial representation of M_θ. But since its generalization to the general case follows exactly the same lines as the proof of the case mentioned above, we shall only state the result and eliminate the proof (see also Proposition 3.2 and Corollary 3.3 of [23]).

PROPOSITION 3.1. *Suppose F is non-archimedean. Given* $f \in V(\nu, \pi, \theta)$ *the integral*

$$\lambda(\nu, \pi, \theta, \chi)(f) = \int_{N'} (f(n' w_\theta), e)\overline{\chi(n')}dn'$$

is convergent and consequently it defines a Whittaker functional $\lambda(\nu, \pi, \theta, \chi)$ *for the space of* $I(\nu, \pi, \theta)$. *It is an entire function of* ν *and furthermore for every* ν *and* π, *there exists a function* $f \in V(\nu, \pi, \theta)$ *for which* $\lambda(\nu, \pi, \theta, \chi)$ f *is non-zero.*

Now suppose $F = \mathbf{R}$. In this case we assume that G is quasi-split. Observe that this includes all the complex groups. By Casselman's subrepresentation theorem, we first find a principal series representation of M, where π appears infinitesimally as one of its subrepresentations. More precisely, write $P_0 \cap M = M_0 A_0 (M \cap U)$, the Langlands decomposition of $P_0 \cap M$ which is a minimal parabolic subgroup of M. Let $(\mathfrak{a}_0)^*_{\mathbf{C}}$ be the complex dual of the Lie algebra of A_0 and denote by $\hat{M}_0$ the group of (unitary) characters of M_0. Fix $\nu_0 \in (\mathfrak{a}_0)^*_{\mathbf{C}}$ and $\eta_0 \in \hat{M}_0$ such that π is infinitesimally equivalent to a subrepresentation of the principal series $I_M(\nu_0, \eta_0)$ of M (cf. [24]). Extend ν_0 and η_0 to characters of the corresponding subgroups of $P_0 \subset G$. Now for $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$ let $\tilde{\nu}$ be its extension to $(\mathfrak{a}_0)^*_{\mathbf{C}}$ defined by

$$\langle \tilde{\nu}, H_\phi(a) \rangle = \langle \nu, H_\theta(a) \rangle \qquad (\forall a \in A_0)$$

and denote by $I(\nu_0 + \tilde{\nu}, \eta_0)$ the corresponding principal series representation of G (cf. [24]). Clearly

$$I(\nu_0 + \tilde{\nu}, \eta_0)_\infty = \operatorname*{Ind}_{P_\theta \uparrow G} (I_M(\nu_0, \eta_0)_\infty \otimes e^{\langle \nu, H_\theta(\ ) \rangle}).$$

Let $\lambda(\nu_0 + \tilde{\nu}, \eta_0)$ be the Whittaker functional for $I(\nu_0 + \tilde{\nu}, \eta_0)_\infty$ defined by analytic continuation of the Whittaker integrals explained in Proposition 1.1 of [24] (originally cf. [9]). Now, the following result (Proposition 3.2 of [24]) establishes the same result for the infinite places.

PROPOSITION 3.2. *Suppose F is archimedean and π is fine. Then the differentiably induced representation $I(\nu, \pi, \theta)_\infty$ is non-degenerate and satisfies the multiplicity one. Furthermore, a Whittaker functional for $I(\nu, \pi, \theta)_\infty$ is given by $\lambda(\nu_0 + \tilde{\nu}, \eta_0)$ which, as a function of ν, is entire on $(\mathfrak{a}_\theta)^*_{\mathbf{C}}$.*

Now take $\theta' \in \{\theta\}$ and choose $w \in W(\theta, \theta')$. Suppose that (2.3) holds for every reduced root $\alpha \in \Sigma^+(\theta)$ for which $w(\alpha) \in \Psi^-$. Then $A(\nu, \pi, w)$ is defined, and for every $f \in V(\nu, \pi, \theta)$, $A(\nu, \pi, w) f \in V(w(\nu), w(\pi), \theta')$. For F non-archimedean, let $\lambda(w(\nu), w(\pi), \theta', \chi)$ be the Whittaker functional for $I(w(\nu), w(\pi), \theta')$ defined by Proposition 3.1, and when F is archimedean, let

$$\lambda(w(\nu), w(\pi), \theta', \chi) = \lambda(w(\nu_0 + \tilde{\nu}), w(\eta_0)).$$

We now prove

THEOREM 3.1. *There exists a complex number $C_\chi(\nu, \pi, \theta, w)$ such that*

$$\lambda(\nu, \pi, \theta, \chi) = C_\chi(\nu, \pi, \theta, w)\lambda(w(\nu), w(\pi), \theta', \chi)\cdot A(\nu, \pi, w). \quad (3.1)$$

Furthermore, as a function of ν, *it is meromorphic on* $(\mathfrak{a}_\theta)^*_{\mathbf{C}}$, *and its value depends only upon the class of* π.

Proof. Choose $\theta_* \subset \theta$ and a supercuspidal representation σ of $M_{\theta_*} \subset M$ such that $\pi \hookrightarrow \mathrm{Ind}_{P_{\theta_*} \uparrow M}\, \sigma$. Then, by Theorems 1.4 and 1.6 of [3] σ is also non-degenerate. Now $I(\nu, \pi, \theta) \hookrightarrow I(\tilde{\nu}, \sigma, \theta_*)$. Let us first define $C_\chi(\nu, \sigma, \theta_*, w)$ for $I(\nu, \sigma, \theta_*)$, where $\nu \in (\mathfrak{a}_{\theta_*})^*_{\mathbf{C}}$. By Theorem 5.4.3.7 of [28], $I(\nu, \sigma, \theta_*)$ is irreducible except for the poles of $\mu(\sigma, \nu)$, provided that $\sigma \otimes q^{\langle \nu, H_{\theta_*}(\,)\rangle}$ is unramified. Consequently for an open dense subset of $(\mathfrak{a}_{\theta_*})^*_{\mathbf{C}}$, $I(\nu, \sigma, \theta_*)$ is irreducible ($\mu(\sigma, \nu)$ is a product of one variable complex functions), and therefore in this subset, $\lambda(w(\nu), w(\sigma), w(\theta_*), \chi)\cdot A(\nu, \sigma, w)$ defines another non-zero Whittaker functional for $I(\nu, \sigma, \theta_*)$. Now the existence of $C_\chi(\nu, \sigma, \theta_*, w)$ follows from Theorem 1.4 and 1.6 of [3] (originally Theorem 2 of [21]), and Proposition 3.2 of this paper. It is clearly meromorphic on $(\mathfrak{a}_{\theta_*})^*_{\mathbf{C}}$. Now, if we apply Proposition 3.2.1 (to be proved in Section 3.2) to $C_\chi(\nu, \sigma, \theta_*, w)$, $w \in W(\theta, \theta')$, which is now defined, we conclude that $C_\chi(\tilde{\nu}, \sigma, \theta_*, w)$ is in fact defined for all ν in an open dense subset of $(\mathfrak{a}_\theta)^*_{\mathbf{C}}$. Suppose for ν in this subset $\lambda(w(\nu), w(\pi), w(\theta), \chi)\cdot A(\nu, \pi, w)$ is zero. Then by inducing in stages so will be $\lambda(w(\tilde{\nu}), w(\sigma), w(\theta_*), \chi)\cdot A(\tilde{\nu}, \sigma, w)$. But this is a contradiction since $C_\chi(\tilde{\nu}, \sigma, \theta_*, w)$ is defined there. Now, we define $C_\chi(\nu, \pi, \theta, w)$ in the same way as $C_\chi(\nu, \sigma, \theta_*, w)$, i.e. by relation (3.1). Then clearly $C_\chi(\nu, \pi, \theta, w) = C_\chi(\tilde{\nu}, \sigma, \theta_*, w)$. The meromorphic continuation of $C_\chi(\nu, \pi, \theta, w)$ now follows from those of other terms in (3.1).

We shall call the number $C_\chi(\nu, \pi, \theta, w)$ the local coefficient attached to ν, π, θ, and w. The reason for this will be cleared in Section 4. In fact we shall show that these are the local coefficients appearing in the functional equations satisfied by certain L-functions (cf. [23]).

3.1. A normalization of intertwining operators. Now, again suppose π is unitary and non-degenerate. Choose $\theta_* \subset \theta$, σ_0, and ν_0 as in Proposition 2.4.2 such that $\pi \hookrightarrow \mathrm{Ind}_{P_{\theta_*} \uparrow M}\,(\sigma_0 \otimes q^{\langle \nu_0, H_{\theta_*}(\,)\rangle})$. Then from Proposition 2.4.2 and Theorem 3.1, we get

PROPOSITION 3.1.1 *Suppose F is non-archimedean. Then* $C_\chi(w(\nu), w(\pi), w(\theta), w^{-1})C_\chi(\nu, \pi, \theta, w) = \gamma_w^{-2}(G/P_{\theta_*})\mu_w(\sigma_0; \tilde{\nu} + \nu_0)$ *for all* $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$.

With the notation as in [7], we also have

PROPOSITION 3.1.2. *Suppose F is archimedean and* π *is in the discrete series. Then*

$$C_\chi(w(\nu), w(\pi), w(\theta), w^{-1})C_\chi(\nu, \pi, \theta, w) = \mu_{w'}(\pi: \nu)\gamma_{w'}{}^{-2}(G/P)$$

for all $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$.

We now prove

PROPOSITION 3.1.3. *For all* $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$ *the identity*

$$C_\chi(w(\nu), w(\pi), w(\theta), w^{-1}) = \overline{C_\chi(-\bar{\nu}, \pi, \theta, w)} \tag{3.1.1}$$

holds.

Proof. We first assume that ν is imaginary so that $I(\nu, \pi, \theta)$ is completely reducible. Then $I(\nu, \pi, \theta)$ has a unique non-degenerate (topologically if $F = \mathbf{R}$) component, which we shall identify with the Whittaker model $W(\nu, \pi, \theta)$ of $I(\nu, \pi, \theta)$, defined by the Whittaker functional $\lambda(\nu, \pi, \theta, \chi)$. The Whittaker model $W(-\bar{\nu}, \pi, \theta)$ may now be considered as the dual of $W(\nu, \pi, \theta)$, if one carries the pairing between $I(\nu, \pi, \theta)$ and $I(-\bar{\nu}, \pi, \theta)$ to them. Since $C_\chi(\nu, \pi, \theta, w)$ is meromorphic, we conclude that for an open dense subset of the imaginary axis, the non-degenerate component of $I(\nu, \pi, \theta)$ is not in the kernel of $A(\nu, \pi, w)$. But now $A(\nu, \pi, w)$ induces a scalar isomorphism $C_\chi(\nu, \pi, \theta, w)$ between $W(\nu, \pi, \theta)$ and $W(w(\nu), w(\pi), w^{-1})$, andtherefore its adjoint, $\overline{C_\chi(\nu, \pi, \theta, w)}$, between $W(-w(\bar{\nu}), w(\pi), w(\theta))$ and $W(-\bar{\nu}, \pi, \theta)$ must be the map induced by the adjoint of $A(\nu, \pi, w)$, i.e. $A(-w(\bar{\nu}), w(\pi), w^{-1})$ (Proposition 2.4.2). But this last map is the scalar $C_\chi(-w(\bar{\nu}), w(\pi), w(\theta), w^{-1})$ and therefore

$$\overline{C_\chi(\nu, \pi, \theta, w)} = C_\chi(-w(\bar{\nu}), w(\pi), w(\theta), w^{-1}).$$

Now the proposition is proved if we change ν to $-\bar{\nu}$ and use the analytic continuation.

COROLLARY. *Suppose* $-\bar{\nu} = \nu$ *and* π *is supercuspidal. Then* $C_\chi(\nu, \pi, \theta, w_\theta)$ *is holomorphic. Furthermore, suppose* ν *is away from the poles of* $A(\nu, \pi, w_\theta)$; *then* $C_\chi(\nu, \pi, \theta, w_\theta)$ *is non-zero.*

Proof. From Propositions 3.1.1, 3.1.2, and 3.1.3 it follows that for $-\bar{\nu} = \nu$

$$|C_\chi(\nu, \pi, \theta, w_\theta)|^2 = \gamma^{-2}(G/P)\mu(\pi; \nu). \tag{3.1.2}$$

By Theorem 20 of [8], it is clear that $\mu(\pi; \nu)$ has no pole for $\nu = -\bar{\nu}$ and therefore $C_\chi(\nu, \pi, \theta, w_\theta)$ is holomorphic there. Now from Proposition 2.4.1 and the previous relation, we get

$$|C_\chi(\nu, \pi, \theta, w_\theta)|^2 A(\nu, \pi, w_\theta) A(-\bar{\nu}, \pi, w_\theta)^* = 1.$$

But for $\nu = -\bar{\nu}$, $A(-\bar{\nu}, \pi, w_\theta)^* = A(\nu, \pi, w_\theta)^*$ and therefore the poles of $A(\nu, \pi, w_\theta)$ and $A(-\bar{\nu}, \pi, w_\theta)^*$ are the same. Consequently if ν is away from the poles of $A(\nu, \pi, w_\theta)$ then $C_\chi(\nu, \pi, \theta, w_\theta) \neq 0$.

Remark. From the Relation (3.1.2) it is clear that $\mu(\pi; \nu) \geq 0$ for $\nu = -\bar{\nu}$.

Now, we shall normalize the intertwining operators. Let

$$\mathcal{A}(\nu, \pi, w) = C_\chi(\nu, \pi, \theta, w) A(\nu, \pi, w).$$

Then we have:

PROPOSITION 3.1.4. *The operators* $\mathcal{A}(\nu, \pi, w)$ *have the following properties:*

(a) $\mathcal{A}(w(\nu), w(\pi), w^{-1})\mathcal{A}(\nu, \pi, w) = 1$
(b) $\mathcal{A}(\nu, \pi, w)^* = \mathcal{A}(-w(\bar{\nu}), w(\pi), w^{-1})$
(c) $\mathcal{A}(\nu, \pi, w)$ *is unitary if* ν *is imaginary, i.e.* $-\bar{\nu} = \nu$.

Proof. (a) follows from Propositions 3.1.1, 3.1.2, and 2.4.2. (b) is a result of Propositions 3.1.3 and 2.4.2. (c) follows from (a) and (b).

3.2. A Factorization of $C_\chi(\nu, \pi, \theta, w)$. With the notation as in Theorem 2.1.1, the intertwining operator $A(\nu, \pi, w)$ is the composite of the rank one intertwining operators $A(\nu_i, \pi_i, w_i)$, $1 \leq i \leq n-1$. We then have:

PROPOSITION 3.2.1.

$$C_\chi(\nu, \pi, \theta, w) = \prod_{i=1}^{n-1} C_\chi(\nu_i, \pi_i, \theta_i, w_i).$$

Proof. By definition, we have

$$\lambda(\nu_i, \pi_i, \theta_i, \chi) f_i = C_\chi(\nu_i, \pi_i, \theta_i, w_i) \lambda(\nu_{i+1}, \pi_{i+1}, \theta_{i+1}, \chi) \cdot A(\nu_i, \pi_i, w_i) f_i$$

for $1 \leq i \leq n-1$, where $f_i \in I(\nu_i, \pi_i, \theta_i)$ is given by

$$f_i = A(\nu_{i-1}, \pi_{i-1}, w_{i-1}) f_{i-1},$$

with f_1 a fixed function in $I(\nu, \pi, \theta)$. Now the proposition follows from the inductive use of the above relation and Theorem 2.1.1.

Factorization of $\epsilon'(s, \pi \times \pi', \chi)$. Suppose $G = GL_{n+m}$ and take $M = GL_n \times GL_m$ as in Section 2.3. Then $i = 1$ $a_1 = m + n$, and $r = \rho_n \otimes \tilde{\rho}_m$, where ρ_n and ρ_m are as in Section 2.3. Now let π_n and π_m be two irreducible admissible representations of $GL_n(F)$ and $GL_m(F)$, respectively. Consider $\pi = \pi_n \otimes \tilde{\pi}_m$ as a representation of $GL_n(F) \times GL_m(F)$.

Let s as before be a complex number and if $M = M_\theta$, let $w_\theta = w_{l,\Delta} w_{l,\theta}$. Now let $\epsilon'(s, \pi_n \times \pi_m, \chi)$ be the local coefficient appearing in the local functional equation of $\pi_n \times \pi_m$ defined by H. Jacquet, I. I. Piatetski Shapiro, and J. A. Shalika in [10]. More precisely

$$\epsilon'(s, \pi_n \times \pi_m, \chi) = \epsilon(s, \pi_n \times \pi_m, \chi) \frac{L(1 - s, \tilde{\pi}_n \times \tilde{\pi}_m)}{L(s, \pi_n \times \pi_m)}.$$

Also let $C_\chi(-2s/(m + n)\cdot\rho_\theta, \pi, \theta, w_\theta)$ be the local coefficient in the previous section. Then as it would be justified later (by means of the functional equation), one expects that

$$C_\chi(-2s/(m + n)\cdot\rho_\theta, \pi, \theta, w_\theta) = \epsilon'(s, \pi_n \times \pi_m, \chi) \qquad (3.2.1)$$

Now, let $m_1 + \cdots + m_r = m$ be a partition of m and put $P_{m_1, \ldots, m_r}$ for the standard parabolic subgroup of GL_m which has $GL_{m_1} \times \cdots \times GL_{m_r}$ as its Levi factor. Suppose

$$\tilde{\pi}_m \hookrightarrow \operatorname*{Ind}_{P_{m_1, \ldots, m_r}(F) \uparrow GL_m(F)} \tilde{\sigma}_1 \otimes \cdots \otimes \tilde{\sigma}_r,$$

where each σ_i, $1 \leq i \leq r$, is an irreducible admissible representation of $GL_{m_i}(F)$. In fact by Jacquet's quotient theorem this is always the case only if $\sigma_1, \ldots, \sigma_r$ are all supercuspidal. One of the interesting questions of Hecke theory is the equality

$$\epsilon'(s, \pi_n \times \pi_m, \chi) = \prod_{i=1}^{r} \epsilon'(s, \pi_n \times \sigma_i, \chi).$$

The proofs given in the context of Hecke theory are rather complicated (cf. [6, 13]). Here we shall use Proposition 3.2.1 to prove a parallel factorization of $C_\chi(-2s/(m+n)\cdot\rho_\theta, \pi, \theta, w_\theta)$. The proof is in fact very simple.

Given i, $1 \leq i \leq r$, let

$$M_i = GL_{m_1} \times \cdots \times GL_{m_{i-1}} \times GL_n \times GL_{m_i} \times \cdots \times GL_{m_r}$$

and

$$\Pi_i = \tilde{\sigma}_1 \otimes \cdots \otimes \tilde{\sigma}_{i-1} \otimes \pi_n \otimes \tilde{\sigma}_i \otimes \cdots \otimes \tilde{\sigma}_r.$$

Then

$$\pi = \pi_n \otimes \tilde{\pi}_m \hookrightarrow \Pi_1.$$

For $1 \leq i < j \leq n+m$, let $\alpha_{i,j}$ denote the corresponding positive root of GL_{n+m}. Set $p_j = n + \Sigma_{i=0}^{j} m_i$ with $m_o = 0$, $0 \leq j \leq m_{r-1}$. Then $\{\alpha_{p_j,p_j+1}\}_{j=0}^{m_r-1}$ is a subset of Δ.

Let $M_1 = M_{\theta_1}$, $\theta_1 \subset \Delta$. More precisely

$$\theta_1 = \Delta - \{\alpha_{p_l,p_l+1} \mid 0 \leq j \leq m_{r-1}\}.$$

Let $\Omega_1 = \theta_1 \cup \{\alpha_{p_o,p_o+1}\}$ and $w_1 = w_{l,\Omega_1}\cdot w_{l,\theta_1}$. Define $\theta_{i+1} = w_i(\theta_i)$, $\Omega_{i+1} = \theta_{i+1} \cup \{\alpha_{p_i,p_i+1}\}$, and $w_{i+1} = w_{l,\Omega_{i+1}}\cdot w_{l,\theta_{i+1}}$. Then $M_i = M_{\theta_i}$. Furthermore the cardinality of $\Sigma^+(\theta)$ is equal to r, and consequently from Lemma 2.1.2 and its corollary follows that $w_\theta = w_r \cdots w_1$. Also observe that $\Pi_{i+1} = w_i(\Pi_i)$ for $1 \leq i \leq r-1$. By inducing in stages it is clear that

$$C_\chi(-2s/(m+n)\cdot\rho_\theta, \pi, \theta, w_\theta) = C_\chi(-2s/(m+n)\cdot\rho_\theta, \Pi_1, \theta, w_\theta)$$

Now from Proposition 3.2.1, it follows that

$$C_\chi(-2s/(m+n)\cdot\rho_\theta, \pi, \theta, w_\theta) = \prod_{i=1}^{r} C_\chi(\nu_i, \Pi_i, \theta_i, w_i).$$

where

$$\nu_{i+1} = w_i(\nu_i), \qquad 1 \leq i \leq r,$$

and

$$\nu_1 = -s/m + n \cdot \sum_{\substack{1 \leq i \leq n \\ n+1 \leq j \leq n+m}} \alpha_{i,j}.$$

Fix i, and choose l and k two integers such that

$$\sum_{j=1}^{i-1} m_j < l \leq \sum_{j=1}^{i-1} m_j + n$$

and

$$\sum_{j=1}^{i-1} m_j + n < k \leq \sum_{j=1}^{i} m_j + n.$$

For the positive root $\alpha = (l, k)$, let

$$h_\alpha = \mathrm{diag}(1, \ldots, 1, \underset{\substack{\downarrow\\ l}}{\overline{\omega}}, 1, \ldots, 1, \underset{\substack{\downarrow\\ k}}{\overline{\omega}^{-1}}, 1, \ldots, 1),$$

where $\overline{\omega}$ is a uniformizing parameter in the ring of integers of F, i.e. $|\overline{\omega}| = q^{-1}$.

Now let $\overline{M}_i = \mathbf{1}_{m_1} \times \cdots \times \mathbf{1}_{m_{i-1}} \times GL_{n+m_i} \times \mathbf{1}_{m_{i+1}} \times \cdots \times \mathbf{1}_{m_r}$. Then $h_\alpha \in \overline{M}_i$ defines a coroot attached to a positive root of $\overline{M}_i$.

Now set $\overline{\Pi}_i = \Pi_i | \overline{M}_i \cap M_i$ and $\overline{\nu}_i = \nu_i | (\mathfrak{a}_{\theta_i})_{\mathbf{C}}^* \cap (\overline{\mathfrak{a}}_0)_{\mathbf{C}}^*$, where $\overline{\mathfrak{a}}_0$ is the real Lie algebra of diagonal elements of $\overline{M}_i$. Finally let $\overline{\theta}_i$ be the subset of elements in θ_i which are not trivial on $\mathfrak{a}_{\theta_i} \cap \overline{\mathfrak{a}}_0$. Each w_i may be considered as an element of the Weyl group of M_i. Then clearly

$$C_\chi(\nu_i, \Pi_i, \theta_i, w_i) = C_\chi(\overline{\nu}_i, \overline{\Pi}_i, \overline{\theta_i}, w_i),$$

where the one on the right hand side defines the same coefficient for $\overline{M}_i$. It is easy to see that $q^{\langle \overline{\nu}_i, H_\alpha \rangle} = q^{-s}$, and therefore one expects that

$$C_\chi(\overline{\nu}_i, \overline{\Pi}_i, \overline{\theta}_i, w_i) = \epsilon'(s, \pi_n \times \sigma_i, \chi).$$

Therefore we have proved:

THEOREM 3.2.1. *Suppose*

$$\epsilon'(s, \pi_n \times \pi_m, \chi) = C_\chi(-2s/(m+n)\cdot\rho_\theta, \pi_n \otimes \tilde{\pi}_m, \theta, w_\theta),$$

so that

$$\epsilon'(s, \pi_n \times \sigma_i, \chi) = C_\chi(\nu_i, \Pi_i, \theta_i, w_i),$$

$1 \leq i \leq r$. *Then*

$$\epsilon'(s, \pi_n \times \pi_m, \chi) = \prod_{i=1}^{r} \epsilon'(s, \pi_n \times \sigma_i, \chi)$$

Computation of $C_\chi(\nu, \pi, \theta, w)$ for real groups. Now suppose $F = \mathbf{R}$ and G is quasi-split (thus any complex group). Further assume that π is fine (e.g. unitary representations for $GL_n(\mathbf{R})$ or any complex group, cf. 1.3). Now by Casselman's subrepresentation theorem choose $\nu_0 \in (\mathfrak{a}_0)^*_{\mathbf{C}}$ and $\eta_0 \in \hat{M}_0$ such that $\pi_K \hookrightarrow I_M(\nu_0, \eta_0)_K$. Now as it was discussed before

$$\operatorname*{Ind}_{P \uparrow G} \pi_\infty \hookrightarrow I(\nu_0 + \tilde{\nu}, \eta_0)_\infty$$

for every $\nu \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$. Then

$$C_\chi(\nu, \pi, \theta, w) = C_\chi(\nu_0 + \tilde{\nu}, \eta_0, \phi, w)$$

and

$$\Sigma_r(\phi, \phi, w) = \{\beta \mid \beta \in \psi^+ - \Sigma_\theta^+, \text{ reduced, } w(\beta) \in \psi^-\}.$$

Given $\alpha \in \psi$, let $\overline{\mathfrak{m}_\alpha}$ be the Lie algebra generated by $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, and their Lie product. Denote by $\overline{M}_\alpha$ the analytic subgroup of G whose Lie algebra is $\overline{\mathfrak{m}_\alpha}$.

Fix i, $1 \leq i \leq n-1$, then with the notation as in Section 2.2, $\overline{M}_{\alpha_i} \subset M_i'$ is a quasi-split semi-simple real group of rank one. Let $\tilde{M}_{\alpha_i}$ be the simply connected covering of $\overline{M}_{\alpha_i}$. There exists a homomorphism ϕ_i from $\tilde{M}_{\alpha_i}$ onto $\overline{M}_{\alpha_i}$. There are two cases to be considered.

Case 1. $\tilde{M}_{\alpha_i} \cong SL_2(F)$, $F = \mathbf{R}$ or $\mathbf{C}$. For $\eta \in \hat{K}^*$ and $s \in \mathbf{C}$, let $\gamma(\eta, s)$ be the local coefficient appearing in the Tate's functional equation [29] defined by relation (4.16) of [23].

Let $\overline{\eta}_{0,\alpha_i} = \eta_{0,i} | \overline{M}_{\alpha_i} \cap M_0$ and $\beta_i = w_1^{-1} \cdots w_{i-1}^{-1}(\alpha_i)$. Then $\overline{\eta}_{0,\alpha_i} \cdot \phi_{\alpha_i} = \overline{\eta}_{0,\beta_i} \cdot \phi_{\beta_i}$, where $\overline{\eta}_{0,\beta_i} = \eta_0 | \overline{M}_{\beta_i} \cap M_0$ which is independent of the decomposition. Here, in the case of $SL_2(\mathbf{C})$, we realize $\tilde{M}_{\alpha_i}$ as a quasi-split real group.

Now from Lemma 4.4 of [23] or computations in [24] it follows that

$$C_\chi((\nu_0 + \tilde{\nu})_i, \eta_{0,i}, \phi, w_i) = \gamma(\overline{\eta}_{0,\beta_i} \cdot \phi_{\beta_i}, \langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle).$$

Case 2. $\tilde{M}_{\alpha_i} = SU(2, 1)$. For the sake of simplicity suppose η_0 is trivial. In this case computation in [26] shows that

$$C_\chi((\nu_0 + \tilde{\nu})_i, 1, \phi, w_i) = \pi^{-1/2} \frac{\Gamma(\frac{1}{4}(4 - \langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle))}{\Gamma(\frac{1}{4}(2 - \langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle))}$$

$$\cdot \gamma_{\mathbf{C}}(1, \tfrac{1}{2}\langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle)$$

where $\gamma_{\mathbf{C}}(1, \frac{1}{2}\langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle)$ is the coefficient of the local Tate functional equation with $F = \mathbf{C}$ explained in Case 1. More precisely

$$\gamma_{\mathbf{C}}(1, \tfrac{1}{2}\langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle) = \frac{(2\pi)^{(1/2)\langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle}}{(2\pi)^{1-(1/2)\langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle}} \cdot \frac{\Gamma(\frac{1}{2}(2 - \langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle))}{\Gamma(\frac{1}{2}\langle \nu_0 + \tilde{\nu}, H_{\beta_i} \rangle)}$$

Now, given $\beta \in \Sigma_r(\phi, \phi, w)$, let $p_\beta = \dim \mathfrak{g}_\beta$, $q = \dim \mathfrak{g}_{2\beta}$, $\rho_\beta = p_\beta + 2q_\beta$. Furthermore, let

$$\gamma_\beta(1, \langle \nu_0 + \tilde{\nu}, H_\beta \rangle) = \gamma(1, \langle \nu_0 + \tilde{\nu}, H_\beta \rangle)$$

if $\tilde{M}_\beta \cong SL_2(\mathbf{R})$ or $SL_2(\mathbf{C})$ as in case 1, and

$$\gamma_\beta(1, \langle \nu_0 + \tilde{\nu}, H_\beta \rangle) = \gamma_{\mathbf{C}}(1, \tfrac{1}{2}\langle \nu_0 + \tilde{\nu}, H_\beta \rangle)$$

otherwise. Then from Proposition 3.2.1 we conclude

THEOREM 3.2.2. *Suppose* $\eta_0 \equiv 1$. *Then*

$$C_\chi(\nu, \pi, \theta, w) = \prod_{\beta \in \Sigma_r(\phi, \phi, w)} \pi^{-q_\beta/2} \cdot \frac{\Gamma(\frac{1}{4}(\rho_\beta - \langle \nu_0 + \tilde{\nu}, H_\beta \rangle))}{\Gamma(\frac{1}{4}(p_\beta - \langle \nu_0 + \tilde{\nu}, H_\beta \rangle))}$$

$$\cdot \gamma_\beta(1, \langle \nu_0 + \tilde{\nu}, H_\beta \rangle)$$

3.3. Relation with the irreducibility of the induced representations. A representation π of M_θ is called *unramified* if $w(\pi) \cong \pi$, $w \in W(\theta, \theta)$, implies that $w = 1$. Let $w_\theta = w_{l,\Delta} \cdot w_{l,\theta}$. Now suppose F is non-archimedean. Then we have

THEOREM 3.3.1. *Suppose* π *is an irreducible non-degenerate unitary supercuspidal representation of* M_θ. *Fix* $\nu_0 \in (\mathfrak{a}_\theta)^*_{\mathbf{C}}$. *Suppose* $\pi \otimes q^{\langle \nu_0, H_\theta(\ ) \rangle}$ *is unramified. Then* $I(\nu_0, \pi, \theta)$ *is irreducible if and only if both*

$C_\chi(\nu, \pi, \theta, w_\theta)$ and $C_\chi(w_\theta(\nu), w_\theta(\pi), w_\theta(\theta), w_\theta{}^{-1})$ are holomorphic at $\nu = \nu_0$.

Proof. Observe first that by Corollary 5.4.2.2 of [28] each of the rank one c-functions and consequently each of the rank one intertwining operators are holomorphic at $\nu = \nu_0$, and therefore $A(\nu_0, \pi, w)$ is defined. Next suppose $I(\nu_0, \pi, \theta)$ is irreducible. Then

$$\lambda(w_\theta(\nu_0), w_\theta(\pi), w_\theta(\theta), \chi)\cdot A(\nu_0, \pi, w)$$

and

$$\lambda(\nu_0, \pi, \theta, \chi)\cdot A(w_\theta(\nu_0), w_\theta(\pi), w_\theta{}^{-1})$$

are both defined and non-zero. This implies the only if part. Conversely suppose $C_\chi(\nu, \pi, \theta, w_\theta)$ and $C_\chi(w_\theta(\nu), w_\theta(\pi), w_\theta(\theta), w^{-1})$ are both holomorphic at $\nu = \nu_0$. Then by Proposition 3.1.1 $\mu(\pi; \nu)$ is holomorphic at $\nu = \nu_0$, and consequently by Theorem 5.4.3.7 of [28] $I(\nu_0, \pi, \theta)$ is irreducible.

COROLLARY. *With the same assumption as in Theorem* 3.3.1 *suppose $C_\chi(\nu, \pi, \theta, w_\theta)$ has a pole at $\nu = \nu_0$. Then $I(w_\theta(\nu_0), w_\theta(\pi), w_\theta(\theta))$ is reducible and the image of $A(\nu_0, \pi, w_\theta)$ in $I(w_\theta(\nu_0), w_\theta(\pi), w_\theta(\theta))$ is degenerate.*

Proof. First observe that $A(\nu_0, \pi, w_\theta)$ is defined since $\pi \otimes q^{\langle \nu_0, H_\theta(\ )\rangle}$ is unramified. Now the corollary follows from the definition of $C_\chi(\nu, \pi, \theta, w_\theta)$ and that it has a pole at $\nu = \nu_0$.

The following proposition follows from the definitions. Here F is any local field (archimedean or non-archimedean).

PROPOSITION 3.3.1. *Let π be an irreducible admissible non-degenerate representation of M_θ.*

a) *Denote by $\mathfrak{A}(\nu, \pi, w)$ the normalized intertwining operator explained in Section* 3.1. *Suppose $\mathfrak{A}(\nu, \pi, w)$ is holomorphic at $\nu = \nu_0$. Then the image of $\mathfrak{A}(\nu_0, \pi, w)$ is non-degenerate* ($w \in W$).

b) *The zeros of $C_\chi(\nu, \pi, \theta, w)$ are among the poles of $A(\nu, \pi, w)$.*

4. Functional equations. For the rest of this paper we shall assume that G is a connected reductive algebraic group which, for the sake of simplicity, we further assume splits over $\mathbf{Q}$.

We let F be a number field. Given a place v of F, we let G_v denote the group of F_v-rational points of G. We use G_F for the group of F-rational points. Let $\mathbf{A}$ be the ring of adèles of F. We write $G_{\mathbf{A}}$ for the corresponding adèlized group. We use same indices to denote the corresponding subgroups.

Throughout the rest of this paper we shall be using the notation and results of [23], and in fact we shall only formulate the last theorem of [23] since all the intermediate steps go through the same way.

Fix $\theta \subset \Delta$, and let P_θ be the corresponding parabolic subgroup of G. Write $P_\theta = M_\theta N_\theta$.

Let π be an irreducible admissible cuspidal representation of $M_{\theta,\mathbf{A}}$ (cf. [17]). Write $\pi = \otimes_v \pi_v$, where each π_v is an irreducible unitary representation of $M_{\theta,v}$. Let ϕ be a function in the space of π, and define ${}^oW$ by

$$ {}^oW(g) = \int_{{}^oU_{\mathbf{A}}/{}^oU_F} \phi(gu)\overline{\chi(u)du}, $$

where the notation is as in [23]. We say π is non-degenerate if for some ϕ, ${}^oW$ is non-zero. Then each π_v is non-degenerate. Suppose ${}^oW = \otimes_v {}^oW_v$. We also assume that ϕ is both $K \cap M_{\theta,\mathbf{A}}$-finite and $\mathcal{Z}_{\theta,\infty}$-finite, where K is the maximal compact subgroup of $G_{\mathbf{A}}$ explained in [23], and $\mathcal{Z}_{\theta,\infty}$ is the center of the universal enveloping algebra of $M_{\theta,\infty}$.

We extend ϕ to $\tilde{\phi}$ on $G_{\mathbf{A}}$ as in [23] and define Φ_s, $s \in \mathbf{C}$, again as in [23]. Now, let

$$ E(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}}) = \sum_{\gamma \in G_F/P_{\theta,F}} \Phi_{-s}(g\gamma) $$

be the corresponding *Eisenstein series*, and define the *Fourier coefficient* of $E(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}})$ by

$$ \underline{\mathrm{E}}_\chi(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}}) = \int_{U_{\mathbf{A}}/U_F} E(s; \tilde{\phi}; gu; P_{\theta,\mathbf{A}})\overline{\chi(u)}du $$

(cf. [23]). The fact that $\underline{\mathrm{E}}_\chi(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}})$ is non-zero for large s will be studied in Section 5 (under the condition that π_v is fine at every archimedean place).

Let S be the smallest set of places which includes all the archimedean ones, and is such that for every $v \notin S$, both π_v and χ_v are unramified (the set of ramified places). For each $v \notin S$, let $L(s, \pi_v, r_i)$ be the corresponding Langlands' L-function explained in Section 2.3. Now, define

$$L_S(s, \pi, r_i) = \prod_{v \notin S} L(s, \pi_v, r_i), \qquad 1 \leq i \leq m.$$

Then by the results in [15], $L_S(s, \pi, r_i)$ is a meromorphic function of s in $\mathbf{C}$. Now using the computations in Section 3 of this paper and the results of [23] we have:

THEOREM 4.1. *Let π be an irreducible admissible non-degenerate cuspidal representation of $M_{\theta,\mathbf{A}}$. Write $\pi = \otimes_v \pi_v$. Assume that for every archimedean place v, π_v is fine. Define $L_S(s, \pi, r_i)$, $1 \leq i \leq m$, as above. Then*

$$\prod_{i=1}^{m} L_S(a_i s, \pi, r_i) = \prod_{v \in S} C_{\chi_v}(-2s\rho_\theta, \pi_v, \theta, w_\theta) \prod_{i=1}^{m} L_S(1 - a_i s, \pi, \tilde{r}_i).$$

Example. As mentioned before, when $G = GL_{n+m}$ and $M = GL_n \times GL_m$, our approach leads to the L-functions considered by H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika (see Section 2.3 here). Therefore Theorem 4.1 provides another proof for their functional equation. Observe that in this case, since π_v is unitary everywhere, it is fine at every archimedean place (cf. [24]).

Now, let G be as in Section 1.2, and let ${}^\circ G_\theta = M_\theta / A_\theta$. Let ρ_θ: $M_\theta \to {}^\circ G_\theta$ be the natural projection. This defines a map ${}^L\rho_\theta$: ${}^L({}^\circ G_\theta) \to {}^L M_\theta$. Let π be an irreducible admissible non-degenerate cuspidal (unitary) representation of ${}^\circ G_{\theta,\mathbf{A}}$. Then $\pi \cdot \rho_\theta$ will be one of $M_{\theta,\mathbf{A}}$. Furthermore for each i, $1 \leq i \leq m$, ${}^\circ r_i = r_i \cdot {}^L\rho_\theta$ is a representation of ${}^L({}^\circ G_\theta)$. Now for every v, $v \in S$, define $L(s, \pi_v, {}^\circ r_i)$ by

$$L(s, \pi_v, {}^\circ r_i) = L(s, (\pi \cdot \rho_\theta)_v, r_i),$$

where $L(s, (\pi \cdot \rho_\theta)_v, r_i)$ is the corresponding unramified Langlands' L-function for $M_{\theta,v}$.

Finally define $L_S(s, \pi, {}^\circ r_i)$, $1 \leq i \leq m$, as before. Then:

THEOREM 4.2. *Let π be an irreducible admissible non-degenerate cuspidal representation of ${}^\circ G_{\theta,\mathbf{A}}$. Write $\pi = \otimes_v \pi_v$. Assume that for every archimedean place v, π_v is fine. Define $L_S(s, \pi, {}^\circ r_i)$, $1 \leq i \leq m$, as above.*

Then

$$\prod_{i=1}^{m} L_S(a_i s, \pi, {}^0 r_i) = \prod_{v \in S} C_{\chi_v}(-2s\rho_\theta, (\pi \cdot \rho_\theta)_v, \theta, w_\theta) \prod_{i=1}^{m} L_S(1 - a_i s, \pi, {}^0\tilde{r}_i).$$

Remark 4.1. There are many cases for which cuspidal representations of ${}^0G_{\theta,\mathbf{A}}$ are known rather than those of $M_{\theta,\mathbf{A}}$. For these cases Theorem 4.2 is very useful (cf. [23]). In the next section we shall consider some new examples of such cases. They are based on an idea due to P. Deligne (cf. [4]).

We now formulate an equivalent form of Theorem 4.1 in terms of global intertwining operators (cf. [17]).

Let ϕ be a function in the space of π. Suppose $\phi = \otimes_v \phi_v$. We extend ϕ to a function $\tilde{\phi}$ on $G_{\mathbf{A}}$ as in [23]. Then $\tilde{\phi} = \otimes_v \tilde{\phi}_v$, where each $\tilde{\phi}_v \in I(-2s\rho_\theta, \pi_v, \theta)$. Furthermore for almost all v, $\tilde{\phi}_v$ is fixed by K_v. Now, we define $A(-2s\rho_\theta, \pi, w_\theta)$ by

$$A(-2s\rho_\theta, \pi, w_\theta)\tilde{\phi} = \otimes_v A(-2s\rho_\theta, \pi_v, w_\theta)\tilde{\phi}_v.$$

Then in fact $A(-2s\rho_\theta, \pi, w_\theta)$ is defined on the restricted tensor product of the spaces $I(-2s\rho_\theta, \pi_v, \theta)$ which we denote by $I(-2s\rho_\theta, \pi, \theta)$. This is the global induced representation from $P_{\theta,\mathbf{A}}$ to $G_{\mathbf{A}}$ (cf. [18]).

Now at each place v, let $\mathcal{A}(-2s\rho_\theta, \pi_v, \theta)$ be the normalized intertwining operator defined in Section 3.1. We then define a global normalized intertwining operator $\mathcal{A}(-2s\rho_\theta, \pi, \theta)$ by

$$\mathcal{A}(-2s\rho_\theta, \pi, w_\theta) = \bigotimes_v \mathcal{A}(-2s\rho_\theta, \pi_v, w_\theta).$$

In fact, given $\tilde{\phi} = \otimes_v \tilde{\phi}_v$, let S be a finite set of places, including the infinite ones, such that $\tilde{\phi}$ is invariant under K_v and χ_v is unramified for all $v \notin S$. Then, we define

$$\mathcal{A}(-2s\rho_\theta, \pi, w_\theta)\tilde{\phi} = \prod_{i=1}^{m} \frac{L_S(1 - a_i s, \pi, r_i)}{L_S(a_i s, \pi, \tilde{r}_i)} \bigotimes_{v \in S} \mathcal{A}(-2s\rho_\theta, \pi_v, w_\theta)\tilde{\phi}_v.$$

Clearly $\mathcal{A}(-2s\rho_\theta, \pi, w_\theta)$ is well defined on $I(-2s\rho_\theta, \pi, \theta)$.

The following theorem is equivalent to Theorem 4.1.

Theorem 4.3. *Let* $A(-2s\rho_\theta, \pi, w_\theta)$ *and* $\mathcal{A}(-2s\rho_\theta, \pi, w_\theta)$ *be the global intertwining operators defined above. Then*

$$\mathcal{A}(-2s\rho_\theta, \pi, w_\theta) = A(-2s\rho_\theta, \pi, w_\theta).$$

In particular

$$\mathcal{A}(-2s\rho_\theta, \pi, w_\theta)\mathcal{A}(2s\rho_{w_\theta(\theta)}, w_\theta(\pi), w_\theta^{-1}) = I.$$

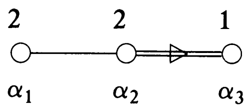
Proof. Let S be a finite set of places, including the infinite ones, such that π_v and χ_v are both unramified outside S. Then the functional equation in Theorem 4.1 implies that

$$\prod_{v \in S} C_{\chi_v}(-2s\rho_\theta, \pi_v, \theta, w_\theta) \prod_{i=1}^{m} \frac{L_S(1 - a_i s, \pi, r_i)}{L_S(a_i s, \pi, \tilde{r}_i)} = 1.$$

Now the theorem follows from the definition of $\mathcal{A}(-2s\rho_\theta, \pi, w_\theta)$. The second statement follows from the same statement for $A(-2s\rho_\theta, \pi, w)$ (cf. [17]).

4.1. Examples; fourth and fifth symmetric power representations of GL_2.

4.1. a. Let G be a simple algebraic group of type B_3 (example (iv) of [15]). As before we assume that G splits over $\mathbf{Q}$. Its Dynkin diagram is given by



Let $\theta = \{\alpha_1, \alpha_2\}$. Then M_θ is of type A_2 and ${}^\circ G_\theta \cong PSL_3$. Let Π be a cuspidal representation of $PGL_3(\mathbf{A})$. Regard Π as a representation of $PSL_3(\mathbf{A})$ by restriction. In this case ${}^L({}^\circ G_\theta) \cong SL_3(\mathbf{C})$, and if (λ_1, λ_2) are the fundamental weights of $SL_3(\mathbf{C})$, then $m = 1$, ${}^\circ r_1 = 2\lambda_1$, and $a_1 = 6$. The irreducible representation ${}^\circ r_1$ may be considered as the second symmetric power of the standard representation ρ_3 of $SL_3(\mathbf{C})$. Then the corresponding L-function is $L(s, \Pi, \mathrm{Sym}^2(\rho_3))$.

There is a distinguished irreducible admissible cuspidal representation of $PGL_3(\mathbf{A})$ whose existence is due to S. Gelbart and H. Jacquet [5]. In fact, let π be a cuspidal representation of $PGL_2(\mathbf{A})$ which is not monomial (cf. [4, 5]). Then π has a lift to a cuspidal representation Π of $PGL_3(\mathbf{A})$. Furthermore the corresponding L-function is equal to $L(s, \pi, \mathrm{Sym}^2(\rho_2))$, where ρ_2 is the standard representation of $SL_2(\mathbf{C})$. Let us now prove the following lemma.

LEMMA 4.1.1. *Let Λ^n denote the n-th exterior power. Then*

a) $\mathrm{Sym}^2(\mathrm{Sym}^2(\rho_2)) = \mathrm{Sym}^4(\rho_2) \oplus (\Lambda^2(\rho_2))^{\otimes 2}$

b) $\mathrm{Sym}^2(\mathrm{Sym}^2(\rho_2)) \otimes \rho_2 = \mathrm{Sym}^5(\rho_2) \oplus \mathrm{Sym}^3(\rho_2) \otimes \Lambda^2(\rho_2) \oplus (\Lambda^2(\rho_2))^{\otimes 2} \otimes \rho_2$,

where

$$V^{\otimes n} = \otimes^n V = \overbrace{V \otimes \cdots \otimes V}^{n}.$$

Proof. By Clebsch-Gordan formula, one has

$$\otimes^2 \mathrm{Sym}^2(\rho_2) = \mathrm{Sym}^4(\rho_2) \oplus \mathrm{Sym}^2(\rho_2) \otimes \Lambda^2(\rho_2) \oplus (\Lambda^2(\rho_2))^{\otimes 2}.$$

But

$$\otimes^2(\mathrm{Sym}^2(\rho_2)) = \mathrm{Sym}^2(\mathrm{Sym}^2(\rho_2)) \oplus \Lambda^2(\mathrm{Sym}^2(\rho_2))$$

and

$$\Lambda^2(\mathrm{Sym}^2(\rho_2)) = \mathrm{Sym}^2(\rho_2) \otimes \Lambda^2(\rho_2).$$

Now comparing the two expressions for $\otimes^2 \mathrm{Sym}^2(\rho_2)$ implies part (a).

For part (b), we use:

$$\mathrm{Sym}^2(\mathrm{Sym}^2(\rho_2)) \otimes \rho_2 = \mathrm{Sym}^4(\rho_2) \otimes \rho_2 \oplus (\Lambda^2(\rho_2))^{\otimes 2} \otimes \rho_2$$

and

$$\mathrm{Sym}^4(\rho_2) \otimes \rho_2 = \mathrm{Sym}^5(\rho_2) \oplus \mathrm{Sym}^3(\rho_2) \otimes \Lambda^2(\rho_2).$$

Now, let ψ: $SL_2(\mathbf{C}) \to SL_3(\mathbf{C})$ be the representation $\mathrm{Sym}^2(\rho_2)$. By Langlands' principle of functoriality there exists a map ψ_* (cf. [5]) such that $\psi_*(\pi)$ is a cuspidal representation of $PGL_3(\mathbf{A})$ and

$$L(s, \pi, r \cdot \psi) = L(s, \psi_*(\pi), r)$$

for any representation r of $SL_3(\mathbf{C})$. In fact $\psi_*(\pi) = \Pi$, where Π is the lift of π explained before. Now, let $r = \mathrm{Sym}^2(\rho_3)$. Then

$$L(s, \pi, \mathrm{Sym}^2(\mathrm{Sym}^2(\rho_2))) = L(s, \Pi, \mathrm{Sym}^2(\rho_3)).$$

But now by part (a), of the lemma, we must have

$$L(s, \Pi, \mathrm{Sym}^2(\rho_3)) = L(s, 1)L(s, \pi, \mathrm{Sym}^4(\rho_2)),$$

where $L(s, 1)$ is the Hecke L-function.

Write $\pi = \otimes_v \pi_v$, and define the local coefficients by

$$\epsilon'(s, \pi_v, \mathrm{Sym}^4(\rho_2), \chi_v) = \frac{1 - q_v^{-(1-s)}}{1 - q_v^{-s}} C_{\chi_v}\left(-\frac{1}{3} s\rho_\theta, \pi_v, \theta, w_\theta\right).$$

Let

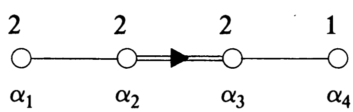
$$\epsilon'_S(s, \pi, \mathrm{Sym}^4(\rho_2), \chi) = \prod_{v \in S} \epsilon'(s, \pi_v, \mathrm{Sym}^4(\rho_2), \chi_v).$$

Then using Hecke's functional equation and Theorem 4.2, we have:

THEOREM 4.1.1. *Let π be an irreducible cuspidal representation of $PGL_2(\mathbf{A})$. Assume that π is not monomial. Define $L_S(s, \pi, \mathrm{Sym}^4(\rho_2))$ as before. Then*

$$L_S(s, \pi, \mathrm{Sym}^4(\rho_2)) = \epsilon'_S(s, \pi, \mathrm{Sym}^4(\rho_2), \chi) L_S(1 - s, \pi, \mathrm{Sym}^4(\tilde{\rho}_2)).$$

4.1.b. Next, let G be a simple algebraic group of type F_4. We again assume that G splits over $\mathbf{Q}$. Its Dynkin diagram is given by



Let $\theta = \{\alpha_1, \alpha_2, \alpha_4\}$. Then M_θ is of type $A_2 \times A_1$ and ${}^\circ G_\theta \cong PSL_3 \times PSL_2$. As before, let π and Π be irreducible admissible cuspidal representations of $PGL_2(\mathbf{A})$ and $PGL_3(\mathbf{A})$, respectively.

The group ${}^L({}^\circ G_\theta)$ is isomorphic to $SL_3(\mathbf{C}) \times SL_2(\mathbf{C})$, and $m = 3$, $a_1 = 21$, $a_2 = 14$, $a_3 = 7$, $r_1 \cong \mathbf{1} \otimes \rho_2$, $r_2 \cong \mathrm{Sym}^2(\tilde{\rho}_3) \otimes \mathbf{1}$, and $r_3 = \mathrm{Sym}^2(\rho_3) \otimes \rho_2$, where ρ_2 and ρ_3 are standard representations of $SL_2(\mathbf{C})$ and $SL_3(\mathbf{C})$, respectively.

Now, suppose π is not monomial and Π is the lift of π as before, i.e. $\Pi = \psi_*(\pi)$. Define the maps

$$\psi \otimes \mathbf{1}: SL_2(\mathbf{C}) \times SL_2(\mathbf{C}) \to SL_3(\mathbf{C}) \times SL_2(\mathbf{C})$$

and

$$\Delta: SL_2(\mathbf{C}) \to SL_2(\mathbf{C}) \times SL_2(\mathbf{C}),$$

where ψ is defined as before and Δ is the diagonal map $\Delta(g) = (g, g)$. Let $\psi_* \otimes \mathbf{1}$ and Δ_* be the corresponding conjectural maps induced by the principle of functoriality. Then for any pair of representations r and R of $SL_2(\mathbf{C})$ and $SL_3(\mathbf{C})$ we have

$$\begin{aligned} L(s, (\psi_* \otimes \mathbf{1})(\pi \times \pi), R \otimes r) &= L(s, \pi \times \pi, (R \otimes r)\cdot(\psi \otimes \mathbf{1})) \\ &= L(s, \Delta_*(\pi), (R \otimes r)\cdot(\psi \otimes \mathbf{1})) \\ &= L(s, \pi, (R \otimes r)\cdot(\psi \otimes \mathbf{1})\cdot\Delta). \end{aligned}$$

Let us first specialize this to $r = \rho_2$ and $R = \mathbf{1}$. We then have

$$\begin{aligned} L(s, \psi_*(\pi) \times \pi, \mathbf{1} \otimes \rho_2) &= L(s, \pi, \rho_2) \\ &= L(s, \pi), \end{aligned}$$

where the last one is Jacquet-Langlands' L-function for GL_2. Next take $r = \mathbf{1}$ and $R = \mathrm{Sym}^2(\tilde{\rho}_3)$. Since $\rho_2 = \tilde{\rho}_2$, we then obtain

$$L(s, \psi_*(\pi) \times \pi, \mathrm{Sym}^2(\tilde{\rho}_3) \otimes \mathbf{1}) = L(s, \mathbf{1})L(s, \pi, \mathrm{Sym}^4(\rho_2))$$

as in 4.1.a. Finally take $r = \rho_2$ and $R = \mathrm{Sym}^2(\rho_3)$. Then using part (b) of Lemma 4.1.1 we conclude

$$L(s,\ \psi_*(\pi) \times \pi,\ \mathrm{Sym}^2(\rho_3) \otimes \rho_2) = L(s,\ \pi,\ \mathrm{Sym}^5(\rho_2))L(s,\ \pi,\ \mathrm{Sym}^3(\rho_2))L(s,\ \pi,\ \rho_2).$$

Let us now define the local coefficients $\epsilon'(s,\ \pi_v,\ \mathrm{Sym}^5(\rho_2),\ \pi_v)$, *where* $\pi = \otimes_v \pi_v$.

Let $\epsilon'(s,\ \pi_v,\ \chi_v)$ be $\epsilon(s,\ \pi_v,\ \chi_v)[L(1-s,\ \tilde{\pi}_v)/L(s,\ \pi_v)]$ as defined by Jacquet-Langlands' [11]. Denote by $\epsilon'(s,\ \pi_v,\ \mathrm{Sym}^3(\rho_2),\ \chi_v)$ the local coefficient defined in [23]. Finally, let $\epsilon'(s,\ \pi_v,\ \mathrm{Sym}^4(\rho_2),\ \chi_v)$ be the local coefficient defined in Theorem 4.1.1. Now, set

$$\begin{aligned}\epsilon'(s,\ \pi_v,\ \mathrm{Sym}^5(\rho_2),\ \chi_v) &= \epsilon'(3s,\ \pi_v,\ \chi_v)^{-1}\epsilon'(2s,\ 1_v)^{-1}\epsilon'(2s,\ \pi_v,\ \mathrm{Sym}^4(\rho_2),\ \chi_v)^{-1}\\ &\cdot \epsilon'(s,\ \pi_v,\ \chi_v)^{-1}\epsilon'(s,\ \pi_v,\ \mathrm{Sym}^3(\rho_2),\ \chi_v)^{-1}\\ &\cdot C_{\chi_v}(-2/7 s\rho_\theta,\ \psi_*(\pi_v) \times \pi_v,\ \theta,\ w_\theta).\end{aligned}$$

Observe that $\mathrm{Sym}^5(\rho_2)$ is a six dimensional irreducible representation of $SL_2(\mathbf{C})$. Now we have:

THEOREM 4.1.2. *Let π be an irreducible cuspidal representation of $PGL_2(\mathbf{A})$. Suppose π is not monomial. Define*

$$L_S(s,\ \pi,\ \mathrm{Sym}^5(\rho_2)) = \prod_{v \notin S} L(s,\ \pi_v,\ \mathrm{Sym}^5(\rho_2))$$

and

$$\epsilon'_S(s,\ \pi,\ \mathrm{Sym}^5(\rho_2),\ \chi) = \prod_{v \in S} \epsilon'(s,\ \pi_v,\ \mathrm{Sym}^5(\rho_2),\ \chi_v),$$

where S is the set of ramified places, and $L(s,\ \pi_v,\ \mathrm{Sym}^5(\rho_2))$ is defined as in Section 2.3 for $v \notin S$. Then:

a) *$L_S(s,\ \pi,\ \mathrm{Sym}^5(\rho_2))$ extends to a meromorphic function of s in $\mathbf{C}$, and*

b) *It satisfies the functional equation*

$$L_S(s,\ \pi,\ \mathrm{Sym}^5(\rho_2)) = \epsilon'_S(s,\ \pi,\ \mathrm{Sym}^5(\rho_2),\ \chi)L_S(1-s,\ \pi,\ \mathrm{Sym}^5(\tilde{\rho}_2)).$$

Remark 4.1.1. Suppose $\theta = \{\alpha_1,\ \alpha_3,\ \alpha_4\}$. Then M_θ is of type $A_1 \times A_2$. Let $\pi \times \Pi$ be a cuspidal representation of ${}^0G_{\theta,\mathbf{A}}$. Then $m = 4$, $a_1 =$

20, $a_2 = 15$, $a_3 = 10$, $a_4 = 5$, $r_1 \cong \mathbf{1} \otimes \tilde{\rho}_3$, $r_2 \cong \rho_2 \otimes \mathbf{1}$, $r_3 \cong \mathrm{Sym}^2(\rho_2) \otimes \rho_3$, and $r_4 \approx \rho_2 \otimes \tilde{\rho}_3$. In particular if $\Pi = \psi_*(\pi)$, one obtains $L(s, \pi, \mathrm{Sym}^4(\rho_2))$ again.

5. Non-vanishing of L-functions. One of the interesting characteristics of this approach is the fact that it provides an explicit relation between the zeros of L-functions and the poles of the Eisenstein series. The following theorem is a consequence of the holomorphy of the Eisenstein series along the imaginary axis. Its special cases have very interesting applications which we shall discuss after the proof of this main theorem.

THEOREM 5.1. a) *Let π be an irreducible admissible non-degenerate cuspidal representation of $M_{\theta,\mathbf{A}}$. Write $\pi = \otimes_v \pi_v$. Assume that every π_v is fine whenever v is archimedean. Let S be a finite set of places including the infinite ones, such that for every $v \notin S$, both π_v and χ_v are unramified. Set*

$$L_S(s, \pi, r_j) = \prod_{v \notin S} L(s, \pi_v, r_j) \qquad (1 \leq j \leq m).$$

Then for every $t \in \mathbf{R}$, the product

$$\prod_{j=1}^{m} L_S(1 + a_j t(-1)^{1/2}, \pi, r_j)$$

is non-zero.

b) *All the statements remain true if we replace $M_{\theta,\mathbf{A}}$ by ${}^\circ G_{\theta,\mathbf{A}}$ and r_j by ${}^\circ r_j$, $1 \leq j \leq m$.*

Proof. Let $E_\chi(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}})$ be the matrix coefficient of the Fourier coefficient $\underline{E}_\chi(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}})$, as it is explained after Lemma 5.1 of [23]. Set

$$I_v = I(-2s\rho_\theta, \pi_v, \theta), \text{ and } \lambda_v = \lambda(-2s\rho_\theta, \pi_v, \theta, \chi_v).$$

Suppose $\tilde{\phi} = \otimes_v \tilde{\phi}_v$ with $\tilde{\phi}_v \in I_v$, and define $W_v(g_v) = \lambda_v(I_v(g_v{}^{-1})\tilde{\phi}_v)$ $g_v \in G_v$. Now using Lemma 2.1 and computations in Section 4 of [23] we conclude that for $\mathrm{Re}(s) > 1/2$.

$$E_\chi(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}}) = \prod_v W_v(g_v),$$

where $g = (g_v)$. In fact when v is archimedean, it is clear that the local component of $E_\chi(s; \tilde{\phi}; g; P_{\theta,\mathbf{A}})$ at v, which is given by Formula (4.2) of [23] (changing s to $-s$), converges absolutely for Re(s) large (it is dominated by the corresponding intertwining integral which converges for Re(s) large). Consequently for Re(s) large it coincides with $\lambda(-2s\rho_\theta, \pi_v, \theta, \chi_v)$ (Relation 3.1 of [24]). Now the analytic continuation of this local component follows from that of $\lambda(-2s\rho_\theta, \pi_v, \theta, \chi_v)$ which is provided by Proposition 3.2. The non-archimedean case follows from Proposition 3.1.

Now take $g = e = (e_v)$, the identity element of $G_\mathbf{A}$, and write

$$E_\chi(s; \tilde{\phi}; e; P_{\theta,\mathbf{A}}) = \prod_{v\in S} W_v(e_v) \cdot \prod_{v\notin S} W_v(e_v).$$

It is a result of W. Casselman and J. A. Shalika [3] that for every $v \notin S$

$$W_v(e_v) = \prod_{j=1}^{m} L(1 + a_j s, \pi_v, r_j)^{-1},$$

and consequently

$$E_\chi(s; \tilde{\phi}; e, P_{\theta,\mathbf{A}}) = \prod_{v\in S} W_v(e_v) \prod_{j=1}^{m} L_S(1 + a_j s, \pi, r_j)^{-1}.$$

Here ϕ_v is so normalized that ${}^\circ W_v(e_v) = 1$ for all $v \notin S$. Suppose $v \in S$. Then from Propositions 3.1 or 3.2 it follows that we may choose ϕ_v (and consequently $\tilde{\phi}_v$) in such a way that $W_v(e_v) \neq 0$. Consequently the zeros of $\prod_{j=1}^{m} L_S(1 + a_j s, \pi, r_j)$ are among the poles of $E_\chi(s; \tilde{\phi}; e; P_{\theta,\mathbf{A}})$. But now by the compactness of $U_\mathbf{A}/U_F$ it is clear that the poles of $E_\chi(s; \tilde{\phi}; e; P_{\theta,\mathbf{A}})$ are among those of $E(s; \tilde{\phi}; e; P_{\theta,\mathbf{A}})$, and therefore the theorem follows from the holomorphy of $E(it, \tilde{\phi}; e; P_{\theta,\mathbf{A}})$ for all $t \in \mathbf{R}$, $i = (-1)^{1/2}$.

Corollary of the proof. *Suppose* Re(s) > ½; *then* ϕ *may be chosen so that* $E_\chi(s; \tilde{\phi}; e; P_{\theta,\mathbf{A}})$ *does not vanish identically.*

We now state some of the consequences of Theorem 5.1.

The following result has been first announced in [25]. It has a very interesting application to the classification of automorphic forms on $GL_n(\mathbf{A})$ (cf. [18]). In fact it can be used to show that if two globally induced representations have the same decomposition factors (for almost all places) then their inducing representations are conjugate by an element of the Weyl group (due to H. Jacquet and J. A. Shalika).

THEOREM 5.2. *Let π and π' be two irreducible admissible cuspidal (non-degenerate) representations of $GL_n(\mathbf{A})$ and $GL_m(\mathbf{A})$, respectively. Write $\pi = \otimes_v \pi_v$ and $\pi' = \otimes \pi_v'$, and let $L(s, \pi_v \times \pi_v')$ be the corresponding Jacquet-Shalika L-function attached to the pair (π_v, π_v'). Set*

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi_v').$$

Then for every $t \in \mathbf{R}$, $L_S(1 + it, \pi \times \pi')$ is non-zero, $i = (-1)^{1/2}$.

Proof. Take $G = GL_{n+m}$, $M_\theta = GL_n \times GL_m$ and apply Theorem 5.1.

Finally the following theorem provides some information concerning the zeros of $L_S(s, \pi, \mathrm{Sym}^3(\rho_2))$ and $L_S(s, \pi, \mathrm{Sym}^4(\rho_2))$ at the line $\mathrm{Re}(s) = 1$.

THEOREM 5.3. *Let π be an irreducible cuspidal representation of $PGL_2(\mathbf{A})$. Suppose π is not monomial. Then the L-functions $L_S(s, \pi, \mathrm{Sym}^3(\rho_2))$ and $L_S(s, \pi, \mathrm{Sym}^4(\rho_2))$ do not vanish on the line $\mathrm{Re}(s) = 1$, except for the second L-function and possibly only at $s = 1$, in which case the zero is at most simple.*

Proof. We first consider $L_S(s, \pi, \mathrm{Sym}^3(\rho_2))$. From

$$\mathrm{Sym}^2(\rho_2) \otimes \rho_2 = \mathrm{Sym}^3(\rho_2) \oplus \rho_2$$

it follows that

$$L_S(s, \pi, \mathrm{Sym}^2(\rho_2) \otimes \rho_2) = L_S(s, \pi, \mathrm{Sym}^3(\rho_2)) L_S(s, \pi, \rho_2).$$

But then by Theorem 5.2, the left hand side which is equal to $L_S(s, \psi_*(\pi) \times \pi)$ does not vanish on the line $\mathrm{Re}(s) = 1$. Now the first part follows if one observes the holomorphy of Jacquet-Langlands' L-function $L_S(s, \pi, \rho_2)$ at $\mathrm{Re}(s) = 1$.

For $L_S(s, \pi, \mathrm{Sym}^4(\rho_2))$, we use Theorem 5.1 to conclude that

$$L_S(1 + 6t(-1)^{1/2}, \mathbf{1}) L_S(1 + 6t(-1)^{1/2}, \pi, \mathrm{Sym}^4(\rho_2))$$

does not vanish for all $t \in \mathbf{R}$. The second part now follows from the fact that $L_S(s, \mathbf{1})$ has a simple pole at $s = 1$ and is holomorphic everywhere else on the line $\mathrm{Re}(s) = 1$.

Remark 5.1. The same kind of result holds for an arbitrary quasi-split group (cf. [3, 26]). More precisely it can be shown that for such groups

$$\prod_{j=1}^{m} L_S(1, \pi, r_j) \neq 0.$$

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