LANGLANDS' CONJECTURE ON PLANCHEREL MEASURES FOR p-ADIC GROUPS

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1. INTRODUCTION

One of the major achievements of Harish-Chandra was a derivation of the Plancherel formula for real and p-adic groups [9,10]. To have an explicit formula, one will have to compute the measures appearing in the formula; the so called Plancherel measures and formal degrees [12]. (For reasons stemming from L-indistinguishability, we would like to distinguish between the formal degrees for discrete series and the Plancherel measures for non-discrete tempered representations, cf. Proposition 9.3 of [29].) While for real groups the Plancherel measures are completely understood [1, 9, 22], until recently little was known in any generality for p-adic groups [29] (except for their rationality and general form due to Silberger [39]). On the other hand any systematic study of the non-discrete tempered spectrum of a p-adic group would very likely have to follow the path of Knapp and Stein [20, 21] and their theory of R-groups. Since the basic reducibility theorems for p-adic groups are available [40, 41], it is the knowledge of Plancherel measures which would be necessary to determine the R-groups. This is particularly evident from the important and the fundamental work of Keys [16, 17, 18] and the work of the author [29, 30, 31].

On the other hand, based on his results on constant terms of Eisenstein series [23], Langlands conjectured that every Plancherel measure must be a product of certain root numbers with the ratios of the corresponding Langlands L-functions at $s = 1$ and $s = 0$. Otherwise said, he suggested that one must be able to normalize the standard intertwining operators by means of certain local root numbers and L-functions [24]. We refer to the introduction of [29] and to [2, 3, 4, 5] for applications of such normalizations. This was further tested for real groups by Arthur [1]. Since in many instances these local factors (especially L-functions which then determine the poles and zeros of Plancherel measures and thus answer reducibility questions) can be explicitly computed, this leads to explicit formulas for Plancherel measures that are not available from any other method [29, 31,

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32]. In fact, except for the the cases coming from minimal parabolics [16, 17, 18], there is not a single example of a Plancherel measure coming from a supercuspidal representation of a non-minimal Levi subgroup of a p-adic group that has been computed in any other way. (We understand that, using Howe-Moy Hecke algebra isomorphisms [11], they can also calculate Plancherel measures in some non-minimal cases.) The main result (Theorem 7.9) of [29] was to prove this conjecture for generic representations of Levi subgroups of quasi-split groups and to set up a program to attack the conjecture in general (Theorem 9.5 of [29] and the conjectural fact that the Plancherel measures are preserved by inner forms [5]). Incidently, we should remark that, although we have not checked carefully, the proof of Proposition 9.6 of [29], when applied to real groups by means of Shelstad's results [38], should basically lead to another proof of Vogan's result on genericity of tempered L-packets for real groups [42].

As a consequence of Theorems 3.5 and 7.9 of [29], a general result (Theorem 8.1 of [29]) was established on reducibility of induced representations from generic supercuspidal representations of maximal Levi subgroups of a quasi-split group. This implied, in particular, that the edge of complementary series (if any) can only have two possible choices, $\widetilde{\alpha}$ and $\frac{1}{2}\widetilde{\alpha}$, no matter what the group or inducing representation.

The first part of this article (Theorems 3.1, 4.2, and 5.1) is aimed at a survey of these results. In the second part we give three examples. The first two examples, described in Propositions 6.1 and 6.2, determine the reducibility of the representations induced from supercuspidal representations of Levi factors of the Siegel parabolic subgroups of Sp_4, PSp_4, and GSp_4, and the parabolic subgroup of an exceptional split group of type G_2 whose Levi subgroup is generated by the short simple root of G_2. Together with Propositions 8.3 and 8.4 of [29] (also Proposition 5.1 of [43] for GSp_4), this completes the analysis of the unitary duals of all the rank two split p-adic groups supported on their maximal parabolics. Propositions 6.1 and 6.2 both seem to be new.

The paper concludes with explicit formulas for the Plancherel measures for $GL(n)$ and $SL(n)$, presented in Proposition 7.1 and Corollary 7.2, respectively. A formula in terms of certain Rankin-Selberg L-functions for $GL(n)$ [13] for these measures has been in print [32] since 1984. But an explicit expression for the Plancherel measures should make it more convenient to calculate R-groups for $SL(n)$ (Remark 7.3).

2. Notation and Preliminaries

Let F be a non-archimedean field of characteristic zero whose ring of integers and maximal ideal are denoted by O and P, respectively. We shall always fix a uniformizing parameter ϖ, i.e., an element of P such that $P = (\varpi)$. Let q denote the number of elements in the residue field O/P. We use $|\cdot|_F$ to denote the absolute value of F. Then $|\varpi|_F = q^{-1}$.

Let $\mathbf{G}$ be a quasi-split connected reductive algebraic group over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ over F and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{T}$ is a maximal torus and $\mathbf{U}$ denotes the unipotent radical of $\mathbf{B}$. For an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{B}$, let $\mathbf{P} = \mathbf{MN}$ be a Levi decomposition. Then $\mathbf{U} \supset \mathbf{N}$. We use G, B, T, U, P, M, and N to denote the corresponding groups of F-rational points.

If $\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$ is the real Lie algebra of the maximal split torus $\mathbf{A}$ of the center of $\mathbf{M}$, we use $H_P : \mathbf{M} \to \mathfrak{a}$ to denote the homomorphism of [35,41] defined by

$$q^{\langle \chi, H_P(m)\rangle} = |\chi(m)|_F.$$

Here $X(\mathbf{M})_F$ is the group of F-rational characters of $\mathbf{M}$.

If $\mathbf{A}_0$ is the maximal F-split torus in $\mathbf{T}$, $\mathbf{A}_0 \supset \mathbf{A}$, let ψ be the set of F-roots of $\mathbf{A}_0$. Then $\psi = \psi_+ \cup \psi_-$, where ψ_+ is the set of positive roots of $\mathbf{A}_0$, i.e., those roots generating $\mathbf{U}$. Let $\Delta \subset \psi_+$ be the set of simple roots. Then $\mathbf{M} = \mathbf{M}_\theta$ is generated by a subset θ of Δ.

Given an irreducible admissible representation σ of M and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, the complex dual of $\mathfrak{a}$, define

$$I(\nu, \sigma) = \operatorname*{Ind}_{MN\uparrow G} \sigma \otimes q^{\langle \nu, H_P(\ )\rangle} \otimes \mathbf{1}.$$

We use $V(\nu, \sigma)$ to denote the space of $I(\nu, \sigma)$ and we let $I(\sigma) = I(0, \sigma)$ and $V(\sigma) = V(0, \sigma)$.

Let $W(\mathbf{A}_0)$ be the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. Fix a $\widetilde{w} \in W(\mathbf{A}_0)$ such that $\widetilde{w}(\theta) \subset \Delta$. Choose a representative $w \in G$ for $\widetilde{w}$. Let $\mathbf{N}_{\widetilde{w}} = \mathbf{U} \cap w\mathbf{N}^- w^{-1}$, where $\mathbf{N}^-$ is the unipotent subgroup of $\mathbf{G}$ opposed to $\mathbf{N}$. Given $f \in V(\nu, \sigma)$, let

$$A(\nu, \sigma, w) f(g) = \int_{N_{\widetilde{w}}} f(w^{-1} n g)\, dn.$$

The integral converges for ν in a certain cone and can be analytically continued to a meromorphic function of ν on all of $\mathfrak{a}^*_{\mathbb{C}}$ (cf. [20, 21, 34, 41]). It intertwines $I(\nu, \sigma)$ with $I(\widetilde{w}(\nu), \widetilde{w}(\sigma))$, where $\widetilde{w}(\sigma)(m') = \sigma(w^{-1} m' w)$, $m' \in M' = wMw^{-1}$. Finally, let $A(\sigma, w) = A(0, \sigma, w)$.

If $\widetilde{w}_0$ is the longest element in the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$ modulo that of $\mathbf{A}_0$ in $\mathbf{M}$, the Plancherel constant $\mu(\nu, \sigma)$ is defined by

$$A(\nu, \sigma, w_0) A(\widetilde{w}_0(\nu), \widetilde{w}_0(\sigma), w_0^{-1}) = \mu(\nu, \sigma)^{-1} \gamma(G/P)^2.$$

Here

$$\gamma(G/P) = \int_{\overline{\mathbf{N}}_{\widetilde{w}_0}(F)} q^{\langle 2\rho_P, H_P(\overline{n})\rangle}\, d\overline{n},$$

where $\overline{\mathbf{N}}_{\widetilde{w}_0} = w_0^{-1} \mathbf{N}_{\widetilde{w}} w_0$. The constant $\mu(\nu, \sigma)$ does not depend on the choice of w_0, nor on that of the defining measures. Also, its dependence on σ is only via the equivalence class of the representation.

Now let ${}^LM$ be the L-group of $\mathbf{M}$. Denote by ${}^L\mathfrak{n}$ the Lie algebra of the L-group ${}^LN$ of $\mathbf{N}$. The group ${}^LM$ acts by adjoint action on ${}^L\mathfrak{n}$. If ${}^L\mathfrak{n}_{\widetilde{w}}$ denotes the Lie algebra of the L-group of $\mathbf{N}_{\widetilde{w}}$, then ${}^L\mathfrak{n}_{\widetilde{w}}$, realized as a subspace of ${}^L\mathfrak{n}$ by $-Ad[w]$, is stable under this adjoint action. Let r and $r_{\widetilde{w}}$ be the adjoint actions of ${}^LM$ on ${}^L\mathfrak{n}$ and ${}^L\mathfrak{n}_{\widetilde{w}}$, respectively.

Let $\rho = \rho_{\mathbf{P}}$ be half the sum of roots whose root spaces generate $\mathbf{N}$. Then, for each α with $X_{\alpha^\vee} \in {}^L\mathfrak{n}$, $\langle 2\rho, \alpha\rangle$ is a positive integer. Let $a_1 < a_2 < \ldots < a_n$ be the distinct values of $\langle 2\rho, \alpha\rangle$. Set

$$V_i = \{X_{\alpha^\vee} \in {}^L\mathfrak{n}_{\widetilde{w}} | \langle 2\rho, \alpha\rangle = a_i\}.$$

Each V_i is invariant under $r_{\widetilde{w}}$. We let $r_{\widetilde{w},i}$ be the restriction of $r_{\widetilde{w}}$ to V_i.

If $\mathbf{P}$ is maximal, then for the non-trivial $\widetilde{w}$ we set $r_{\widetilde{w}} = r$. We then set $r_i = r_{\widetilde{w},i}$; thus $r = \bigoplus_{i=1}^{m} r_i$ with each r_i irreducible (cf. [35]). Let $\alpha \in \Delta$ identify the unique reduced root of $\mathbf{A}$ in $\mathbf{N}$. Set $\widetilde{\alpha} = \langle \rho, \alpha\rangle^{-1}\rho$, an element of $\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$. Observe that, for each i, $1 \le i \le m$,

$$V_i = \{X_{\beta^\vee} \in {}^L\mathfrak{n} | \langle \widetilde{\alpha}, \beta\rangle = i\},$$

and therefore each V_i is an eigenspace for the action of the connected center of ${}^LM^0$, the connected component of ${}^LM$.

When $F = \mathbb{R}$ we have similar definitions for which we use the same notation.

Fix a non-trivial additive character ψ_F of F. We shall now define a generic character χ of U. Let $\alpha \in \Delta$. If E_α is the smallest extension of F over which the rank one subgroup of $\mathbf{G}$ generated by α splits, we let $\chi | U_\alpha = \psi_F \cdot \mathrm{Tr}_{E_\alpha/F}$. By restriction χ is also a generic character of U^0 which we still denote by χ. Here $\mathbf{U}^0 = \mathbf{U} \cap \mathbf{M}$.

The representation σ is called χ-generic if it can be realized on a space of functions W^0 satisfying $W^0(um) = \chi(u)W^0(m)$. Changing the splitting on $\mathbf{G}$ (or, said in other words, up to L-indistinguishability) every irreducible admissible generic representation is χ-generic with respect to such a χ (cf. [29]). Moreover by Conjecture 9.4 of [29] and Section 6 of [42] every tempered L-packet contains such a representation. Generic representations are thus much more fundamental than once thought.

Next, suppose $\sigma \subset \operatorname{Ind}_{M_\theta N_\theta \uparrow M} \sigma_1 \otimes \mathbf{1}$, where, for each $\theta \subset \Delta$, $\mathbf{M}_\theta\mathbf{N}_\theta$, is a parabolic subgroup of $\mathbf{M}$ and σ_1 is an irreducible admissible representation of M_θ. Let $\theta' = \widetilde{w}(\theta) \subset \Delta$ and fix a reduced decomposition $\widetilde{w} = \widetilde{w}_{n-1} \ldots \widetilde{w}_1$ as in Lemma 2.1.1 of [34]. Then, for each j, there exists a unique root $\alpha_j \in \Delta$ such that $\widetilde{w}_j(\alpha_j) < 0$. For each j, $2 \le j \le n-1$, let $\overline{w}_j = \widetilde{w}_{j-1} \ldots \widetilde{w}_i$. Set $\overline{w}_1 = 1$. Moreover let $\Omega_j = \theta_j \cup \{\alpha_j\}$, where $\theta_1 = \theta$, $\theta_n = \theta'$, and $\theta_{j+1} = \widetilde{w}_j(\theta_j)$, $1 \le j \le n-1$. Then the group $\mathbf{M}_{\Omega_j}$ contains $\mathbf{M}_{\theta_j}\mathbf{N}_{\theta_j}$ as a maximal parabolic subgroup and $\overline{w}_j(\sigma_1)$ is a representation of M_{θ_j}. The L-group ${}^LM_\theta$ acts on V_i. Given an irreducible component of this action, there exists a unique j, $1 \leq j \leq n-1$, which, under $\overline{w}_j$, makes this component equivalent to an irreducible constituent of the action of ${}^L M_j$ on the Lie algebra of ${}^L N_{\theta_j}$ ($\mathbf{M}_{\theta_j}\mathbf{N}_{\theta_j}$ is a maximal parabolic subgroup of $\mathbf{M}_{\Omega_j}$). We denote by $i(j)$ the index of this subspace of the Lie algebra of ${}^L N_{\theta_j}$. Finally let S_i denote the set of all such j's; S_i is, in general, a proper subset of $1 \leq j \leq n-1$. We refer to [36] for several examples.

Langlands' conjecture on Plancherel measures is global by nature. In other words the most important property of the standard normalization is that the normalizing factors can be related globally. For this reason one needs statements about groups over global fields and global L-functions. For this we need further preparation.

Let K be a number field and fix a place v of K. If $\mathbf{G}$ is a quasi-split connected reductive group over K, we use $\mathbf{G} \times_K K_v$ to denote $\mathbf{G}$ as a group over K_v, the completion of K at v. Next suppose $\mathbb{A}_K$ is the ring of adeles of K. Fix a non-trivial character $\psi = \otimes_v \psi_v$ of $\mathbb{A}_K$, trivial on K, and define a character $\chi = \otimes_v \chi_v$ of $\mathbf{U}(\mathbb{A}_K)$ as before, where $\mathbf{U}$ is the unipotent radical of a Borel subgroup of $\mathbf{G}$.

Let $\pi = \otimes_v \pi_v$ be a cusp form on $M = \mathbf{M}(\mathbb{A}_K)$. We shall say π is globally χ-generic if there exists a function φ in the space of π such that

$$\int_{\mathbf{U}^0(K)\backslash \mathbf{U}^0(\mathbb{A}_K)} \varphi(u)\overline{\chi(u)}du \neq 0,$$

where $\mathbf{U}^0 = \mathbf{U} \cap \mathbf{M}$.

Now assume v is such that ψ_v, $\mathbf{G} \times_K K_v$, and π_v are all unramified. If ρ is an analytic representation of ${}^L M$, we define ρ_v by $\rho_v = \rho \cdot \eta_v$, where η_v is the natural map $\eta_v : {}^L M_v \to {}^L M$. Here ${}^L M_v$ is the L-group of $\mathbf{M} \times_K K_v$. Let $L(s, \pi_v, \rho_v)$ be the local Langlands L-function attached to π_v and ρ_v (cf. [7,25]). If S is a finite set of places of K outside of which everything is unramified, we set

$$L_S(s, \pi, \rho) = \prod_{v \notin S} L(s, \pi_v, \rho_v).$$

The first aim of this paper is to define these L-functions at the other places (Section 7 of [29]).

3. The Fundamental Theorem

The standard normalization of intertwining operators as conjectured by Langlands is done by means of certain L-functions and root numbers [24]. Crucial among their properties is that, whenever the representation becomes a local component of a global cusp form, the factors must be those satisfying the corresponding functional equation. In fact this is the only way one can globally relate normalizing factors on different local groups to each other. This is important in many deep applications of the trace formula [2, 3, 4, 19, 30]. To define these local factors, the following theorem was proved in [29] (Theorem 3.5 of [29]).

Theorem 3.1. *Given a local field F of characteristic zero and a quasi-split connected reductive algebraic group $\mathbf{G}$ over F containing an F-parabolic subgroup $\mathbf{P} = \mathbf{MN}$, $\mathbf{N} \subset \mathbf{U}$, let $r_{\widetilde{w}} = \bigoplus_{i=1}^{m} r_{\widetilde{w},i}$ be the adjoint action of ${}^L M$ on ${}^L \mathfrak{n}_{\widetilde{w}}$ as in Section 2. Then, for every irreducible admissible χ-generic representation σ of M, there exists m complex functions $\gamma_i(s, \sigma, \psi_F, \widetilde{w})$, $s \in \mathbb{C}$, satisfying the following properties:*

1) If F is archimedean or σ has an Iwahori fixed vector, let $\varphi' : W'_F \to {}^L M$ be the homomorphism attached to σ, where W'_F is the Deligne-Weil group of F. Denote by $\varepsilon(s,\ r_{\widetilde{w},i} \cdot \varphi',\ \psi_F)$ and $L(s,\ r_{\widetilde{w},i} \cdot \varphi')$, the Artin root number and L-function attached to $r_{\widetilde{w},i} \cdot \varphi'$, respectively. Then

$$\gamma_i(s, \sigma, \psi_F, \widetilde{w}) = \varepsilon(s,\ r_{\widetilde{w},i} \cdot \varphi',\ \psi_F) L(1-s,\ \widetilde{r}_{\widetilde{w},i} \cdot \varphi')/L(s,\ r_{\widetilde{w},i} \cdot \varphi').$$

2) For each i, $1 \le i \le m$,

$$\gamma_i(s, \sigma, \psi_F, \widetilde{w}) \gamma_i(1-s, \widetilde{\sigma}, \overline{\psi}_F, \widetilde{w}) = 1, \text{ and} \tag{3.1}$$
$$\gamma_i(s, \sigma, \psi_F, \widetilde{w}) = \gamma_i(s + s_0, \sigma_0, \psi_F, \widetilde{w}),$$

where $\sigma = \sigma_0 \otimes q^{\langle s_0 \widetilde{\alpha}, H_P(\ )\rangle}$ if F is nonarchimedean.

3) Inductive property: Suppose $\sigma \subset \operatorname{Ind}_{M_\theta N_\theta \uparrow M} \sigma_1 \otimes 1$, where $\mathbf{M}_\theta \mathbf{N}_\theta$ is a parabolic subgroup of $\mathbf{M}$ and σ_1 is an irreducible admissible χ-generic representation of M_θ. Write $\widetilde{w} = \widetilde{w}_{n-1} \ldots \widetilde{w}_1$ and for each j, $2 \le j \le n-1$, let $\overline{w}_j = \widetilde{w}_{j-1} \ldots \widetilde{w}_1$ and $\overline{w}_1 = 1$. Then for each j, $\overline{w}_j(\sigma_1)$ is a representation of M_{θ_j}. If, for each $j \in S_i$, $\gamma_{i(j)}(s, \overline{w}_j(\sigma_1), \psi_F, \widetilde{w}_j)$, $1 \le i \le m$, denotes the corresponding factor, then

$$\gamma_i(s, \sigma, \psi_F, \widetilde{w}) = \prod_{j \in S_i} \gamma_{i(j)}(s, \overline{w}_j(\sigma_1), \psi_F, \widetilde{w}_j).$$

4) Functional equations: Let K be a number field and $\mathbf{G}$ a quasi-split connected reductive algebraic group over K. Let $\mathbf{P} = \mathbf{MN}$, $\mathbf{N} \subset \mathbf{U}$, be a maximal K-parabolic subgroup of $\mathbf{G}$. Fix a non-degenerate character $\chi = \bigotimes_v \chi_v$ of $U = \mathbf{U}(\mathbb{A}_K)$, trivial on K and defined by a non-trivial character $\psi = \bigotimes_v \psi_v$ of $K \setminus \mathbb{A}_K$. Let $\pi = \bigotimes_v \pi_v$ be a globally χ-generic cusp form on $\mathbf{M}(\mathbb{A}_K)$. Finally, if r is the adjoint action of ${}^L M$ on ${}^L \mathfrak{n}$, write $r = \bigoplus_{i=1}^{m} r_i$. Then $r_v = \bigoplus_{i=1}^{m} r_{i,v}$ is the adjoint action of ${}^L M_v$, where $r_v = r \cdot \eta_v$, $r_{i,v} = r_i \cdot \eta_v$, and $\eta_v : {}^L M_v \to {}^L M$ is the natural map. Let S be a finite set of places of K such that, for $v \notin S$, $\mathbf{G} \times_K K_v$, π_v, and χ_v are all unramified. Then

$$L_S(s, \pi, r_i) = \prod_{v \in S} \gamma_i(s, \pi_v, \psi_v) L_S(1-s, \pi, \widetilde{r}_i),$$

for every i, $1 \le i \le m$, where $\gamma_i(s, \pi_v, \psi_v) = \gamma_i(s, \pi_v, \psi_v, \widetilde{w}_0)$.

Moreover, conditions (1), (3), and (4) determine the γ_i uniquely.

Remark 3.2. The factors $\gamma_i(s, \sigma, \psi_F, \widetilde{w})$ are all defined locally by means of local coefficients (see the next remark). But to prove their properties and uniqueness one has to employ global methods—more precisely, functional equations. In fact, local proofs are available only for $\mathbf{G} = GL(n)$, and even in that case they are fairly deep and complicated [13, 32]. Thus one must again observe how powerful global methods can be in answering local questions.

Remark 3.3. To make this survey short, we have suppressed the important role played by the theory of local coefficients which was developed in [29, 34, 37].

4. Local Factors and Langlands Conjecture

When $F = \mathbb{R}$ our local factors are those of Artin [27, 37] as defined in [26] (part 1 of Theorem 3.1). In this case the Langlands Conjecture has been verified by Arthur [1]. Therefore for the remainder of this article we shall assume F is non-archimedean.

We first define our local root numbers and L-functions. Start with a maximal parabolic $\mathbf{P}$ and fix an irreducible tempered χ-generic representation σ of M. From now on we use $\gamma(s,\sigma,r_i,\psi_F)$ to denote $\gamma_i(s,\sigma,\psi_F)$, $1 \le i \le m$.

For each i, let $P_{\sigma,i}(t)$ be the unique polynomial satisfying $P_{\sigma,i}(0) = 1$ such that $P_{\sigma,i}(q^{-s})$ has the same zeros as $\gamma(s,\sigma,r_i,\psi_F)$, i.e., $P_{\sigma,i}(t)$ is the unique numerator of $\gamma(s,\sigma,r_i,\psi_F)$ (which is a rational function of q^{-s}) satisfying $P_{\sigma,i}(0) = 1$. Define the L-functions attached to σ, r_i, and $\widetilde{r}_i$ as

$$L(s,\sigma,r_i) = P_{\sigma,i}(q^{-s})^{-1} \quad \text{and} \quad L(s,\sigma,\widetilde{r}_i) = P_{\widetilde{\sigma},i}(q^{-s})^{-1}.$$

They do not depend on ψ_F. Then by (3.1)

$$\gamma(s,\sigma,r_i,\psi_F)L(s,\sigma,r_i)/L(1-s,\sigma,\widetilde{r}_i)$$

is a monomial in q^{-s} which we denote by $\varepsilon(s,\sigma,r_i,\psi_F)$, the root number attached to σ and r_i. Thus

$$\gamma(s,\sigma,r_i,\psi_F) = \varepsilon(s,\sigma,r_i,\psi_F)L(1-s,\sigma,\widetilde{r}_i)/L(s,\sigma,r_i).$$

The definition of L and ε for non-tempered σ is then a consequence of inductive property (3) and the Langlands classification for irreducible admissible representations of p-adic groups (due to Borel-Wallach and Silberger).

The following natural conjecture serves two purposes: On the one hand it provides one of the conditions on normalizing factors demanded by Arthur (Condition R_7 of [1]; it was not among the original conditions conjectured by Langlands in [24]; also see [29]). On the other hand, using inductive property (3), it allows us to prove the multiplicative properties of these factors in general. This is of great interest in the theory of automorphic L-functions. We refer to [36] for an account of this and several examples.

Conjecture 4.1. *If σ is tempered, then each $L(s,\sigma,r_i)$ is holomorphic for $Re(s)>0$.*

It is enough to prove this for σ in the discrete series and $\mathbf{P}$ maximal.

By Proposition 7.2 and 7.3 of [29] the conjecture is a theorem if $m=1$; $m=2$ and

$$L(s,\sigma,r_2)=\prod_j(1-a_jq^{-s})^{-1}\quad(\alpha_j\in\mathbb{C}),$$

(possibly empty) with $|\alpha_j|=1$; m is arbitrary but σ is unitary supercuspidal; and finally $\mathbf{G}$ is a simple classical group, $\mathbf{M}=\mathbf{H}\times GL(n)$ for some classical group H, and σ is at least supercuspidal on one of the factors H or $GL_n(F)$ (Theorem 5.5 of [36]).

Now assume $\mathbf{P}=\mathbf{MN}$, $\mathbf{N}\subset\mathbf{U}$, is any standard parabolic subgroup of $\mathbf{G}$. Fix $\widetilde{w}\in W(\mathbf{A}_0)$ such that $\widetilde{w}(\theta)\subset\Delta$, $\mathbf{P}=\mathbf{P}_\theta$. Let σ be an irreducible unitary χ-generic representation of M. We fix a reduced decomposition $\widetilde{w}=\widetilde{w}_{n-1}\dots\widetilde{w}_1$ and set

$$L(s,\sigma,r_{\widetilde{w}})=\prod_{j=1}^{n-1}\prod_{i=1}^{m_j}L(s,\overline{w}_j(\sigma),r_{\widetilde{w}_j,i})$$

and

$$\varepsilon(s,\sigma,r_{\widetilde{w}},\psi_F)=\prod_{j=1}^{n-1}\prod_{i=1}^{m_j}\varepsilon(s,\overline{w}_j(\sigma),r_{\widetilde{w}_j,i},\psi_F).$$

They are both independent of the decomposition of $\widetilde{w}$.

We shall now normalize the intertwining operator $A(\sigma,w)$ in the way conjectured by Langlands [24]. Let

$$\mathcal{A}(\sigma,w)=\varepsilon(0,\sigma,\widetilde{r}_{\widetilde{w}},\psi_F)L(1,\sigma,\widetilde{r}_{\widetilde{w}})L(0,\sigma,\widetilde{r}_{\widetilde{w}})^{-1}A(\sigma,w),$$

where the right hand side is defined as a limit.

Theorem 4.2. *(Langlands' conjecture). The normalized operator $\mathcal{A}(\sigma,w)$ satisfies:*

a) $\mathcal{A}(\sigma,w_1w_2)=\mathcal{A}(\widetilde{w}_2(\sigma),w_1)\mathcal{A}(\sigma,w_2)$, and
b) $\mathcal{A}(\sigma,w)=\mathcal{A}(\widetilde{w}(\sigma),w^{-1})$, i.e. $\mathcal{A}(\sigma,w)$ is unitary.

Remark 4.3. Theorem 4.2 is clearly equivalent to a formula for Plancherel measures in terms of L-functions and root numbers (Corollary 3.6 of [29]).

5. Reducibility of Induced Representations

One of the consequences of Theorems 3.1 and 4.2 is a general result on reducibility of representations induced from supercuspidal generic representations of Levi factors of maximal parabolic subgroups of any quasi-split group in terms of polynomials $P_{\sigma,i}$, $i=1,2$. More precisely, even the equality (3.1) of Theorem 3.1 allows us to determine the edge of complementary series coming from maximal parabolic subgroups and generic supercuspidal representations of such groups. They turn out to take only two values (if any), $\widetilde{\alpha}$ or $\frac{1}{2}\widetilde{\alpha}$, no matter what the representation and the group are. As mentioned above, such a result is possible only because of the identity (3.1) which provides the only important unknown in the formula for the Plancherel measure obtained by Silberger [39]. This identity is quite deep and is proved by global methods. Its local proof in this generality seems far from reach at present (cf. [13, 32] and the remark at the end of Section 3 of [29] for $GL(n)$). Finally we should remark that in many cases the Levi subgroups are products of A-type groups for which supercuspidal representations are always generic and our assumptions on genericity is automatically satisfied. As a consequence of this, in the next section we shall obtain those parts of the unitary duals of all the rank two split p-adic groups which are supported on their maximal parabolic subgroups.

Let $\mathbf{P}=\mathbf{MN}$ be a maximal parabolic subgroup of a quasi-split connected reductive algebraic group over a p-adic field F. Let σ be an irreducible unitary supercuspidal χ-generic representation of M. Then by Lemma 7.5 of [29], $P_{\sigma,i}\equiv 1$ for $3\leq i\leq m$. Moreover $I(\sigma)$ is irreducible and σ is ramified, i.e. $\widetilde{w}_0(\sigma)\cong\sigma$ if and only if $P_{\sigma,i}(1)=0$ for exactly one of the values $i=1$ or 2 (Corollary 7.6 of [29]). The following is Theorem 8.1 of [29].

Theorem 5.1. *Let $\mathbf{P}=\mathbf{MN}$ be a maximal parabolic subgroup of $\mathbf{G}$, where $\mathbf{G}$ is a quasi-split connected reductive algebraic group over a p-adic field. Let σ be an irreducible unitary supercuspidal χ-generic representation of M. Assume σ is ramified and $I(\sigma)$ is irreducible. Choose a unique i, $i=1$ or 2, such that $P_{\sigma,i}(1)=0$. Then:*

a) *For $0<s<\frac{1}{i}$, $I(s\widetilde{\alpha},\sigma)$ is irreducible and in the complementary series.*
b) *$I(\widetilde{\alpha}/i,\sigma)$ is reducible with a unique χ-generic subrepresentation which is in the discrete series. Its Langlands quotient is never generic. It is a preunitary, non-tempered representation.*
c) *For $s>\frac{1}{i}$, $I(s\widetilde{\alpha},\sigma)$ is irreducible and not in the complementary series.*

If σ is ramified and $I(\sigma)$ is reducible, then no $I(s\widetilde{\alpha},\sigma)$, $s>0$, is preunitary. They are all irreducible. In particular the edge of complementary series (if any) is always either $\widetilde{\alpha}$ or $\frac{1}{2}\widetilde{\alpha}$.

We now state the following corollary, expressing our results in terms of points of reducibility of induced representations.

Corollary 5.2. *Let σ be an irreducible unitary supercuspidal χ-generic representation of M. Then $I(s\widetilde{\alpha},\sigma)$ is irreducible unless some unramified twist of σ is ramified. Assume σ is ramified and $I(\sigma)$ is irreducible. Then the only point of reducibility for $I(s\widetilde{\alpha},\sigma)$ in the region $s>0$ occurs at either $s=1/2$ or $s=1$. If $I(\sigma)$ is reducible, then $s=0$ gives the only point of reducibility.*

Remark 5.3. Both results are valid if σ is generic with respect to any other generic character of U.

6. Examples: Unitary Duals of Rank Two Split Groups

Since unitary duals of all the rank one quasi-split groups are completely determined [17], we can apply Theorem 5.1 to a general rank two split group and obtain that part of the unitary dual which is supported on the maximal parabolic subgroups.

We first let $\mathbf{G}$ be either Sp_4, PSp_4, or GSp_4 over a non-archimedean field F. We let $\mathbf{P} = \mathbf{MN}$ be such that $\mathbf{M}$ is generated by the short simple root α. If $\mathbf{G} = Sp_4$, then $\mathbf{M} = GL_2$. For $\mathbf{G} = PSp_4$, $\mathbf{M} = PGL_2 \times GL_1$. Otherwise it is $GL_2 \times GL_1$. Let σ be an irreducible unitary supercuspidal representation of $M = \mathbf{M}(F)$. Then $\sigma = \sigma_1 \otimes \chi$, where σ_1 is an irreducible unitary supercuspidal representation of $GL_2(F)$ and χ is a character of F^*. We disregard χ if $G = Sp_4(F)$ or $PSp_4(F)$. For $\mathbf{G} = PSp_4$, $\mathbf{M}(F) = GL_2(F)/\{\pm 1\}$ and σ will be a representation of $GL_2(F)$ trivial on $\{\pm 1\}$. Let ω be the central character of σ_1. The element $\widetilde{w}_0$ can be chosen to be $\widetilde{w}_\beta \widetilde{w}_\alpha \widetilde{w}_\beta$, where β is the long simple root. Suppose $\mathbf{G} = Sp_4$ or PSp_4. Then $\widetilde{w}_0(\sigma) \cong \sigma$ if and only if $\widetilde{\sigma}_1 \cong \sigma_1$. In particular, $\omega^2 = 1$. Otherwise, i.e. if $\mathbf{G} = GSp_4$, $\widetilde{w}_0(\sigma) \cong \sigma$ if and only if $\omega = 1$. Thanks are due to David Goldberg for pointing out this difference which was carelessly overlooked in the first version of this paper.

It is instructive to observe that if σ_1 is an irreducible unramified principal series (class one) representation of $GL_2(F)$ defined by the pair (μ_1, μ_2) of unramified characters of F^* and χ is unramified, then the corresponding semi-simple conjugacy class [7,25] in the L-group $GSp_4(\mathbb{C})$ of GSp_4 can be represented by

$$A(I(\sigma)) = \mathrm{diag}(\chi^2(\varpi), \mu_2^{-1}\chi^2(\varpi), \mu_1^{-1}\chi^2(\varpi), \mu_1^{-1}\mu_2^{-1}\chi^2(\varpi)),$$

where ϖ is a uniformizing parameter for F.

However, if $\mathbf{G} = Sp_4$ and $\sigma = \sigma_1$ as above, then ${}^L G = PSp_4(\mathbb{C})$ and

$$\begin{aligned} A(I(\sigma)) = \mathrm{diag}(&(\mu_1\mu_2(\varpi))^{1/2},\ (\mu_1\mu_2^{-1}(\varpi))^{1/2}, \\ &(\mu_1^{-1}\mu_2(\varpi))^{1/2},\ (\mu_1^{-1}\mu_2^{-1}(\varpi))^{1/2}) \pmod{\pm 1}, \end{aligned}$$

where the choice of the square root is irrelevant as long as it is consistent for all the entries, i.e. $(\mu_1\mu_2^{-1}(\varpi))^{1/2} = \mu_2^{-1}(\varpi) \cdot (\mu_1\mu_2(\varpi))^{1/2}$ and so on. (That $A(I(\sigma)) = \nu \cdot \mathrm{diag}(\mu_1\mu_2(\varpi), \mu_2(\varpi), \mu_1(\varpi), 1)$ with some $\nu \in \mathbb{C}^*$ follows immediately from definitions, using roots and coroots of Sp_4. To show that $\nu = \chi^2\mu_1^{-1}\mu_2^{-1}(\varpi)$, one uses the decomposition

$$\mathrm{diag}(\eta\alpha, \eta\beta, \beta^{-1}, \alpha^{-1}) = \lambda \cdot \mathrm{diag}(ab, a, b, 1),$$

where $a = \alpha\eta\beta$, $b = \alpha\beta^{-1}$, and $\lambda = \alpha^{-1}$. The character (μ_1, μ_2, χ) of (α, β, η) is then equal to the character (μ_1', μ_2', χ') of (a, b, λ), where $\mu_1' = \chi$, $\mu_2' = \chi\mu_2^{-1}$, and $\chi' = \chi^2\mu_1^{-1}\mu_2^{-1}$. Applying the central cocharacter to

$A(I(\sigma))$ then implies $\nu = \chi^2\mu_1^{-1}\mu_2^{-1}(\varpi)$.) Thus, for example, the adjoint action r of ${}^LM$ on ${}^L\mathfrak{n}$ is simply $r = \rho_2 \oplus \wedge^2\rho_2$, where ρ_2 is the standard representation of $GL_2(\mathbb{C})$. Then

$$P_{\sigma,1}(q^{-s}) = L(s,\sigma_1,\rho_2)^{-1} = 1,$$

while

$$P_{\sigma,2}(q^{-s}) = L(s,\sigma_1,\wedge^2\rho_2)^{-1} = L(s,\omega)^{-1}.$$

We recall that

$$L(s,\omega)^{-1} = 1 - \omega(\varpi)q^{-s}$$

if ω is unramified and is identically one otherwise. Suppose $\mathbf{G} = Sp_4$ or PSp_4. If ω is unramified, $\omega^2 = 1$ with $\omega \neq 1$ implies $\omega(\varpi) = -1$ and therefore $P_{\sigma,2}(1) = 2 \neq 0$. The same is true if ω is ramified. Thus in both cases $I(\sigma)$ is reducible and thus there are no complementary series.

Otherwise, i.e. if $\omega = 1$, then $P_{\sigma,2}(1) = 0$ and therefore $I(\sigma)$ is irreducible. Here we also include GSp_4. Then index i of Theorem 5.1 is $i = 2$. Half the sum of positive roots in $\mathbf{N}$ is $\rho = \frac{3}{2}(\alpha+\beta)$ and therefore $\widetilde{\alpha}$ is in fact equal to $\alpha+\beta$. It is then clear from the definition of H_P that

$$\begin{aligned} q^{\langle s\widetilde{\alpha}, H_P(m)\rangle} &= |(\alpha+\beta)(m)|^s \\ &= |\det(m_1)|^s \cdot |\lambda|^s \\ &= |\det(m_1)|^s \cdot |\det(m)|^{s/2}, \end{aligned}$$

where $m = (m_1, \widetilde{\lambda})$ with $m_1 \in GL_2(F)$, $\widetilde{\lambda} = \mathrm{diag}(\lambda,\lambda,1,1)$, $\lambda \in F^*$, and $\det(m) = \lambda^2$ (the determinant as an element in $GL_4(F)$). If $\mathbf{G} = Sp_4$ or PSp_4, we then set $\lambda = 1$. Finally observe that

$$I(s\widetilde{\alpha},\sigma) \cong I((\sigma_1 \otimes \nu^s)\otimes\chi) \otimes |\det(\ )|^s,$$

where $\nu(m_1) = |\det(m_1)|$. But now it follows from Theorem 5.1 that $I((\sigma_1\otimes\nu^s)\otimes\chi)$ is irreducible unless $s = \pm\frac{1}{2}$. Moreover $I((\sigma_1\otimes\nu^s)\otimes\chi)$ has a unique generic special subrepresentation and a unique non-tempered non-generic preunitary quotient. We have thus proved:

Proposition 6.1. *a) Suppose $G = GSp_4(F)$ for F a non-archimedean field. Let α and β be the short and the long simple roots of $\mathbf{G}$, respectively, and denote by $\mathbf{P} = \mathbf{MN}$ the maximal parabolic subgroup in which $\mathbf{M}$ is generated by α, $\mathbf{M} \cong GL_2 \times GL_1$. Fix an irreducible unitary supercuspidal representation $\sigma = \sigma_1 \otimes \chi$ of $M = \mathbf{M}(F)$, where σ_1 is a supercuspidal unitary representation of $GL_2(F)$ with central character ω and χ is a unitary character of F^*. Then $I(\sigma)$ is always irreducible. The representation $I(\sigma_1\nu^s\otimes\chi)$ is reducible if and only if $\omega = 1$ and $s = \pm\frac{1}{2}$, where ν denotes $\nu = |\det(\ )|$ for $GL_2(F)$. The representation $I(\sigma_1\nu^{1/2}\otimes\chi)$ has a unique* *generic special subrepresentation and a unique irreducible preunitary non-tempered non-generic quotient. For $0 < s < 1/2$, all the representations $I(\sigma_1\nu^s \otimes \chi)$ are in the complementary series and $s = 1/2$ is their end point.*

b) For $G = Sp_4(F)$ or $PSp_4(F)$, the representation $I(\sigma)$ is reducible if and only if $\sigma \cong \widetilde{\sigma}$ (thus $\omega^2 = 1$) and $\omega \neq 1$. Suppose $\omega = 1$ so that $I(\sigma)$ is irreducible. Then $I(\sigma\nu^s)$ is reducible if and only if $s = \pm 1/2$. The representation $I(\sigma\nu^{1/2})$ has a unique generic special subrepresentation and a unique irreducible preunitary non-tempered non-generic quotient. For $0 < s < 1/2$, all the representations $I(\sigma\nu^s)$ are in the complementary series and $s = 1/2$ is their end point.

c) The Plancherel measure $\mu(s\widetilde{\alpha}, \sigma)$ is given by the formula

$$\mu(s\widetilde{\alpha}, \sigma) = \gamma(G/P)^2 q^{n(\sigma_1)} \frac{(1-\omega(\varpi)q^{-2s})(1-\omega(\varpi)^{-1}q^{2s})}{(1-\omega(\varpi)q^{-1-2s})(1-\omega(\varpi)^{-1}q^{-1+2s})}$$

if ω is unramified, and by

$$\mu(s\widetilde{\alpha}, \sigma) = \gamma(G/P)^2 q^{n(\sigma_1)+n(\omega)}$$

otherwise. $n(\sigma_1)$ and $n(\omega)$ are the conductors of σ_1 and ω, respectively.

Proof. One only needs prove part c). By Corollary 3.6 of [29] one must calculate

$$\varepsilon(s, \sigma_1, \rho_2, \psi_F)\varepsilon(2s, \omega, \psi_F)\varepsilon(-s, \widetilde{\sigma}_1, \rho_2, \overline{\psi}_F)\varepsilon(-2s, \omega^{-1}, \overline{\psi}_F). \tag{6.1.1}$$

Observe that the defining Euclidean measures for intertwining operators are self dual with respect to ψ_F. Therefore, if ψ_F is unramified, then these measures must be the standard ones, i.e. O must have measure one with respect to every direction on the Lie algebra of $\mathbf{N}$. We shall now assume ψ_F is unramified. If one uses [14,45] one immediately sees that

$$\varepsilon(s, \sigma_1, \rho_2, \psi_F) = c(\sigma_1, \psi_F) q^{-n(\sigma_1)s}$$

and

$$\varepsilon(s, \omega, \psi_F) = c(\omega, \psi_F) q^{-n(\omega)s},$$

where $c(\sigma_1, \psi_F)$ and $c(\omega, \psi_F)$ are two non-zero complex numbers. Proposition 7.8 of [29] now implies $c(\widetilde{\sigma}_1, \overline{\psi}_F) = \overline{c(\sigma_1, \psi_F)}$. Thus (6.1.1) equals

$$|c(\sigma_1, \psi_F)c(\omega, \psi_F)|^2.$$

By (3.1) this equals $q^{n(\sigma_1)+n(\omega)}$, proving Proposition 6.1. □

Next let $\mathbf{G}$ be a split group of exceptional type G_2. Let $\mathbf{P} = \mathbf{MN}$ be the parabolic subgroup for which $\mathbf{M}$ is generated by the short simple root α. Then $\mathbf{M} = GL(2)$. Let β be the long simple root of $\mathbf{G}$. The isomorphism $\mathbf{M} \cong GL(2)$ is such that $H_\alpha(t) = \mathrm{diag}(t, t^{-1})$, $H_{3\alpha+2\beta}(t) = \mathrm{diag}(t,t)$, $H_\beta(t) = \mathrm{diag}(1,t)$, $H_{3\alpha+\beta} = \mathrm{diag}(t,1)$, $H_{\alpha+\beta}(t) = \mathrm{diag}(t,t^2)$, and $H_{2\alpha+\beta}(t) = \mathrm{diag}(t^2,t)$. Let σ be an irreducible unitary supercuspidal representation of $M = GL_2(F)$. Then $\widetilde{w}_0(\sigma) \cong \sigma$ implies $\sigma \cong \widetilde{\sigma}$.

The adjoint action r of ${}^LM$ on ${}^L\mathfrak{n}$ is $r = \rho_2 \oplus \wedge^2\rho_2 \oplus \rho_2 \otimes \wedge^2\rho_2$, where ρ_2 is the standard representation of $GL_2(\mathbb{C})$. Again

$$P_{\sigma,1}(q^{-s}) = L(s,\sigma,\rho_2)^{-1} = 1$$

and

$$P_{\sigma,2}(q^{-s}) = L(s,\omega)^{-1}.$$

If $\sigma \cong \widetilde{\sigma}$ with $\omega \neq 1$, then $I(\sigma)$ is reducible and there is no more reducibility or complementary series. Now assume $\sigma \cong \widetilde{\sigma}$ and $\omega = 1$. Then $I(\sigma)$ is irreducible. The index i is again $i = 2$. The value of $\widetilde{\alpha}$ is $3\alpha + 2\beta$ and

$$q^{\langle s\widetilde{\alpha}, H_\theta(m)\rangle} = |(3\alpha+2\beta)(m)|^s.$$

Let $m = \mathrm{diag}(\det m, 1)\cdot m_0$ with $m_0 \in SL_2(F)$. Writing $\mathrm{diag}(\det m, 1) = H_{3\alpha+\beta}(\det m)$ then implies $(3\alpha+2\beta)(m) = \det m$. But now it follows from Theorem 5.1 that $I(\sigma\otimes\nu^s)$ is irreducible unless $s = \pm 1/2$, $\nu = |\det(\ )|$. Moreover $I(\sigma\otimes\nu^{1/2})$ has a unique generic special subrepresentation and a unique irreducible preunitary non-tempered non-generic quotient. Thus:

Proposition 6.2. *a) Let $\mathbf{G}$ be an exceptional split group of type G_2. Assume the Levi factor $\mathbf{M}$ of $\mathbf{P} = \mathbf{MN}$ is generated by the short root of $\mathbf{G}$. Let σ be an irreducible unitary supercuspidal representation of M. Then σ is ramified if and only if $\sigma \cong \widetilde{\sigma}$. Assume $\sigma \cong \widetilde{\sigma}$ but $\omega \neq 1$. Then $I(\sigma)$ is reducible and there are no complementary series. Now suppose $\sigma \cong \widetilde{\sigma}$ but $\omega = 1$. Then $I(\sigma)$ is irreducible. Moreover $I(\sigma\otimes\nu^s)$ is irreducible unless $s = \pm 1/2$. The representation $I(\sigma\otimes\nu^{1/2})$ has a unique generic discrete series subrepresentation and a unique irreducible preunitary non-tempered Langlands quotient. All the representations $I(\sigma\otimes\nu^s)$, $0<s<1/2$, are in the complementary series and $s = 1/2$ is the edge of complementary series.*

b) The Plancherel measure $\mu(s\widetilde{\alpha},\sigma)$ is given by the formula

$$\mu(s\widetilde{\alpha},\sigma) = \gamma(G/P)^2 q^{n(\sigma)+n(\sigma\otimes\omega)} \frac{(1-\omega(\varpi)q^{-2s})(1-\omega(\varpi)^{-1}q^{2s})}{(1-\omega(\varpi)q^{-1-2s})(1-\omega(\varpi)^{-1}q^{-1+2s})}$$

if ω is unramified, and by

$$\mu(s\widetilde{\alpha},\sigma) = \gamma(G/P)^2 q^{n(\sigma)+n(\omega)+n(\sigma\otimes\omega)}$$

otherwise. Here $n(\sigma)$, $n(\omega)$, and $n(\sigma\otimes\omega)$ are the corresponding conductors.

Remark 6.3. Together with Propositions 8.3 and 8.4, and Remark 8.5 of [29] (also see [43] for $\mathbf{G} = GSp_4$ with $\mathbf{P}$ equal to the non-Siegel maximal parabolic subgroup of $\mathbf{G}$), this leads to a complete analysis of the unitary duals of all the rank two split groups supported at their maximal parabolic subgroups. The complete unitary dual of $GSp_4(F)$ is the subject matter of a forthcoming paper of M. Tadic, a sequal to his work with P. Sally [28]. Finally we refer to the Corollary to Proposition 6.2 and Proposition 1.1 of [31] for a formula for the Plancherel measure for the exceptional group $\mathbf{G}$ of type G_2 when $\mathbf{M}$ is generated by its long simple root.

Remark 6.4. Although the results of Propositions 6.1 and 6.2 seem to follow the same simple pattern in terms of the inducing representations, this is definitely not the case in general. This is evident from Proposition 8.3 of [29], where $\mathbf{G}$ is of type G_2 and $\mathbf{P}$ is the other maximal parabolic subgroup.

Remark 6.5. It is knowledge of the polynomials $P_{\sigma,1}$ and $P_{\sigma,2}$—or, said in other terms, the L-functions $L(s,\sigma,r_1)$ and $L(s,\sigma,r_2)$—which allows us to obtain such precise results on reducibility of $I(s\tilde{\alpha},\sigma)$. While there are many cases where these factors are not known, there are instances in which they can be predicted. For example, based on the results of [15], we believe there may be no complementary series coming from the supercuspidal representations of the Levi subgroup $GL_n(F)$ of the group $Sp_{2n}(F)$ whenever $n > 1$ is odd. In other words no reducibility can happen off the unitary axis in this case. This is just an example of a case where both L-functions are identically equal to 1.

Remark 6.6. In general when σ is supercuspidal, the polynomials $P_{\sigma,1}$ and $P_{\sigma,2}$ are such that the operator

$$P_{\sigma,1}(q^{-s})P_{\sigma,2}(q^{-2s})A(s\tilde{\alpha},\sigma,w_0)$$

is holomorphic and non-zero for all values of s [34]. Therefore local calculation of these polynomials rests upon the knowledge of poles of intertwining operators, a subject in which the method of Olšanskiĭ [46] seems to be useful [47]. One deep and surprising consequence of Theorem 5.1 is that understanding the poles not only determines the reducibility on the unitary axis, but also off it.

7. Explicit Formulas for the Plancherel Measures for $GL(n)$ and $SL(n)$

When $\mathbf{G} = GL(n)$, Langlands' conjecture was proved by the author in [32,33]. Consequently Plancherel measures were given in terms of certain Rankin-Selberg L-functions and root numbers [13] for $GL(n)$. The purpose of this section is to use the results of [13] and [32] to give explicit formulas for the measures.

It is enough to compute Plancherel measures when $\mathbf{P}$ is maximal. Thus let $\mathbf{G}$ equal $GL(n+m)$, where n and m are positive integers, and let $\mathbf{M}$ equal $GL(n) \times GL(m)$. Assume $\sigma = \sigma_1 \otimes \sigma_2$ is a tempered representation of M. Let $L(s,\sigma_1 \times \sigma_2)$ and $\varepsilon(s,\sigma_1 \times \sigma_2, \psi_F)$ denote the Rankin-Selberg L-function and root number attached to σ_1 and σ_2 by Jacquet, Piatetski-Shapiro, and Shalika in [13], respectively. Then by Theorem 5.1 of [32] and the validity of Conjecture 4.1 in this case,

$$L(s,\sigma_1 \times \sigma_2) = L(s,\sigma_1 \otimes \sigma_2, \rho_m \otimes \widetilde{\rho}_n)$$

and

$$\varepsilon(s,\sigma_1 \times \sigma_2, \psi_F) = \varepsilon(s,\sigma_1 \otimes \sigma_2, \rho_m \otimes \widetilde{\rho}_n, \psi_F),$$

where ρ_m and ρ_n are the standard representations of $GL_m(\mathbb{C})$ and $GL_n(\mathbb{C})$, respectively, and the factors on the right are those defined in Section 4. Now, either using our results or Theorem 6.1 of [32], the Plancherel constant $\mu(s\widetilde{\alpha}, \sigma_1 \otimes \sigma_2)$ satisfies:

$$\mu(s\widetilde{\alpha}, \sigma_1 \otimes \sigma_2) = \gamma(G/P)^2 q^{n(\widetilde{\sigma}_1 \times \sigma_2)} \frac{L(1+s, \sigma_1 \times \widetilde{\sigma}_2)}{L(s,\sigma_1 \times \widetilde{\sigma}_2)} \cdot \frac{L(1-s, \widetilde{\sigma}_1 \times \sigma_2)}{L(-s, \widetilde{\sigma}_1 \times \sigma_2)},$$

where $n(\widetilde{\sigma}_1 \times \sigma_2)$ is an integer defined by

$$\varepsilon(s, \widetilde{\sigma}_1 \times \sigma_2, \psi_F) = c(\widetilde{\sigma}_1 \times \sigma_2) q^{-n(\widetilde{\sigma}_1 \times \sigma_2)s}.$$

The constant $c(\widetilde{\sigma}_1 \times \sigma_2)$ is a non-zero complex number. As an example consider the case $n = 1$ and $\sigma_2 = \mathbf{1}$. Then by Theorem 5.1 of [32], the integer $n(\sigma_1 \times \mathbf{1})$ is the conductor of σ_1 (cf.[14]). The purpose of this section is to calculate these L-functions explicitly in order to obtain explicit formulas for $\mu(s\widetilde{\alpha}, \sigma_1 \otimes \sigma_2)$.

By the product formula for Plancherel measures and Proposition 8.4 of [13], we may assume σ_1 and σ_2 are both in the discrete series. By [6,44], there exist two integers a and t with $at = m$ and an irreducible unitary supercuspidal representation π_0 of $GL_a(F)$ such that if $\pi_i = \pi_0 \otimes \nu^{\frac{t+1}{2}-i}$, $1 \leq i \leq t$, then σ_1 is the unique discrete series constituent $\sigma(\pi_1, \ldots, \pi_t)$ of the representation induced from $\pi_1 \otimes \pi_2 \cdots \otimes \pi_t$. Similarly, choose integers b and u with $bu = n$ and an irreducible unitary supercuspidal representation ρ_0 of $GL_b(F)$ such that $\sigma_2 = \sigma(\rho_1, \ldots, \rho_u)$, $\rho_j = \rho_0 \otimes \nu^{\frac{u+1}{2}-j}$, $1 \leq j \leq u$.

Assume $n \leq m$. Then Theorem 8.2 of [13] implies

$$L(s, \sigma_1 \times \sigma_2) = \prod_{j=1}^{u} L(s, \pi_1 \times \rho_j).$$

Now assume σ_1 and σ_2 are two irreducible supercuspidal representations of $GL_m(F)$ and $GL_n(F)$, respectively. Using equations (2.4.1) and (2.4.2) and Theorem 2.7 of [13], one sees that $L(s, \sigma_1 \times \sigma_2) = 1$ unless $\sigma_1 \cong \widetilde{\sigma}_2 \otimes \nu^{s_0}$ with a complex number s_0 which is not necessarily unique. Then

$$L(s, \sigma_1 \times \sigma_2) = L(s + s_0, \widetilde{\sigma}_2 \times \sigma_2).$$

Now assume $\sigma_2 \cong \sigma_2 \otimes \eta$, where $\eta \in \hat{F}^*$ is unramified and η denotes $\eta \cdot \det$. Then $\eta^n = 1$. The set of all such η is a cyclic group of order r, $r|n$. This is true because each η is determined by $\eta(\varpi)$ and finite subgroups of $\mathbb{C}^*$ are cyclic. From equation (2.4.1) of [13] it is clear that $L(s, \sigma_2 \times \tilde{\sigma}_2)^{-1}$ divides $(1 - q^{-ns})$, and therefore poles of $L(s, \sigma_2 \times \tilde{\sigma}_2)$ are all simple. Moreover, if η is such that $\sigma_2 \cong \sigma_2 \otimes \eta$, then $q^s = \eta(\varpi)$ is in fact a pole of $L(s, \sigma_2 \times \tilde{\sigma}_2)$. This is clear from the fact that the residue at every such pole is non-zero (see the proof of Proposition 1.2 of [8]). It now follows from the simplicity of the poles of $L(s, \sigma_2 \times \tilde{\sigma}_2)$ that

$$L(s, \sigma_2 \times \tilde{\sigma}_2) = (1 - q^{-rs})^{-1},$$

where, as above, r is the order of the cyclic group of unramified characters η satisfying $\sigma_2 \cong \sigma_2 \otimes \eta$.

With notation as in the case of discrete series, our discussion implies

$$L(s, \pi_1 \times \rho_j) = 1$$

unless $\rho_0 \cong \tilde{\pi}_0 \otimes \nu^{s_0}$ with a pure imaginary number s_0. In this case

$$\rho_j \cong \tilde{\pi}_1 \otimes \nu^{s_0 + 1/2(t+u) - j}$$

and

$$\tilde{\rho}_j \cong \tilde{\pi}_1 \otimes \nu^{-s_0 + 1/2(t-u) - 1 + j}.$$

Therefore

$$L(s, \pi_1 \times \tilde{\rho}_j) = (1 - q^{rs_0 - r/2(t-u) + r - rj} \cdot q^{-rs})^{-1},$$

while

$$L(s, \tilde{\pi}_1 \times \rho_j) = (1 - q^{-rs_0 + r/2(t-u) - r + rj} \cdot q^{-rs})^{-1},$$

where r is the order of the cyclic group of unramified characters η satisfying $\pi_0 \cong \pi_0 \otimes \eta$. Observe that s_0 is not unique but q^{-rs_0} is. We thus have:

Proposition 7.1. *Let σ_1 and σ_2 be two discrete series representations of $GL_m(F)$ and $GL_n(F)$, respectively. Choose positive integers a, t, b, and u with $at = m$ and $bu = n$, and irreducible unitary supercuspidal representations π_0 and ρ_0 of $GL_a(F)$ and $GL_b(F)$, respectively, such that $\sigma_1 = \sigma(\pi_1, \ldots, \pi_t)$ and $\sigma_2 = \sigma(\rho_1, \ldots, \rho_u)$, where $\pi_i = \pi_0 \otimes \nu^{\frac{t+1}{2} - i}$, $1 \le i \le t$, and $\rho_j = \rho_0 \otimes \nu^{\frac{u+1}{2} - j}$, $1 \le j \le u$. Then*

$$\mu(s\tilde{\alpha}, \sigma_1 \otimes \sigma_2) = \gamma(G/P)^2 q^{n(\tilde{\sigma}_1 \times \sigma_2)}$$

unless $\rho_0 \cong \pi_0 \otimes \nu^{s_0}$ (and therefore $a = b$) for some pure imaginary number s_0, in which case

$$\mu(s\tilde{\alpha}, \sigma_1 \otimes \sigma_2) = \gamma(G/P)^2 q^{n(\tilde{\sigma}_2 \times \sigma_2)} \cdot \prod_{j=1}^{u} \frac{(1 - q^{rs_0 - r/2(t-u) + r - jr} \cdot q^{-rs})(1 - q^{-rs_0 + r/2(t-u) - r + jr} \cdot q^{rs})}{(1 - q^{rs_0 - r/2(t-u) - jr} \cdot q^{-rs})(1 - q^{-rs_0 + r/2(t-u) + jr} \cdot q^{rs})},$$

if $n \leq m$. Otherwise, i.e. if $n \geq m$, one must change the role of the triple (b, u, ρ_0) with (a, t, π_0). Here r is the order of the cyclic group of all the unramified characters η satisfying $\pi_0 \cong \pi_0 \otimes \eta$. In particular, if σ_1 and σ_2 are both supercuspidal and $\sigma_2 \cong \sigma_1 \otimes \nu^{s_0}$, then

$$\mu(s\widetilde{\alpha}, \sigma_1 \otimes \sigma_2) = \gamma(G/P)^2 q^{n(\widetilde{\sigma}_1 \times \sigma_2)} \frac{(1 - q^{rs_0} \cdot q^{-rs})(1 - q^{-rs_0} q^{rs})}{(1 - q^{rs_0 - r} \cdot q^{-rs})(1 - q^{-rs_0 + r} \cdot q^{rs})}.$$

Suppose $\mathbf{G} = SL(r)$. Let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of G which we may assume to be standard. Let σ be an irreducible tempered representation of M. Then let $\widetilde{\mathbf{G}} = GL(r)$. There exists a standard parabolic subgroup $\widetilde{\mathbf{P}} = \widetilde{\mathbf{M}}\mathbf{N}$ of $\widetilde{\mathbf{G}}$ such that $\mathbf{M} = \widetilde{\mathbf{M}} \cap \mathbf{G}$. If $\mathbf{P}$ is maximal, then so is $\widetilde{\mathbf{P}}$. By Lemma 1.1 of [30] there exists an irreducible tempered representation $\widetilde{\sigma}$ of $\widetilde{M}$ such that $\sigma \subset \widetilde{\sigma}|M$. Moreover, if $\widetilde{\sigma}_1$ is another such representation of $\widetilde{M}$, then there exists a character $\eta \in \hat{F}^*$ such that $\widetilde{\sigma}_1 \cong \widetilde{\sigma} \otimes \eta$. If σ is in the discrete series, then so is $\widetilde{\sigma}$. Clearly

$$A(\widetilde{\nu}, \widetilde{\sigma}, w)|I(\nu, \sigma) = A(\nu, \sigma, w)$$

and therefore

$$\mu(\nu, \sigma) = \mu(\widetilde{\nu}, \widetilde{\sigma}),$$

where $\widetilde{\nu}$ is any extension of ν from $\mathfrak{a}_{\mathbb{C}}^*$ to $\widetilde{\mathfrak{a}}_{\mathbb{C}}^*$ with obvious notation. We therefore have:

Corollary 7.2. *Let $\mathbf{P} = \mathbf{MN}$ be a standard maximal parabolic subgroup of $SL(r)$. Let σ be a discrete series representation of M. Choose two positive integers m and n, $m + n = r$, such that $\mathbf{M} = GL(r) \cap \widetilde{\mathbf{M}}$, where $\widetilde{\mathbf{M}} = GL(m) \times GL(n)$. Choose a pair of discrete series representations σ_1 and σ_2 of $GL_m(F)$ and $GL_n(F)$ such that $\sigma \subset \widetilde{\sigma}|M$, where $\widetilde{\sigma} = \sigma_1 \otimes \sigma_2$. Then*

$$\mu(s\widetilde{\alpha}, \sigma) = \mu(s\widetilde{\alpha}, \sigma_1 \otimes \sigma_2),$$

where $\mu(s\widetilde{\alpha}, \sigma_1 \otimes \sigma_2)$ are given by the formulas in Proposition 7.1. It is independent of the possible choices of σ_1 and σ_2.

Remark 7.3. Using Proposition 7.1 and Corollary 7.2, it must now be a combinatorial problem to obtain R-groups and therefore determine the number of components of the representation $I(\sigma)$, where σ is in the discrete series. This must take care of the non-discrete part of the tempered spectrum of $SL_r(F)$.

REFERENCES

1. J. Arthur, *Intertwining operators and residues I: weighted characters*, J. Funct. Anal. 84 (1989), 19–84.

2. ______, *On some problems suggested by the trace formula*, Lie Group Representations II, Lecture Notes in Math, Vol. 1041, Springer-Verlag, Berlin-Heidelberg-New York, 1983, pp. 1–49.

3. ______, *Unipotent automorphic representations: Conjectures*, Orbites Unipotentes et Representations, II. Groupes p-adic et réels, Société Mathématique de France, Astérisque, Nos. 171–172, 1989, pp. 13–71.

4. ______, *Unipotent automorphic representations: Global motivations*, Proceedings of the Conference on Automorphic Forms, Shimura Varieties, and L-functions, Vol. I (L. Clozel and J.S. Milne, eds.), in *Perspectives in Mathematics*, vol. 10, Academic Press, 1990, pp. 1–75.

5. J. Arthur and L. Clozel, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula,* Annals of Math. Studies, Vol. 120, Princeton University Press, Princeton, 1989.

6. I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups I,* Ann. Scient. Éc. Norm. Sup. **10** (1977), 441–472.

7. A. Borel, *Automorphic L-functions*, Proc. Sympos. Pure Math., AMS, 33, II (1979), 27–61.

8. S. Gelbart and H. Jacquet, *A relation between automorphic representations of* $GL(2)$ *and* $GL(3)$, Ann. Scient. Éc. Norm Sup. **11** (1978), 471–542.

9. Harish-Chandra, *Harmonic analysis on real reductive groups III. The Maass-Selberg relation and the Plancherel formula*, Annals of Math. **104** (1976), 117–201.

10. ______, *The Plancherel formula for reductive p-adic groups*, reprinted in Collected Papers, Springer Verlag, Berlin-Heidelberg-New York, 1984, pp. 353–367.

11. R. Howe and A. Moy, *Hecke algebra isomorphisms for* GL_n *over a p-adic field*, Jour. of Algebra **131** (1990), 388–424.

12. D. Jabon, D. Keys, and A. Moy, *An explicit Plancherel formula for* $U(2,1)$, preprint.

13. H. Jacquet, I.I. Piatetski-Shapiro, and J.A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), 367–464.

14. ______, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981), 199–214.

15. H. Jacquet and J.A. Shalika, *Exterior square L-functions*, Proceedings of the Conference on Automorphic Forms, Shimura Varieties, and L-functions, Vol. II (L. Clozel and J.S. Milne, eds.), in *Perspectives in Mathematics*, vol. 11, Academic Press, 1990, pp. 143–226.

16. D. Keys, *On the decomposition of reducible principal series representation of p-adic Chevalley groups*, Pacific J. Math. **101** (1982), 351–388.

17. ______, *Principal series representations of special unitary groups over local field*, Comp. Math. **51** (1984), 115–130.

18. ______, *L-indistinguishability and R-groups for quasi-split groups: Unitary groups of even dimension*, Ann. Scient. Éc. Norm. Sup. **20** (1987), 31–64.

19. D. Keys and F. Shahidi, *Artin L-functions and normalization of intertwining operators*, Ann. Scient. Éc. Norm. Sup. **21** (1988), 67–89.

20. A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups*, Annals of Math. **93** (1971), 489–578.

21. ______, *Intertwining operators for semisimple groups, II*, Invent. Math. **60** (1980), 9–84.

22. A. W. Knapp and G. J. Zuckerman, *Classification of irreducible tempered representations of semisimple groups*, Annals of Math. **116** (1982), 389–455.

23. R. P. Langlands, *Euler Products*, Yale University Press, New Haven, 1971.

24. ______, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Math., Vol 544, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

25. ———, *Problems in the theory of automorphic forms*, Lec. Notes in Math., Vol. 170, Springer-Verlag, Berlin-Heidelberg-New York, pp. 18–86.

26. ———, *On Artin's L-functions*, Rice University Studies **56** (1970), 23–28.

27. ———, *On the classification of irreducible representations of real algebraic groups*, Representation Theory and Harmonic Analysis on Semisimple Lie Groups (P.J. Sally, Jr. and D.A. Vogan, eds.), Mathematical Surveys and Monographs, AMS, Vol 31, 1989, pp. 101–170.

28. P. J. Sally and M. Tadic, *On representations of non-archimedean symplectic groups*, preprint.

29. F. Shahidi, *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Annals of Math. **132** (1990), 273–330.

30. ———, *Some results on L-indistinguishability for $SL(r)$*, Canadian J. Math. **35** (1983), 1075–1109.

31. ———, *Third symmetric power L-functions for $GL(2)$*, Comp. Math. **70** (1989), 245–273.

32. ———, *Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$*, Amer. J. Math. **106** (1984), 67–111.

33. ———, *Local coefficients and normalization of intertwining operators for $GL(n)$*, Comp. Math. **48** (1983), 271–295.

34. ———, *On certain L-functions*, Amer. J. Math. **103** (1981), 297–356.

35. ———, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Annals of Math. **127** (1988), 547–584.

36. ———, *On multiplicativity of local factors*, in Festschrift in honor of I.I. Piatetski-Shapiro, Part II, Israel Mathematical Conference Proceedings **3** (1990), 279–289.

37. ———, *Local coefficients as Artin factors for real groups*, Duke Math. J. **52** (1985), 973–1007.

38. D. Shelstad, *L-indistinguishability for real groups*, Math. Ann. **259** (1982), 385–430.

39. A. Silberger, *Special representations of reductive p-adic groups are not integrable*, Annals of Math. **111** (1980), 571–587.

40. ———, *The Knapp-Stein dimension theorem for p-adic groups*, Proc. Amer. Math. Soc. **68** (1978), 243–246.

41. ———, *Introduction to Harmonic Analysis on Reductive p-adic Groups* Math. Notes, Vol. 23, Princeton University Press, Princeton, 1979.

42. D. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), 75–98.

43. J. L. Waldspurger, *Un exercice sur $GSp(4, F)$ et les représentations de Weil*, Bull. Soc. Math. France **115** (1987), 35–69.

44. A. V. Zelevinsky, *Induced representations of reductive p-adic groups II: on irreducible representations of $GL(n)$*, Ann. Scient. Eć. Norm. Sup. **13** (1980), 165–210.

45. W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **206** (1973), 311–318.

46. G. I. Olšanskiĭ, *Intertwining operators and complementary series*, Math. USSR Sbornik. **22** (1974), 217–255.

47. B. Tamir, *On L-functions and intertwining operators for unitary groups*, preprint (1990).

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