OUTER LIPSCHITZ CLASSIFICATION OF NORMAL PAIRS OF HÖLDER TRIANGLES

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Abstract. A normal pair of Hölder triangles is the union of two normally embedded Hölder triangles satisfying some natural conditions on the tangency orders of their boundary arcs. It is a special case of a surface germ, a germ at the origin of a two-dimensional closed semialgebraic (or, more general, definable in a polynomially bounded o-minimal structure) subset of $\mathbb{R}^n$. Classification of normal pairs considered in this paper is a step towards outer Lipschitz classification of definable surface germs. In the paper [5] we introduced a combinatorial invariant of the outer Lipschitz equivalence class of normal pairs, called $\sigma\tau$-pizza, and conjectured that it is complete: two normal pairs of Hölder triangles with the same $\sigma\tau$-pizzas are outer Lipschitz equivalent. In this paper we prove that conjecture and define realizability conditions for the $\sigma\tau$-pizza invariant. Moreover, only one of the two pizzas in the $\sigma\tau$-pizza invariant, together with some admissible permutations related to σ and τ, is sufficient for the existence and uniqueness, up to outer Lipschitz equivalence, of a normal pair of Hölder triangles.

1. Introduction

All sets, functions and maps in this paper are germs at the origin of $\mathbb{R}^n$ definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the field of exponents $\mathbb{F}$. The simplest (and most important in applications) examples of such structures are real semialgebraic and (global) subanalytic sets, with $\mathbb{F} = \mathbb{Q}$.

There are two natural metrics on a connected set $X \subset \mathbb{R}^n$: the *inner metric*, where the distance between two points of X is the length of a shortest path in X connecting these points, and the *outer metric*, where the distance between two points of X is their distance in $\mathbb{R}^n$. A set X is called *normally embedded* (see [3]) if its inner and outer metrics are equivalent. There are three natural equivalence relations associated with these metrics. Two sets X and Y are *inner* (resp., *outer*) Lipschitz equivalent if there is an inner (resp., outer) bi-Lipschitz homeomorphism $h : X \to Y$. The sets X and Y are *ambient* Lipschitz equivalent if the homeomorphism $h : X \to Y$ can be extended to a bi-Lipschitz homeomorphism H of the ambient space. The ambient equivalence is stronger than the outer equivalence, and the outer equivalence is stronger then the inner equivalence. Finiteness theorems of Mostowski [10] and Valette [12] show that there are finitely many ambient Lipschitz equivalence classes in any definable family.

This paper is a part of a research project with the ultimate goal of outer Lipschitz classification of *surface germs*, definable two-dimensional closed germs at the origin.

Inner Lipschitz geometry of surface germs is relatively simple. The building block of the inner Lipschitz classification of surface germs is a *β-Hölder triangle* (see Definition 2.7). A combinatorial model for the inner Lipschitz equivalence class of a surface germ

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X is a *Canonical Hölder Complex*, based on a decomposition of X into Hölder triangles and isolated arcs (see [2]).

Outer Lipschitz geometry of surface germs is much more complicated. A special case of a surface germ is the union of a Hölder triangle T and a graph of a Lipschitz function f defined on T. Two such surface germs are outer Lipschitz equivalent when two functions are Lipschitz contact equivalent. This relates outer Lipschitz geometry of surface germs with the Lipschitz geometry of functions. A complete invariant of the Lipschitz contact equivalence of Lipschitz functions, called *minimal pizza*, was defined in [4]. Informally, a pizza for a Lipschitz function f on a normally embedded Hölder triangle T is a decomposition of T into "pizza slices," subtriangles T_i of T, such that the order of f on each arc $\gamma \subset T_i$ depends linearly on the tangency order of γ with a boundary arc of T_i. A pizza is minimal if the union of any two adjacent pizza slices is not a pizza slice.

In this paper we consider *normal pairs* of Hölder triangles defined in [5], surface germs $X = T \cup T' \subset \mathbb{R}^n$, where T and T' are normally embedded Hölder triangles satisfying natural conditions (17) on the tangency orders of their boundary arcs. Let $f : T \to \mathbb{R}$ and $g : T' \to \mathbb{R}$ be Lipschitz functions defined as the distances in $\mathbb{R}^n$ from the points in each of these two triangles to the other triangle. The first question is whether X is outer Lipschitz equivalent to the union of T and the graph of the distance function f. Simple examples (see Figure 1) show that the answer may be negative. Another natural question is whether the minimal pizzas of f and g are equivalent. It was shown in [5] that the answer to this question is also negative. The minimal pizzas for f and g may even have different numbers of pizza slices (see Examples 7.8 and 7.9 below).

In paper [5] we defined the $\sigma\tau$*-pizza* invariant of a normal pair (T, T') of Hölder triangles, consisting of the minimal pizzas Λ and Λ' associated with the distance functions f and g on T and T', characteristic permutation σ and characteristic correspondence τ, where σ and τ detect relations between certain elements of the pizzas Λ and Λ' as follows. The spaces of arcs of T and T' contain outer Lipschitz invariant subsets, called *maximum zones* (see Definition 3.2). The permutation σ in [5, Definition 4.5] (see Proposition 3.3 and Definition 3.4 below) is encoding a canonical one-to-one correspondence between the maximum zones of Λ and Λ'. Two pizza slices T_i and T'_j of Λ and Λ' are called *transversal* if the distance functions of the pair (T_i, T'_j) are equivalent to the distances from the points of one of these triangles to a boundary arc of another one. Otherwise T_i and T'_j are called *coherent* (see Definition 4.1). There is a canonical one-to-one correspondence τ between coherent pizza slices of Λ and Λ' (see [5, Definition 4.8] and Definition 4.6 below).

The main result of [5] states that $\sigma\tau$-pizzas of outer Lipschitz equivalent normal pairs of Hölder triangles are combinatorially equivalent. In this paper we show that the converse is also true: The pizzas Λ and Λ', together with the characteristic permutation σ and characteristic correspondence τ, constitute a *complete invariant* of the outer Lipschitz equivalence class of normal pairs of Hölder triangles. Moreover, given any one of the two pizzas, and given the permutation σ and correspondence τ (more precisely, given σ and a permutation ϖ associated with σ and τ, see Definition 7.14) satisfying some explicit admissibility conditions, a normal pair (T, T') with the given pizza, σ and τ exists and is unique up to outer Lipschitz equivalence. The admissibility conditions are necessary: they are satisfied by the $\sigma\tau$-pizza invariant of any normal pair (T, T').

Section 2 of this paper introduces the main tools of our study. We consider the Valette link [11] of a germ X, the set $V(X)$ of arcs in X parameterized by the distance to the origin. It has a structure of a non-archimedean metric space, where the metric is defined by the tangency order of arcs.

The main advantage of working in $V(X)$, instead of X itself, is the possibility to define outer Lipschitz invariant sets of arcs in $V(X)$, while the only outer Lipschitz invariant arcs in X are Lipschitz singular arcs (see Definition 2.11). We describe the properties of sets and maps in the spaces of arcs corresponding to the geometric properties of surface germs and their bi-Lipschitz maps. In particular, we give another proof of the theorem of Fernandes (see [8]) that a map between two germs is outer bi-Lipschitz if, and only if, the corresponding map between their Valette links is an isometry. We also introduce a notion of *combinatorial normal embedding* of a Hölder triangle, in terms of its Valette link, and prove that combinatorial normal embedding is equivalent to normal embedding.

We remind the definition of a pizza from [4], and define an *abstract pizza* (see Definition 2.19) as a combinatorial encoding of an equivalence class of a pizza associated with a non-negative Lipschitz function on a normally embedded Hölder triangle. We show that a pizza, unique up to combinatorial equivalence, can be recovered from an abstract pizza (see Theorem 2.25). Thus a minimal abstract pizza represents a contact equivalence class of a non-negative Lipschitz function.

At the end of Section 2 we reformulate the notion of pizza in terms of *pizza zones*, Lipschitz invariant subsets of the Valette link of a normally embedded Hölder triangle T, where the boundary arcs of pizza slices of a minimal pizza associated with a Lipschitz function on T may be selected.

In Section 3 we define *maximal exponent zones* (or simply *maximum zones*), the pizza zones of the minimal pizzas Λ and Λ' on T and T', associated with the distance functions f and g, where the orders of these functions attain local maxima, and introduce the *characteristic permutation* σ of a normal pair (T, T') encoding one-to-one correspondence between the maximum zones of Λ and Λ'.

In Section 4 we define *transversal* and *coherent* pizza slices of the pizzas Λ and Λ' and introduce the correspondence τ between their coherent pizza slices. This is a *signed* correspondence: since pizza slices of Λ and Λ' have orientations induced by the orientations of T and T', the action of τ may either preserve or reverse that orientation. To understand relations between σ and τ, we investigate combinatorial and metric properties of the $\sigma\tau$-pizza invariant defined in [5]. Note that σ and τ are of a rather different nature: the permutation σ is encoding a one-to-one correspondence between maximum zones of pizzas Λ and Λ', which are some of their pizza zones, while τ is a one-to-one correspondence between coherent pizza slices of Λ and Λ', which cannot in general be extended to a one-to-one correspondence between their pizza zones. We start with the permutation υ and the sign function s on coherent pizza slices induced by τ (see Definition 4.7). The triple (σ, υ, s) defined for a pair (T, T') satisfies *allowability conditions* (see Definition 4.22) which can be formulated in terms of the pizza Λ on T, based on the properties of sets of pizza slices of Λ called *caravans* (see Definition 4.12). A triple (σ, υ, s) is *allowable* when these conditions are satisfied.

To combine σ with υ, we consider the disjoint union $\mathcal{K}$ of the sets of maximum zones and coherent pizza slices of Λ, and the corresponding set $\mathcal{K}'$ for Λ'. These two sets have the same number of elements K. Ordering them according to orientation of T and T', we define the *combined characteristic permutation* ω of the pair (T, T') on a set of K elements (see Definition 4.19) and show that, given a pizza Λ on T and an allowable triple (σ, υ, s), the permutation ω can be uniquely determined, even when T' and Λ' are not known. This construction is important for the realization theorem (Theorem 7.25) in Section 7.

In Section 5 we consider prove that two normal pairs (T, T') and (S, S') of Hölder triangles are outer Lipschitz equivalent if, and only if, their $\{\sigma\tau\}$-pizza invariants are combinatorially equivalent.

In Section 6 we introduce a combinatorial notion of *blocks* that allows us to establish relations between the metric (pizzas) and combinatorial (σ and τ) parts of the $\{\sigma\tau\}$-pizza invariant, which are necessary and sufficient for the existence and uniqueness, up to outer Lipschitz equivalence, of a normal pair of Hölder triangles. We do this first in the *totally transversal* case, when there are no coherent pizza slices.

For a totally transversal pizza Λ associated with a non-negative Lipschitz function f on a Hölder triangle T, a family $\lambda_0, \ldots, \lambda_{n-1}$ of n arcs in $V(T)$, ordered according to orientation of T, which are either the boundary arcs of T or belong to maximum zones of Λ, such that each maximum zone contains exactly one of these arcs, is called a *supporting family* (see Definition 6.10). The pizza Λ is completely determined by the exponents $\beta_{ij} = tord(\lambda_i, \lambda_j)$ and $q_i = ord_{\lambda_i} f$ associated with a supporting family.

Let π be a permutation of the set $[n] = \{0, \ldots, n-1\}$ of n elements. A *segment* of $[n]$ is a non-empty set of consecutive indices $\{i, \ldots, k\}$. A segment B of $[n]$ is called a *block* of π if the set $\pi(B)$ is also a segment of $[n]$ (not necessarily in increasing order). Each non-empty subset J of $[n]$ is contained in a unique minimal block $B_\pi(J)$ of π. A permutation π of $[n]$ is called *admissible* with respect to a Lipschitz function f on T (or with respect to a minimal pizza Λ on T associated with f) if it satisfies the *block conditions*: $\beta_{ij} \le \beta_{ik}$ for all $k \in B_\pi(\{i, j\})$ (condition (24) in Theorem 6.6).

Given a totally transversal pizza Λ with a supporting family of n arcs on a Hölder triangle T, associated with a non-negative function f on T, and an admissible with respect to f permutation π of $[n]$, there exists a unique, up to outer Lipschitz equivalence, totally transversal normal pair (T, T') realizing the pizza Λ and the permutation π of $[n]$ compatible with the characteristic permutation σ of the pair (T, T') on the family of maximum zones.

In Section 7 we establish the existence and uniqueness, up to outer Lipschitz equivalence, of a general normal pair of Hölder triangles, based on admissibility conditions, including the block conditions, for an analog ϖ of the permutation π. In this section we define *pre-pizza* (see Definition 7.3) obtained from a minimal pizza by removing *non-essential* pizza zones where the action of σ and τ is not defined, and *twin pre-pizza* (see Definition 7.6) obtained from a pre-pizza by adding "twin arcs" to ensure that τ is one-to-one and compatible with σ on the expanded set of arcs. In the *totally transversal* case, when there are no coherent pizza slices, pre-pizza is determined by the maximum zones and twin pre-pizza is the same as pre-pizza. The minimal pizza, unique up to combinatorial equivalence, can be recovered from the corresponding pre-pizza, and a pre-pizza can be recovered from the corresponding twin pre-pizza. Using definitions of pre-pizza and twin pre-pizza, we define admissibility conditions for the permutation ϖ. The construction of ϖ is based on the combined characteristic permutation ω defined in Section 4. The admissibility conditions for ϖ include the allowability conditions for the triple (σ, υ, s) and the block conditions (39). The main result of this section (and also the main realization result of the paper) is similar to the realization theorem (Theorem 6.14) for the totally transversal case in Section 6.

Given a pizza Λ and an admissible permutation ϖ, one can construct a normally embedded triangle T', such that (T, T') is a normal pair of Hölder triangles with the pizza Λ associated with the distance function $f(x) = dist(x, T')$ on T, and ϖ is compatible with the characteristic permutation σ and characteristic correspondence τ of the pair (T, T').

Some remarks about the figures. We try to illustrate the "dynamics" of the link of a surface germ X, dependence of the intersection of X with a small sphere on the radius of the sphere. Accordingly, in our figures X is a curve, a pair of Hölder triangles is the union of two intervals, and some arcs in X are marked as points. We try to imitate the non-archimedean metric on $V(X)$ by representing the arcs with higher tangency order by the nearby points in the plane, and mark some important zones as shaded disks. We hope it will bring some intuition to the reader.

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2. Preliminaries

Definition 2.1. A germ X at the origin inherits two metrics from the ambient space: the *inner metric* where the distance between two points of X is the length of the shortest path connecting them inside X, and the *outer metric* with the distance between two points of X being their distance in the ambient space. A germ X is *normally embedded* if its inner and outer metrics are equivalent.

For a point $x \in X$ and a subset $Y \subset X$ we define the *outer distance* $dist(x, Y) = \inf_{y \in Y} |x - y|$, and the *inner distance* $idist(x, Y)$ as the infimum of the lengths of paths connecting x with points $y \in Y$.

A *surface germ* is a closed germ X at the origin such that $\dim_{\mathbb{R}} X = 2$.

Definition 2.2. An *arc* in $\mathbb{R}^n$ is (a germ at the origin of) a mapping $\gamma : [0, \epsilon) \to \mathbb{R}^n$ such that $\gamma(0) = \mathbf{0}$. Unless otherwise specified, we suppose that arcs are parameterized by the distance to the origin, i.e., $|\gamma(t)| = t$. We usually identify an arc γ with its image in $\mathbb{R}^n$. The *Valette link* of X is the set $V(X)$ of all arcs $\gamma \subset X$.

Definition 2.3. Let $f \not\equiv 0$ be a Lipschitz function defined on an arc γ, such that $f(\mathbf{0}) = 0$. The *order* of f on γ is the exponent $q = ord_\gamma f \in \mathbb{F}_{\geq 1}$ such that $f(\gamma(t)) = ct^q + o(t^q)$ as $t \to 0$, where $c \neq 0$. If $f \equiv 0$ on γ, then $ord_\gamma f = \infty$.

Definition 2.4. The *tangency order* between two arcs γ and γ' is defined as $tord(\gamma, \gamma') = ord_\gamma |\gamma(t) - \gamma'(t)|$. The tangency order between an arc γ and a set of arcs $Z \subset V(X)$ is defined as $tord(\gamma, Z) = \sup_{\lambda \in Z} tord(\gamma, \lambda)$. The tangency order between two subsets Z and Z' of $V(X)$ is defined as $tord(Z, Z') = \sup_{\gamma \in Z} tord(\gamma, Z')$. Similarly, $itord_X(\gamma, \gamma')$, $itord_X(\gamma, Z)$ and $itord_X(Z, Z')$ denote the tangency orders with respect to the inner metric. The distance $\xi(\gamma, \gamma') = 1/tord(\gamma, \gamma')$ between arcs in X defines a *non-archimedean metric* ξ on $V(X)$.

Proposition 2.5. (See [8].) *Let $(X, 0)$ and $(Y, 0)$ be definable germs at the origin of $\mathbb{R}^n$. A definable homeomorphism $\Phi : (X, 0) \to (Y, 0)$ preserving the distance to the origin is bi-Lipschitz if, and only if, for any two arcs $\gamma_1, \gamma_2 \in V(X)$ one has*

$$tord(\gamma_1, \gamma_2) = tord(\Phi(\gamma_1), \Phi(\gamma_2)). \tag{1}$$

Proof. We present a proof, slightly different from the proof given in [8]. It is similar to the proof of the main result of [6]. If Φ is bi-Lipschitz then (1) is obviously satisfied. Let us show that Φ is a Lipschitz map when (1) is satisfied.

Consider the set $W \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined as

$$W = \{(x_1, x_2, z) : x_1 \in X,\ x_2 \in X,\ 0 < z < 1,\ |x_1 - x_2| < z|\Phi(x_1) - \Phi(x_2)|\} \tag{2}$$

Note that, since Φ preserves the distance to the origin, the set W does not contain any points with $x_1 = 0$ or $x_2 = 0$. If Φ is not a Lipschitz map, then the closure of $W_c = W \cap \{z = c\}$ contains the point $(0, 0, c)$, for $0 < c < 1$. By the arc selection lemma, the set $W \cup \{0, 0, 0\}$ contains an arc $\gamma(z) = (\gamma_1(z), \gamma_2(z), z)$ parameterized by $z \geq 0$, such that $\lim_{z\to 0}(\gamma_1(z), \gamma_2(z)) = (0, 0)$ and $|\gamma_1(z) - \gamma_2(z)| < z|\Phi(\gamma_1(z)) - \Phi(\gamma_2(z))|$. Then projection of γ to $\mathbb{R}^n \times \mathbb{R}^n$ along the z-axis is an arc $\Gamma(z) = (\gamma_1(z), \gamma_2(z)) \subset X \times X$ parameterized by z, such that

$$|\gamma_1(z) - \gamma_2(z)| < z|\Phi(\gamma_1(z)) - \Phi(\gamma_2(z))|. \tag{3}$$

Let $t_1 = |\gamma_1(z)|$, $t_2 = |\gamma_2(z)|$ and $t = \sqrt{t_1^2 + t_2^2} = |\Gamma(z)|$, thus $z = Ct^\beta + o(t^\beta)$ for some $\beta > 0$ and $C > 0$. Since Φ preserves the distance to the origin, we have $|\Phi(\gamma_1(z))| = t_1$, $|\Phi(\gamma_2(z))| = t_2$ and $|\Phi(\Gamma(z))| = t$. Since $|\gamma_1(z) - \gamma_2(z)| \leq t_1 + t_2 \leq 2t$ by triangle inequality, and $|\Phi(\gamma_1(z)) - \Phi(\gamma_2(z))| \leq 2t$, (3) implies that

$$|\gamma_1(z) - \gamma_2(z)| < 2zt = 2Ct^{\beta+1} + o(t^{\beta+1}). \tag{4}$$

Since $|t_1 - t_2| \leq |\gamma_1(z) - \gamma_2(z)| = o(t)$, this implies that $\alpha = tord(\gamma_1, \gamma_2) \geq \beta + 1 > 1$. Since Φ preserves the tangency orders of arcs, we have $tord(\Phi(\gamma_1), \Phi(\gamma_2)) = \alpha$, thus $|\Phi(\gamma_1(z)) - \Phi(\gamma_2(z))| = C_1 z^\alpha + o(z^\alpha)$ for some $C_1 > 0$, in contradiction with (3). This finishes the proof that Φ is a Lipschitz map.

The same arguments applied to Φ^{-1} show that Φ^{-1} is also a Lipschitz map. □

Definition 2.6. For $\beta \in \mathbb{F}$, $\beta \geq 1$, the *standard β-Hölder triangle* is (a germ at the origin of) the set

$$T_\beta = \{(u, v) \in \mathbb{R}^2 \mid u \geq 0,\ 0 \leq v \leq u^\beta\}. \tag{5}$$

The arcs $\{u \geq 0,\ v = 0\}$ and $\{u \geq 0,\ v = u^\beta\}$ are the *boundary arcs* of T_β.

Definition 2.7. A *β-Hölder triangle* is (a germ at the origin of) a set $T \subset \mathbb{R}^n$ that is inner bi-Lipschitz homeomorphic to the standard β-Hölder triangle (5). The number $\beta = \mu(T) \in \mathbb{F}$ is called the *exponent* of T. The arcs γ_1 and γ_2 of T mapped to the boundary arcs of T_β by an inner bi-Lipschitz homeomorphism are the *boundary arcs* of T (notation $T = T(\gamma_1, \gamma_2)$). All other arcs of T are its *interior arcs*. The set of interior arcs of T is denoted by $I(T)$. An arc $\gamma \subset T$ is *generic* if $itord(\gamma, \gamma_1) = itord(\gamma, \gamma_2)$. The set of generic arcs of T is denoted by $G(T)$.

Definition 2.8. Two normally embedded Hölder triangles are called *transversal* if there is a boundary arc $\tilde\gamma$ of T and a boundary arc $\tilde\gamma'$ of T' such that $tord(\tilde\gamma, \gamma') = tord(\gamma', T)$ for any arc γ' of T' and $tord(\tilde\gamma', \gamma) = tord(\gamma, T')$ for any arc γ of T.

Definition 2.9. Let $T = \bigcup T_i$ be a Hölder triangle decomposed into normally embedded subtriangles $T_i = T(\lambda_{i-1}, \lambda_i)$, such that $T_i \cap T_{i+1} = \{\lambda_i\}$. We say that T is *combinatorially normally embedded* if any two triangles T_i and T_j are transversal (see Definition 2.8) and

$$tord(\lambda_i, \lambda_j) = \min(tord(\lambda_i, \lambda_k), tord(\lambda_k, \lambda_j)) \ \text{ for } \ i < k < j. \tag{6}$$

Proposition 2.10. *If a Hölder triangle T is combinatorially normally embedded, then it is normally embedded.*

Proof. If T_i is a β_i-Hölder triangle, where $\beta_i = tord(\lambda_{i-1}, \lambda_i)$, then (6) implies that $tord(\lambda_i, \lambda_j) = \min\limits_{k:i<k\leq j} \beta_k = itord(\lambda_i, \lambda_j)$ for $i < j$ by the non-archimedean property of the tangency order.
If T is not normally embedded, then there are arcs γ and γ' in T such that $itord(\gamma, \gamma') < tord(\gamma, \gamma')$. These two arcs cannot belong to the same triangle $T_i \subset T$, since T_i is normally embedded. Suppose that $\gamma \subset T_i$ and $\gamma' \subset T_j$, where $i < j$. Let $T_0 = T(\gamma, \gamma') \subset T$. If $j = i+1$, then $T_i \cup T_j$ is a Hölder triangle. Since T_i and T_j are transversal, there is a normally embedded Hölder triangle T' such that $T_0 \subset T_i \cup T_j \subset T'$. This implies that T_0 is a normally embedded Hölder triangle, a contradiction. Thus we may assume that $i < j-1$. If T' is a normally embedded Hölder triangle such that $T_i \cup T_j \subset T'$, let $T'_0 = T'(\gamma, \gamma')$ be a β'-Hölder subtriangle of T' bounded by γ and γ'. Then the following four cases are possible:
(a) $\lambda_i \cup \lambda_{j-1} \subset T'_0$, **(b)** $\lambda_i \cup \lambda_j \subset T'_0$, **(c)** $\lambda_{i-1} \cup \lambda_{j-1} \subset T'_0$, **(d)** $\lambda_{i-1} \cup \lambda_j \subset T'_0$.
In case **(a)** we have

$$\begin{aligned}\beta' &= tord(\gamma, \gamma') = itord_{T'}(\gamma, \gamma') = \min(tord(\gamma, \lambda_i), itord_{T'}(\lambda_i, \lambda_{j-1}), tord(\lambda_{j-1}, \gamma')) \\ &= \min(tord(\gamma, \lambda_i), tord(\lambda_i, \lambda_{j-1}), tord(\lambda_{j-1}, \gamma')) = itord_T(\gamma, \gamma'),\end{aligned} \tag{7}$$

where $tord(\gamma, \gamma') = itord_{T'}(\gamma, \gamma')$ and $tord(\lambda_i, \lambda_{j-1}) = itord_{T'}(\lambda_i, \lambda_{j-1})$ since T' is normally embedded, a contradiction.
In case **(b)** we have

$$\begin{aligned}\beta' &= tord(\gamma, \gamma') = itord_{T'}(\gamma, \gamma') = \min(tord(\gamma, \lambda_i), itord_{T'}(\lambda_i, \lambda_j), tord(\lambda_j, \gamma')) \\ &\leq \min(tord(\gamma, \lambda_i), tord(\lambda_i, \lambda_{j-1}), tord(\lambda_{j-1}, \gamma')) = itord_T(\gamma, \gamma'),\end{aligned} \tag{8}$$

as $tord(\lambda_i, \lambda_j) \leq tord(\lambda_i, \lambda_{j-1})$ and $tord(\lambda_{j-1}, \gamma') \geq tord(\lambda_{j-1}, \lambda_j) \geq tord(\lambda_i, \lambda_j) = itord_{T'}(\lambda_i, \lambda_j)$, again a contradiction.
Cases **(c)** and **(d)** are similar. This completes the proof of Proposition 2.10. □

Definition 2.11. Let X be a surface germ. An arc $\gamma \in V(X)$ is *Lipschitz non-singular* if there exists a normally embedded Hölder triangle $T \subset X$ such that $\gamma \in I(T)$ and $\gamma \not\subset \overline{X \setminus T}$. Otherwise, γ is *Lipschitz singular*. A Hölder triangle T is *non-singular* if any arc $\gamma \in I(T)$ is Lipschitz non-singular.

Definition 2.12. For a Lipschitz function f on a Hölder triangle T, let

$$Q_f(T) = \bigcup_{\gamma \in V(T)} ord_\gamma f. \tag{9}$$

Remark 2.13. It was shown in [4, Lemma 3.3] that $Q_f(T)$ is either a point or a closed interval in $\mathbb{F} \cup \{\infty\}$.

Definition 2.14. A Hölder triangle T is *elementary* with respect to a Lipschitz function f on T if, for any $q \in Q_f(T)$ and any two arcs γ and γ' in T such that $ord_\gamma f = ord_{\gamma'} f = q$, the order of f is q on any arc in the Hölder triangle $T(\gamma, \gamma') \subset T$.

Definition 2.15. Let T be a normally embedded Hölder triangle and f a Lipschitz function on T. For an arc $\gamma \subset T$, the *width* $\mu_T(\gamma, f)$ of γ with respect to f is the infimum of exponents of Hölder triangles $T' \subset T$ containing γ such that $Q_f(T')$ is a point. For

$q \in Q_f(T)$ let $\mu_{T,f}(q)$ be the set of exponents $\mu_T(\gamma, f)$, where γ is any arc in T such that $ord_\gamma f = q$. It was shown in [4] that, for each $q \in Q_f(T)$, the set $\mu_{T,f}(q)$ is finite. This defines a multivalued *width function* $\mu_{T,f} : Q_f(T) \to \mathbb{F} \cup \{\infty\}$. If T is an elementary Hölder triangle with respect to f, then the function $\mu_{T,f}$ is single valued. When f is fixed, we write $\mu_T(\gamma)$ and μ_T instead of $\mu_T(\gamma, f)$ and $\mu_{T,f}$.

The *depth* $\nu_T(\gamma, f)$ of an arc γ with respect to f is the infimum of exponents of Hölder triangles $T' \subset T$ such that $\gamma \in G(T')$ and $Q_f(T')$ is a point. By definition, $\nu_T(\gamma, f) = \infty$ when there are no such triangles T'.

Definition 2.16. Let $T = T(\gamma_1, \gamma_2)$ be normally embedded β-Hölder triangle and f a Lipschitz function on T. We say that T is a *pizza slice* associated with f if it is elementary with respect to f, either $Q_f(T)$ is not a point and $\mu_{T,f}(q) = aq + b$ is an affine function on $Q_f(T)$, or $Q_f(T) = \{q\}$ is a point and $\mu_{T,f}(q) = \beta$ is a single exponent. If T is a pizza slice and $Q_f(T)$ is not a point, then $a \neq 0$, and one of the boundary arcs of T (either $\tilde{\gamma} = \gamma_1$ or $\tilde{\gamma} = \gamma_2$) such that $\mu_T(\tilde{\gamma}, f) = \max_{q \in Q_f(T)} \mu_{T,f}(q)$ is called the *supporting arc* of T with respect to f.

Proposition 2.17. *Let T be a normally embedded β-Hölder triangle which is a pizza slice associated with a non-negative Lipschitz function f, such that $Q = Q_f(T)$ is not a point. Then $\beta \le \mu_{T,f}(q) \le \max(q, \beta)$ for all $q \in Q$. If $\tilde{\gamma}$ is the supporting arc of T with respect to f, then $\mu_T(\gamma, f) = tord(\tilde{\gamma}, \gamma)$ for all arcs $\gamma \subset T$ such that $\mu_T(\gamma, f) < \mu_T(\tilde{\gamma}, f)$. In particular, $\min_{q \in Q} \mu_{T,f}(q) = \beta$.*

Proof. Since f is a Lipschitz function, we obtain $\beta \le \mu_{T,f}(q) \le \max(q, \beta)$. Using Lemma 3.3 from [4] we get $\mu_T(\gamma, f) = tord(\tilde{\gamma}, \gamma)$. □

Definition 2.18. (See [4, Definition 2.13].) Let f be a non-negative Lipschitz function on a normally embedded β-Hölder triangle $T = T(\gamma_1, \gamma_2)$ oriented from γ_1 to γ_2. A *pizza* on T associated with f is a decomposition $\Lambda = \{T_\ell\}_{\ell=1}^p$ of T into Hölder triangles $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ ordered according to the orientation of T, such that $\lambda_0 = \gamma_1$ and $\lambda_p = \gamma_2$ are the boundary arcs of T, $T_\ell \cap T_{\ell+1} = \lambda_\ell$ for $0 < \ell < p$, and each triangle T_ℓ is a pizza slice associated with f. The pizza Λ comes with the following *toppings*:
- Exponents $q_\ell = ord_{\lambda_\ell} f$ for $0 \le \ell \le p$;
- Exponents $\beta_\ell = \mu(T_\ell)$ for $1 \le \ell \le p$;
- Intervals of exponents $Q_\ell = [q_{\ell-1}, q_\ell]$ for $1 \le \ell \le p$;
- Affine functions $\mu_\ell(q)$ on Q_ℓ for $1 \le \ell \le p$, such that $\mu_\ell(q) \le q$ for all $q \in Q_\ell$ and $\min_{q \in Q_\ell} \mu_\ell(q) = \beta_\ell$, or $\mu_\ell(q_\ell) = \beta_\ell$ is a single exponent when $Q_\ell = \{q_\ell\}$ is a point;
- Exponents $\nu_\ell = \nu_T(\lambda_\ell, f)$, where $\nu_0 = \nu_p = \infty$, $\nu_\ell = \max(\mu_\ell(q_\ell), \mu_{\ell+1}(q_\ell))$ for $0 < \ell < p$.

The pizza Λ is *minimal* if $T_{\ell-1} \cup T_\ell$ is not a pizza slice for any $\ell > 1$.

Definition 2.19. (See [4, Definition 2.12].) An *abstract pizza* $\mathcal{P}$ on a set $\{0, \ldots, p\}$ is a finite sequence $\{q_\ell\}_{\ell=0}^p$, where $q_\ell \in \mathbb{F}_{\ge 1} \cup \{\infty\}$, and a finite collection $\{\beta_\ell, Q_\ell, \mu_\ell\}_{\ell=1}^p$, where $\beta_\ell \in \mathbb{F}_{\ge 1}$, $Q_\ell = [q_{\ell-1}, q_\ell] \subset \mathbb{F}_{\ge 1} \cup \{\infty\}$ is either a point or a closed interval, $\mu_\ell : Q_\ell \to \mathbb{F} \cup \{\infty\}$ is an affine function, non-constant when Q_ℓ is not a point, such that $\mu_\ell(q) \le q$ for all $q \in Q_\ell$ and $\min_{q \in Q_\ell} \mu_\ell(q) = \beta_\ell$. If $Q_\ell = \{q_\ell\}$ is a point, then $\mu_\ell(q_\ell) = \beta_\ell$ is a single exponent. For $0 \le \ell \le p$, exponents ν_ℓ are defined as follows: $\nu_0 = \nu_p = \infty$ and $\nu_\ell = \max(\mu_\ell(q_\ell), \mu_{\ell+1}(q_\ell))$ for $0 < \ell < p$.

An abstract pizza is *reducible* if $p > 1$ and one of the following conditions is satisfied:

(a) $Q_\ell = Q_{\ell+1} = \{q_\ell\}$ for some $\ell < p$.

(b) $Q_\ell = \{q_\ell\} \neq Q_{\ell+1}$ and $\beta_\ell \geq \mu_{\ell+1}(q_\ell)$ for some $\ell < p$.
(c) $Q_\ell \neq \{q_\ell\} = Q_{\ell+1}$ and $\beta_{\ell+1} \geq \mu_\ell(q_\ell)$ for some $\ell < p$.
(d) $Q_\ell \neq \{q_\ell\} \neq Q_{\ell+1}$ and $Q_\ell \cap Q_{\ell+1} = \{q_\ell\}$ for some $\ell < p$, and the function $\mu(q)$ on $Q_\ell \cup Q_{\ell+1}$, such that $\mu(q) = \mu_\ell(q)$ for $q \in Q_\ell$ and $\mu(q) = \mu_{\ell+1}(q)$ for $q \in Q_{\ell+1}$, is affine.
An abstract pizza is *minimal* if it is not reducible.

Remark 2.20. We do not need the signs $\pm$ that are part of the definition in [4], since only non-negative functions are considered in this paper.

Lemma 2.21. *Let $\mathcal{P} = \{\{q_\ell\}_{\ell=0}^p,\ \{\beta_\ell, Q_\ell, \mu_\ell\}_{\ell=1}^p,\ \{\nu_\ell\}_{\ell=0}^p\}$, where $p > 1$, be an abstract pizza. If $q_{\ell-1} = q_\ell$ and $\min(\nu_{\ell-1}, \nu_\ell) = \beta_\ell$ for some $\ell > 0$, then $\mathcal{P}$ is reducible.*

Proof. According to Definition 2.19, an abstract pizza is reducible if one of the conditions (a)-(d) is satisfied. If $q_{\ell-1} = q_\ell$ and $\min(\nu_{\ell-1}, \nu_\ell) = \beta_\ell$, then one of conditions (a)-(c) is satisfied, thus the abstract pizza is reducible. $\square$

Definition 2.22. Given a pizza $\Lambda = \{T_\ell\}_{\ell=1}^p$, where $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$, associated with a non-negative Lipschitz function f on an oriented β-Hölder triangle T, the *corresponding abstract pizza* $\mathcal{P}$ is defined by setting $q_\ell = ord_{\lambda_\ell} f$, $\beta_\ell = \mu(T_\ell) = tord(\lambda_{\ell-1}, \lambda_\ell)$, $Q_\ell = Q_f(T_\ell)$, $\mu_\ell(q) = \mu_{T_\ell, f}(q)$, $\nu_\ell = \nu_T(\lambda_\ell, f)$. The abstract pizza $\mathcal{P}$ is minimal if, and only if, the pizza Λ is minimal.

Conversely, any abstract pizza as in Definition 2.19 is associated with the pizza of a non-negative Lipschitz function on a standard β-Hölder triangle T_β, where $\beta = \min_\ell \beta_\ell$ (see Theorem 2.25 below). To construct such a function, we start with the definition of a standard pizza slice.

Definition 2.23. Given exponents β and $\underline{q}$ in $\mathbb{F}_{\geq 1}$, exponent $\tilde{q} \neq \underline{q}$ in $\mathbb{F}_{\geq 1} \cup \{\infty\}$, and a non-constant affine function $\mu(q) = a(q - \alpha)$ from $Q = [\underline{q}, \tilde{q}]$ to $\mathbb{F} \cup \{\infty\}$, such that $\mu(q) \leq q$ for all $q \in Q$ and $\min_{q \in Q} \mu(q) = \mu(\underline{q}) = \beta \geq 1$, let

$$\phi = \phi_{\beta, \underline{q}, \tilde{q}, \mu}(u, v) = u^\alpha v^{1/a} \tag{10}$$

be a function on the β-Hölder triangle $T_{\beta,\kappa} = \{u \geq 0,\ u^\kappa \leq v \leq u^\beta\}$. Here $\kappa = \max_{q \in Q} \mu(q) = \mu(\tilde{q}) > \beta$ and $u^\kappa \equiv 0$ if $\kappa = \infty$. Let $h_{\beta,\kappa}$ be a bi-Lipschitz mapping $(u, v) \mapsto \big(u, u^\kappa + v(1 - u^{\kappa-\beta})\big)$ from the standard β-Hölder triangle T_β to $T_{\beta,\kappa}$, and let

$$\psi = \psi_{\beta, \underline{q}, \tilde{q}, \mu}(u, v) = \phi_{\beta, \underline{q}, \tilde{q}, \mu}(h(u, v)). \tag{11}$$

The pair $(T_\beta, \psi_{\beta, \underline{q}, \tilde{q}, \mu})$ is called the *standard pizza slice* associated with β, $\underline{q}$, $\tilde{q}$ and μ. When $\underline{q} = \tilde{q}$, thus $Q = \{\tilde{q}\}$ is a point and $\mu(\tilde{q}) = \beta$ is a single exponent, the standard pizza slice is defined as (T_β, ψ), where

$$\psi = \psi_{\beta, \underline{q}, \tilde{q}, \mu}(u, v) = u^{\tilde{q}}. \tag{12}$$

Lemma 2.24. *The function $\psi = \psi_{\beta, \underline{q}, \tilde{q}, \mu}(u, v)$ in (11) is Lipschitz, and T_β is a pizza slice associated with ψ, with the affine width function $\mu(q) = a(q - \alpha)$ defined on $Q_\psi(T_\beta) = [\underline{q}, \tilde{q}]$.*

Proof. Since $h_{\beta,\kappa}$ is (the germ at the origin of) a bi-Lipschitz homeomorphism $T_\beta \to T_{\beta,\kappa}$, it is enough to show that $\phi = u^\alpha v^{1/a}$ in (10) is a Lipschitz function on $T_{\beta,\kappa}$. Moreover, as the family of arcs $\gamma_\mu = \{v = u^\mu\} \subset T_{\beta,\kappa}$, for $\mu \in \mathbb{F}$, $\kappa \geq \mu \geq \beta$, is dense in $T_{\beta,\kappa}$, it is enough to show that $d\phi(u, u^\mu)/du = (\alpha + \frac{\mu}{a}) u^{\alpha + \frac{\mu}{a} - 1}$, and $\partial\phi/\partial v = \frac{1}{a} u^\alpha v^{\frac{1}{a} - 1}$ restricted to

γ_μ, are uniformly in μ bounded in a neighborhood of the origin. Since $\mu(q) = a(q-\alpha)$, we have $q(\mu) = \alpha + \frac{\mu}{a}$. We consider three cases, depending on the value of a.

Case $0 < a \leq 1$. Note that, when $\kappa = \infty$, we have $T_{\beta,\kappa} = T_\beta$ and $\psi = \phi = u^\alpha v^{1/a}$. Since q may be arbitrary large in that case, condition $1 \leq \beta \leq \mu(q) \leq q$ for all $q \in Q$ implies that this is possible only when $0 < a \leq 1$. As the minimal value of $q(\mu)$ is $q(\beta) = \alpha + \frac{\beta}{a}$, we have $\alpha \geq q\left(1 - \frac{1}{a}\right) \geq \left(\alpha + \frac{\beta}{a}\right)\left(1 - \frac{1}{a}\right)$. Thus $\alpha \geq \beta\left(1 - \frac{1}{a}\right)$ and $\alpha + \frac{\mu}{a} - 1 \geq \beta + \frac{\mu - \beta}{a} - 1 \geq 0$. Thus implies that $d\phi(u, u^\mu)/du = (\alpha + \frac{\mu}{a})u^{\alpha + \frac{\mu}{a} - 1}$ is either monotone increasing function of u for $u > 0$, or a constant $\alpha + \frac{\mu}{a} = 1$ when $\mu = \beta = 1$ and $\alpha = 1 - \frac{1}{a}$. Since this function is bounded as a function of μ for a fixed positive $u < 1$, it is uniformly bounded on a neighborhood of the origin in T_β. This proves also that $d\phi(u, u^\mu)/du$ is bounded on $T_{\beta,\kappa} \subset T_\beta$ when $\kappa \neq \infty$. Similarly, $\partial\phi/\partial v = \frac{1}{a}u^\alpha v^{\frac{1}{a} - 1}$ restricted to γ_μ is $\frac{1}{a}u^{\alpha + \frac{\mu}{a} - \mu} \leq u^{(\mu - \beta)\left(\frac{1}{a} - 1\right)}$ when $0 \leq u \leq 1$, as $\alpha + \frac{\mu}{a} - \mu \geq (\mu - \beta)\left(\frac{1}{a} - 1\right)$.

Case $a > 1$. In that case $\kappa \neq \infty$, thus $\mu \leq \kappa$ is bounded. Condition $\mu(q) \leq q$ is equivalent to $\alpha \geq q(1 - \frac{1}{a})$ for all $q \in Q$. Since the maximal value of $q \in Q$ is $q(\kappa) = \alpha + \frac{\kappa}{a}$, this implies $\alpha \geq \left(\alpha + \frac{\kappa}{a}\right)\left(1 - \frac{1}{a}\right)$, thus $\alpha \geq \kappa\left(1 - \frac{1}{a}\right)$. We have $d\phi(u, u^\mu)/du = \left(\alpha + \frac{\mu}{a}\right)u^{\alpha + \frac{\mu}{a} - 1}$, which is bounded since $\alpha + \frac{\mu}{a} = q(\mu) \leq q(\kappa) = \alpha + \frac{\kappa}{a}$ and $\alpha + \frac{\mu}{a} - 1 = q(\mu) - 1 \geq 0$. Similarly, $\partial\phi/\partial v = \frac{1}{a}u^\alpha v^{\frac{1}{a} - 1}$ restricted to γ_μ is $\frac{1}{a}u^{\alpha + \frac{\mu}{a} - \mu}$ which is bounded when $0 \leq u \leq 1$, as $\alpha + \frac{\mu}{a} - \mu \geq (\kappa - \mu)\left(1 - \frac{1}{a}\right) \geq 0$.

Case $a < 0$. In that case $\kappa \neq \infty$, thus $\mu \leq \kappa$ is bounded. Condition $\mu(q) \leq q$ is equivalent to $\alpha \leq q(1 - \frac{1}{a})$ for all $q \in Q$. Since the minimal value of $q \in Q$ is $q(\kappa) = \alpha + \frac{\kappa}{a}$, this implies $\alpha \leq \left(\alpha + \frac{\kappa}{a}\right)\left(1 - \frac{1}{a}\right)$, thus $\alpha \geq \kappa\left(1 - \frac{1}{a}\right)$. We have $d\phi(u, u^\mu)/du = \left(\alpha + \frac{\mu}{a}\right)u^{\alpha + \frac{\mu}{a} - 1}$, which is bounded, as $\alpha + \frac{\mu}{a} = q(\mu) \leq q(\beta) = \alpha + \frac{\beta}{a}$ and $\alpha + \frac{\mu}{a} - 1 = q(\mu) - 1 \geq 0$. Similarly, $\partial\phi/\partial v = \frac{1}{a}u^\alpha v^{\frac{1}{a} - 1}$ restricted to γ_μ is $\frac{1}{a}u^{\alpha + \frac{\mu}{a} - \mu}$ which is bounded when $0 \leq u \leq 1$, as $\alpha + \frac{\mu}{a} - \mu \geq (\kappa - \mu)\left(1 - \frac{1}{a}\right) \geq 0$. □

Theorem 2.25. *Given an abstract pizza $\mathcal{P} = \{\{q_\ell\}_{\ell=0}^p, \{\beta_\ell, Q_\ell, \mu_\ell\}_{\ell=1}^p\}$, there is a non-negative Lipschitz function $f : T_\beta \to \mathbb{R}$, where $\beta = \min_\ell(\beta_\ell)$, such that $\mathcal{P}$ is the abstract pizza corresponding to the pizza $\Lambda = \{T_\ell\}_{\ell=1}^p$ on T_β associated with f.*

Proof. The standard β-Hölder triangle T_β can be decomposed into β_ℓ-Hölder triangles $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$, where $1 \leq \ell \leq p$, by the arcs $\lambda_\ell = \{u \geq 0,\ v = v_\ell(u)\}$, where $0 \leq \ell \leq p$, $v_0(u) \equiv 0$ and $v_p(u) = u^\beta$. For each $\ell = 1, \ldots, p$, let $\mu_\ell(q) = a_\ell q + b_\ell$.

If $a_\ell = 0$ then $Q_\ell = \{q_\ell\}$ is a point and $f|_{T_\ell} = u^{q_\ell}$.

If $a_\ell > 0$ and $q_{\ell-1} > q_\ell$, or $a_\ell < 0$ and $q_{\ell-1} < q_\ell$, thus $\mu_\ell(q_\ell) = \beta$ is the minimal value of $\mu_\ell(q)$ and $\mu_\ell(q_{\ell-1}$ is its maximal value, we define a bi-Lipschitz homeomorphism $H_\ell^+ : T_\ell \to T_{\beta_\ell}$,

$$H_\ell^+(u, v) = \left(u, u^{\beta_\ell}(v - v_{\ell-1}(u))/(v_\ell(u) - v_{\ell-1}(u))\right), \tag{13}$$

such that $H_\ell^+(\lambda_{\ell-1}) = \{v \equiv 0\}$ and $H_\ell^+(\lambda_\ell) = \{v = u_\ell^\beta\}$, and set

$$f(u, v)|_{T_\ell} = \psi_\ell(H_\ell^+(u, v)) \tag{14}$$

where $\psi_\ell = \psi_{\beta_\ell, q_\ell, q_{\ell-1}, \mu_\ell}$ is the standard pizza slice (11). In this case, $\lambda_{\ell-1}$ is the supporting arc $\tilde{\lambda}_\ell$ of T_ℓ with respect to f, $f|_{\lambda_{\ell-1}} = u^{q_{\ell-1}}$ and $f|_{\lambda_\ell} = u^{q_\ell}$.

If $a_\ell > 0$ and $q_{\ell-1} < q_\ell$, or $a_\ell < 0$ and $q_{\ell-1} > q_\ell$, thus $\mu_\ell(q_{\ell-1}) = \beta$ is the minimal value of $\mu_\ell(q)$ and $\mu_\ell(q_\ell)$ is its maximal value, we define a bi-Lipschitz homeomorphism $H_\ell^- : T_\ell \to T_{\beta_\ell}$,

$$H_\ell^-(u, v) = \left(u, u^{\beta_{\ell-1}}(v - v_\ell(u))/(v_{\ell-1}(u) - v_\ell(u))\right), \tag{15}$$

such that $H^-_\ell(\lambda_\ell) = \{v \equiv 0\}$ and $H^-_\ell(\lambda_{\ell-1}) = \{v = u^\beta_\ell\}$, and set

$$f(u,v)|_{T_\ell} = \psi_\ell(H^-_\ell(u,v)) \tag{16}$$

where $\psi_\ell = \psi_{\beta_\ell,q_{\ell-1},q_\ell,\mu_\ell}$ is the standard pizza slice (11). In this case, λ_ℓ is the supporting arc $\tilde\lambda_\ell$ of T_ℓ with respect to f, $f|_{\lambda_{\ell-1}} = u^{q_{\ell-1}}$ and $f|_{\lambda_\ell} = u^{q_\ell}$.

Since the standard pizza slice (11) and (12) is a Lipschitz function, and $f|_{\lambda_\ell} = u^{q_\ell}$ for $0 \le \ell \le p$, the function f is a Lipschitz function on T_β, such that $\mathcal{P}$ is the abstract pizza corresponding to the pizza associated with f. □

Definition 2.26. Two pizzas are called *combinatorially equivalent* if the corresponding abstract pizzas are the same.

Proposition 2.27. (See [4, Theorem 4.9].) *Two non-negative Lipschitz functions f and g on a normally embedded Hölder triangle T are contact Lipschitz equivalent if, and only if, minimal pizzas on T associated with f and g are combinatorially equivalent.*

Definition 2.28. (See [7, Definitions 2.36 and 2.40].) Let X be a surface germ. A non-empty set of arcs $Z \subset V(X)$ is called a *zone* if, for any two arcs $\gamma_1 \ne \gamma_2$ in Z, there exists a non-singular Hölder triangle $T = T(\gamma_1,\gamma_2)$ such that $V(T) \subset Z$. A *singular zone* is a zone $Z = \{\gamma\}$ consisting of a single arc γ. A zone Z is *normally embedded* if, for any two arcs $\gamma_1 \ne \gamma_2$ in Z, there exists a normally embedded Hölder triangle $T = T(\gamma_1,\gamma_2)$ such that $V(T) \subset Z$. The *order* of a zone Z is defined as $\mu(Z) = \inf_{\gamma,\gamma' \in Z} tord(\gamma,\gamma')$. If Z is a singular zone, then $\mu(Z) = \infty$. A zone Z is called a β-zone when $\mu(Z) = \beta$.

Definition 2.29. (See [7, Definition 2.43].) A β-zone Z is *closed* if there are two arcs γ and γ' in Z such that $tord(\gamma,\gamma') = \beta$. Otherwise, Z is an *open* zone. A zone $Z \subset V(X)$ is *perfect* if, for any two arcs γ and γ' in Z, there exists a Hölder triangle $T \subset X$ such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T. By definition, a singular zone is closed perfect.

Remark 2.30. (see [5, Lemma 2.29]) A zone $Z \subset V(X)$ is perfect if, and only if, for any two arcs $\{\gamma_1,\gamma_2\} \in Z$ there exists an inner bi-Lipschitz map $\Psi : X \to X$, such that $\Psi(\gamma_1) = \gamma_2$ and Ψ is identity on $V(X) \setminus Z$. If Z and Z' are disjoint closed perfect zones, then $itord(Z,Z') = itord(\gamma,\gamma')$ for any arcs $\gamma \in Z$ and $\gamma' \in Z'$.

Definition 2.31. (See [5, Definition 2.22].) Let $f : T \to \mathbb{R}$ be a Lipschitz function on a normally embedded Hölder triangle T. A zone $Z \subset V(T)$ is a *q-order zone* for f if $ord_\gamma f = q$ for any arc $\gamma \in Z$. A q-order zone for f is *maximal* if it is not a proper subset of any other q-order zone for f. The *width zone* $W_T(\gamma,f)$ of an arc $\gamma \subset T$ with respect to f is the maximal q-order zone for f containing γ, where $q = ord_\gamma f$. The order of $W_T(\gamma,f)$ is $\mu_T(\gamma,f)$. The *depth zone* $D_T(\gamma,f)$ of an arc $\gamma \subset T$ with respect to f is the union of zones $G(T')$ for all triangles $T' \subset T$ such that $\gamma \in G(T')$ and $Q_f(T')$ is a point. The order of $D_T(\gamma,f)$ is $\nu_T(\gamma,f)$. By definition, $D_T(\gamma,f) = \{\gamma\}$ and $\nu_T(\gamma,f) = \infty$ when there are no such triangles T'.

Lemma 2.32. (See [5, Lemma 2.23].) *If f is a Lipschitz function on a normally embedded Hölder triangle T, then the width zone $W_T(\gamma,f)$ is closed for any arc $\gamma \subset T$.*

Definition 2.33. (See [5, Definition 2.24].) Let T be a normally embedded Hölder triangle and f a Lipschitz function on T. If $Z \subset V(T)$ is a zone, we define $Q_f(Z)$ as the set of all exponents $ord_\gamma f$ for $\gamma \in Z$. The zone Z is *elementary* with respect to f if the set of arcs $\gamma \in Z$ such that $ord_\gamma f = q$ is a zone for each $q \in Q_f(Z)$.

For $\gamma \in Z$ and $q = ord_\gamma f$, the *width* $\mu_Z(\gamma, f)$ of γ with respect to f is the infimum of exponents of Hölder triangles T' containing γ such that $V(T') \subset Z$ and $Q_f(T')$ is a point. The *width zone* $W_Z(\gamma, f)$ of γ with respect to f is the maximal subzone of Z containing γ such that $q = ord_\lambda f$ for all arcs $\lambda \subset W_Z(\gamma, f)$. The order of $W_Z(\gamma, f)$ is $\mu_Z(\gamma, f)$. For $q \in Q_f(Z)$ let $\mu_{Z,f}(q)$ be the set of exponents $\mu_Z(\gamma, f)$, where $\gamma \in Z$ is any arc such that $ord_\gamma f = q$. It follows from Lemma 3.3 from ([4]) that, for each $q \in Q_f(Z)$, the set $\mu_{Z,f}(q)$ is finite. This defines a multivalued *width function* $\mu_{Z,f} : Q_f(Z) \to \mathbb{F} \cup \{\infty\}$. If Z is an elementary zone with respect to f then the function $\mu_{Z,f}$ is single valued.

We say that Z is a *pizza slice zone* associated with f if it is elementary with respect to f, $Q_f(Z)$ is a closed interval in $\mathbb{F} \cup \{\infty\}$ and, unless $Q_f(Z)$ is a point, $\mu_{Z,f}(q) = aq + b$ is an affine function on $Q_f(Z)$.

Lemma 2.34. (See [5, Lemma 2.25].) *Let f be a Lipschitz function on a normally embedded Hölder triangle T. Let γ be an interior arc of T, so that $T = T' \cup T''$ and $T' \cap T'' = \{\gamma\}$. Then either $\mu_{T'}(\gamma, f) = \mu_{T''}(\gamma, f)$ and $\nu_T(\gamma, f) = \mu_T(\gamma, f) = \mu_{T'}(\gamma, f) = \mu_{T''}(\gamma, f)$, or $\nu_T(\gamma, f) = \max(\mu_{T'}(\gamma, f), \mu_{T''}(\gamma, f)) > \mu_T(\gamma, f)$. In both cases, $D_T(\gamma, f)$ is a closed perfect zone.*

Lemma 2.35. (See [5, Lemma 2.28].) *Let f be a non-negative Lipschitz function on a normally embedded Hölder triangle $T = T(\gamma_1, \gamma_2)$, oriented from γ_1 to γ_2. There exists a unique finite family $\{D_\ell\}_{\ell=0}^p$ of disjoint zones $D_\ell \subset V(T)$ with the following properties:*
1. *The singular zones $D_0 = \{\gamma_1\}$ and $D_p = \{\gamma_2\}$ are the boundary arcs of T.*
2. *$D_\ell = D_T(\gamma, f)$ is a closed perfect ν_ℓ-zone, where $\nu_\ell = \nu_T(\gamma, f)$ for any arc $\gamma \in D_\ell$. In particular, D_ℓ is a q_ℓ-order zone for f, where $q_\ell = ord_\gamma f$. Moreover, D_ℓ is a maximal perfect q_ℓ-order zone for f: if $Z \supseteq D_\ell$ is a perfect q_ℓ-order zone for f, then $Z = D_\ell$.*
3. *Any choice of arcs $\lambda_\ell \in D_\ell$ defines a minimal pizza $\Lambda = \{T_\ell\}_{\ell=1}^p$, where $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ on T associated with f.*
4. *Any minimal pizza on T associated with f can be obtained as a decomposition $\{T_\ell\}_{\ell=1}^p$ of T defined by some choice of arcs $\lambda_\ell \in D_\ell$ for $\ell = 0, \dots, p$.*

Definition 2.36. The zones D_ℓ in Lemma 2.35 are called *pizza zones* of a minimal pizza on T associated with f.

Corollary 2.37. *Let $\{D_\ell\}_{\ell=0}^p$ be the family of pizza zones of a minimal pizza $\Lambda = \{T_\ell\}_{\ell=1}^p$ on T associated with f, where $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$, as in Lemma 2.35. For each $\ell = 1, \dots, p$, the set $Y_\ell = D_{\ell-1} \cup D_\ell \cup V(T_\ell)$ is a pizza slice zone associated with f, independent of the choice of arcs $\lambda_\ell \in D_\ell$. Moreover, Y_ℓ is a maximal pizza slice zone: if $Y \supseteq Y_\ell$ is a pizza slice zone associated with f, then $Y = Y_\ell$.*

Definition 2.38. If $T = T(\gamma_1, \gamma_2)$ is a Hölder triangle oriented from γ_1 to γ_2, then its Valette link $V(T)$ is totally ordered: for two arcs $\lambda_1 \neq \Lambda_2$ in $V(T)$, we define $\lambda_1 \prec \lambda_2$ when orientation of T induces orientation from λ_1 to λ_2 on $T(\lambda_1, \lambda_2)$. This order of $V(T)$ defines a partial order on the set of zones in $V(T)$: for two zones $Z_1 \neq Z_2$ in $V(T)$, we define $Z_1 \prec Z_2$ when either $Z_2 \setminus Z_1$ is a zone and $\lambda_1 \prec \lambda_2$ for any arcs $\lambda_1 \in Z_1$ and $\lambda_2 \in Z_2 \setminus Z_1$, or when $Z_1 \setminus Z_2$ is a zone and $\lambda_1 \prec \lambda_2$ for any arcs $\lambda_1 \in Z_1 \setminus Z_2$ and $\lambda_2 \in Z_2$. If Λ is a pizza on T, then this partial order defines a total order on the set of all pizza zones D_ℓ and Y_ℓ of Λ, consistent with the total order $\lambda_{\ell-1} \prec T_\ell \prec \lambda_\ell$ on the disjoint union of the sets of arcs and pizza slices of Λ induced by orientation of T. If Z is a zone in $V(T)$ and T_ℓ is a pizza slice of Λ, we write $Z \prec T_\ell$ or $T_\ell \prec Z$ when $Z \prec V(T_\ell)$ or $V(T_\ell) \prec Z$.

3. Maximal exponent zones and the permutation σ

Definition 3.1. If T and T' are normally embedded Hölder triangles, then a pair of arcs (γ,γ'), where $\gamma \subset T$ and $\gamma' \subset T'$, is called *normal* when $tord(\gamma,T') = tord(\gamma,\gamma') = tord(\gamma',T)$. A pair (T,T') of normally embedded Hölder triangles $T = T(\gamma_1,\gamma_2)$ and $T' = T(\gamma'_1,\gamma'_2)$, oriented from γ_1 to γ_2 and from γ'_1 to γ'_2 respectively, is called a *normal pair* if (γ_1,γ'_1) and (γ_2,γ'_2) are normal pairs, i.e., the following condition is satisfied:

$$tord(\gamma_1,T') = tord(\gamma_1,\gamma'_1) = tord(\gamma'_1,T), \quad tord(\gamma_2,T') = tord(\gamma_2,\gamma'_2) = tord(\gamma'_2,T). \tag{17}$$

Definition 3.2. (See [5, Definition 4.2].) Let $\Lambda = \{T_\ell\}_{\ell=1}^p$, where $T_\ell = T(\lambda_{\ell-1},\lambda_\ell)$ is a β_ℓ-Hölder triangle, be a minimal pizza on a normally embedded Hölder triangle $T = T(\gamma_1,\gamma_2)$ oriented from γ_1 to γ_2, associated with a non-negative Lipschitz function f defined on T. Let $\{D_\ell\}_{\ell=0}^p$ be the pizza zones of Λ, ordered according to the orientation of T, and let $q_\ell = tord(D_\ell,T') = ord_{\lambda_\ell} f$. A pizza zone D_ℓ is called a *maximal exponent zone* for f (or simply a *maximum zone*) and $\lambda_\ell \in D_\ell$ is called a *maximum arc* of Λ if one of the following inequalities holds:

$$\begin{aligned} \ell = 0, &\quad \beta_1 < q_0 \ge q_1, \\ \ell = p, &\quad \beta_p < q_p \ge q_{p-1}, \\ 0 < \ell < p, &\quad \max(\beta_\ell,\beta_{\ell+1}) < q_\ell \ge \max(q_{\ell-1},q_{\ell+1}). \end{aligned} \tag{18}$$

If a zone D_ℓ is not a maximum zone, it is called a *minimal exponent zone* for f (or simply a *minimum zone*) and an arc λ_ℓ of Λ is called a *minimum arc* if either $\ell = 0$ and $q_0 \le q_1$, or $0 < \ell < p$ and $q_\ell \le \min(q_{\ell-1},q_{\ell+1})$, or $\ell = p$ and $q_p \le q_{p-1}$. In particular, each boundary arc of T is either a maximum zone or a minimum zone.

If $\mathcal{P}$ is a minimal abstract pizza corresponding to Λ (see Definition 2.22) then ℓ is called a *maximum index* of $\mathcal{P}$ if conditions (18) are satisfied.

If (T,T') is a normal pair of Hölder triangles with distance functions $f(x) = dist(x,T')$ and $g(x') = dist(x',T)$, then maximum and minimum zones $D'_{\ell'} \subset V(T')$ and arcs $\lambda'_{\ell'}$ of a minimal pizza $\Lambda' = \{T'_{\ell'}\}_{\ell'=1}^{p'}$ on T' associated with g, where $T'_{\ell'} = T(\lambda'_{\ell'-1},\lambda'_{\ell'})$, are defined similarly, replacing T with T' and f with g.

Proposition 3.3. (See [5, Proposition 4.4].) *Let (T,T'), where $T = T(\gamma_1,\gamma_2)$ and $T' = T(\gamma'_1,\gamma'_2)$, be a normal pair of Hölder triangles. Let $\mathcal{M} = \{M_i\}_{i=1}^m$ and $\mathcal{M}' = \{M'_j\}_{j=1}^{m'}$ be the sets of maximum zones in $V(T)$ and $V(T')$ for minimal pizzas Λ and Λ' associated with the distance functions $f(x) = dist(x,T')$ and $g(x') = dist(x',T)$, ordered according to the orientations of T and T'. Let $\bar q_i = tord(M_i,T')$ and $\bar q'_j = tord(M'_j,T)$. Then $m' = m$, and there is a one-to-one correspondence $j = \sigma(i)$ between the sets $\mathcal{M} = \{M_i\}_{i=1}^m$ and $\mathcal{M}' = \{M'_j\}_{j=1}^m$, such that $\mu(M'_j) = \mu(M_i)$ and $tord(M_i,M'_j) = \bar q_i = \bar q'_j$ for $j = \sigma(i)$. If $\{\gamma_1\} = M_1$ is a maximum zone of Λ, then $\{\gamma'_1\} = M'_1$ is a maximum zone of Λ' and $\sigma(1) = 1$. If $\{\gamma_2\} = M_m$ is a maximum zone of Λ, then $\{\gamma'_2\} = M'_m$ is a maximum zone of Λ' and $\sigma(m) = m$.*

Definition 3.4. (See [5, Definition 4.5].) The permutation σ of the set $[m] = \{1,\dots,m\}$ in Proposition 3.3 is called the *characteristic permutation* of the normal pair (T,T') of Hölder triangles.

Remark 3.5. Since the Hölder triangles T and T' are normally embedded and the zones M_i and M'_i are closed perfect, Remark 2.30 implies that $\xi(M_i, M_j) = 1/tord(M_i, M_j)$ and $\xi'(M'_i, M'_j) = 1/tord(M'_i, M'_j)$ (see Definition 2.4) define non-archimedean metrics ξ and ξ' on the sets $\mathcal{M}$ and $\mathcal{M}'$ of the maximum zones of Λ and Λ'. The characteristic permutation σ defines an isometry $\sigma : \mathcal{M} \to \mathcal{M}'$ with respect to these metrics.

4. Transversal and coherent pizza slices, the correspondence τ

Definition 4.1. (See [5, Definition 4.6].) Let f be a non-negative Lipschitz function on a normally embedded β-Hölder triangle T, and let $\Lambda = \{T_\ell\}_{\ell=1}^p$, where $T_\ell = (\lambda_{\ell-1}, \lambda_\ell)$, be a minimal pizza on T associated with f. Let $\{D_\ell\}_{\ell=0}^p$ be the pizza zones (see Lemma 2.35) such that $\lambda_\ell \in D_\ell$. We have $\mu(D_\ell) = \nu(\lambda_\ell)$ and $q_\ell = ord_{\lambda_\ell} f$. Let $Y_\ell = D_{\ell-1} \cup V(T_\ell) \cup D_\ell$ be the maximal pizza slice zones associated with f (see Corollary 2.37). An interior pizza zone D_ℓ is called *transversal* if $q_\ell = \mu(D_\ell)$ and *coherent* otherwise. A pizza slice T_ℓ, and a pizza slice zone Y_ℓ, are called *transversal* if either $\ell = p = 1$ and $q_0 = q_1 \le \beta$ or $Q_\ell = [q_{\ell-1}, q_\ell]$ is not a point and the width function μ_ℓ on Q_ℓ (see Definition 2.15) satisfies $\mu_\ell(q) \equiv q$. Otherwise, T_ℓ and Y_ℓ are called *coherent*. A boundary arc $\check{\gamma}$ of T is called *transversal* if $\mu(\check{\gamma}) = ord_f(\check{\gamma})$. The function f is called *totally transversal* if all pizza slices T_ℓ of a minimal pizza associated with f are transversal. Alternatively, f is totally transversal if either $ord_\gamma f < \beta$ or $ord_\gamma f = \mu_T(\gamma, f)$ for all arcs $\gamma \in V(T)$.
If (T, T') is a normal pair of Hölder triangles, $\Lambda = \{T_\ell\}_{\ell=1}^p$ is the minimal pizza on T associated with $f(x) = dist(x, T')$ and $\Lambda' = \{T'_{\ell'}\}_{\ell'=1}^{p'}$ is a minimal pizza on T' associated with $g(x') = dist(x', T)$, then transversal pizza zones $D'_{\ell'}$ of Λ', transversal and coherent pizza slices $T'_{\ell'}$, maximal pizza slice zones $Y'_{\ell'}$, and affine functions $\mu'_{\ell'}(q)$ on $Q'_{\ell'}$, are defined similarly, replacing T with T' and f with g.

Definition 4.2. A normal pair (T, T') of Hölder triangles, where $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$, with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ on T and T', respectively, is called *totally transversal* if the distance function f on T is totally transversal. It follows from [5, Proposition 4.7] that a pair (T, T') is totally transversal if, and only if, the distance function g on T' is totally transversal.

Let $T = T(\gamma_1, \gamma_2)$ be a normally embedded β-Hölder triangle and f a totally transversal non-negative Lipschitz function on T. Let $\mathcal{M} = \{M_i\}_{i=1}^m$, be the set of maximum zones in $V(T)$ of a minimal pizza Λ on T associated with f. We may assume that at least one maximum zone exists, as $m = 0$ appears only in a trivial case when $ord_\gamma f \le \beta$ for all $\gamma \subset T$. Let $n = m - 1$ when both boundary arcs of T are maximum zones, $n = m$ when only one of the boundary arcs of T is a maximum zone, $n = m + 1$ otherwise. Selecting $n + 1$ arc $\theta_0, \ldots, \theta_n$ so that $\theta_0 = \gamma_1$, $\theta_n = \gamma_2$, and there is exactly one arc θ_j in each maximum zone M_i, we define a decomposition of T into n Hölder triangles $\bar{T}_j$. Here $i = j + 1$ if $\{\gamma_1\}$ is a maximum zone and $i = j$ otherwise. Let $\bar{q}_j = ord_{\theta_j} f$ and $\bar{\beta}_j = \mu(\bar{T}_j)$. Then

$$\begin{aligned} &\bar{q}_0 > \bar{\beta}_1 \text{ when } \{\gamma_1\} \text{ is a maximum zone, otherwise } \bar{q}_0 = \bar{\beta}_1, \\ &\bar{q}_n > \bar{\beta}_n \text{ when } \{\gamma_2\} \text{ is a maximum zone, otherwise } \bar{q}_n = \bar{\beta}_n, \\ &\bar{q}_j > \max(\bar{\beta}_j, \bar{\beta}_{j+1}) \ \text{ for } \ 0 < j < n. \end{aligned} \tag{19}$$

Lemma 4.3. *Let $T = T(\gamma_1, \gamma_2)$ be a normally embedded β-Hölder triangle oriented from γ_1 to γ_2. Given exponents $\{\bar\beta_j\}_{j=1}^n$ such that $\min_j(\bar\beta_j) = \beta$, and exponents $\{\bar q_j\}_{j=0}^n$ satisfying (19), there exists a totally transversal non-negative Lipschitz function f on T, unique up to contact Lipschitz equivalence, such that exponents $\bar\beta_j$ and $\bar q_j$ are determined by the maximum zones of a minimal pizza on T associated with f.*

Proof. Given exponents $\{\bar\beta_j\}_{j=1}^n$ such that $\min_j(\bar\beta_j) = \beta$, and exponents $\{\bar q_j\}_{j=0}^n$ satisfying (19), let $p = 2n$ if $\bar q_0 > \bar\beta_1$ and $\bar q_n > \bar\beta_n$, $p = 2n - 2$ if $\bar q_0 = \bar\beta_1$ and $\bar q_n = \bar\beta_n$, $p = 2n - 1$ otherwise. Then one can define a pizza $\{T_\ell\}_{\ell=1}^p$ on T as follows. Consider decomposition $\{\bar T_j\}_{j=1}^n$ of T into $\bar\beta_j$-Hölder triangles $\bar T_j = T(\bar\theta_{j-1}, \bar\theta_j)$. Decompose each Hölder triangle $\bar T_j$, for $1 < j < n$, into two $\bar\beta_j$-Hölder triangles $T_j^- = T(\bar\theta_{j-1}, \lambda_j)$ and $T_j^+ = T(\lambda_j, \bar\theta_j)$ by a generic arc $\lambda_j \subset \bar T_j$. Decompose similarly $\bar T_1$ if $\bar q_0 > \bar\beta_1$ and $\bar T_n$ if $\bar q_n > \bar\beta_n$. This defines a decomposition of T into p Hölder triangles. If the new arcs λ_j are assigned exponents $q_j = \bar\beta_j$, and the new Hölder triangles $T_j^\pm$ are assigned the width functions $\mu_j^\pm(q) \equiv q$ on the intervals $Q_j^\pm$, where $Q_j^- = [\bar q_{j-1}, q_j]$ and $Q_j^+ = [\bar q_j, q_j]$, then this decomposition becomes a minimal pizza for a totally transversal function f on T, such that the arcs $\bar\theta_j$ either are the boundary arcs of T or belong to maximum zones of the pizza, and the arcs λ_j belong to minimum zones of the pizza which are not the boundary arcs of T. According to Theorem 2.25, such a function f exists and is unique up to contact Lipschitz equivalence. □

Definition 4.4. Let (T, T'), where $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$, be two normally embedded β-Hölder triangles satisfying (17), such that T is a pizza slice associated with $f(x) = dist(x, T')$ and $tord(T, T') = tord(\gamma_1, \gamma_1') > \beta$. The pair (T, T') is called *positively oriented* if either T is oriented from γ_1 to γ_2 and T' from γ_1' to γ_2', or if T is oriented from γ_2 to γ_1 and T' from γ_2' to γ_1'. Otherwise, the pair (T, T') is called *negatively oriented*.

Proposition 4.5. (See [5, Proposition 4.7].) *Let (T, T') be a normal pair of Hölder triangles, and let T_ℓ, D_ℓ, Q_ℓ, μ_ℓ, $T'_{\ell'}$, $D'_{\ell'}$, $Q'_{\ell'}$, $\mu'_{\ell'}$ be as in Definition 4.1. Then, for each index ℓ such that the pizza slice T_ℓ is coherent, there is a unique index $\ell' = \tau(\ell)$ such that $Q'_{\ell'} = Q_\ell$, $\mu'_{\ell'}(q) \equiv \mu_\ell(q)$ and one of the following two conditions holds:*

$$\begin{aligned}(20)\quad & tord(D_\ell, T') = tord(D_\ell, D'_{\ell'}) = tord(D'_{\ell'}, T),\\ & \qquad tord(D_{\ell-1}, T') = tord(D_{\ell-1}, D'_{\ell'-1}) = tord(D'_{\ell'-1}, T);\end{aligned}$$

$$\begin{aligned}(21)\quad & tord(D_\ell, T') = tord(D_\ell, D'_{\ell'-1}) = tord(D'_{\ell'-1}, T),\\ & \qquad tord(D_{\ell-1}, T') = tord(D_{\ell-1}, D'_{\ell'}) = tord(D'_{\ell'}, T).\end{aligned}$$

Definition 4.6. (See [5, Definition 4.8].) Let (T, T') be a normal pair of Hölder triangles, and let Λ and Λ' be minimal pizzas on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ respectively. According to Proposition 4.5, the sets $\mathcal{L}$ and $\mathcal{L}'$ of coherent pizza slices of Λ and Λ' have the same number of elements, and there is a canonical one-to-one *characteristic correspondence* $\tau : \ell' = \tau(\ell)$ between these two sets. A pair $(T_\ell, T'_{\ell'})$ of coherent pizza slices, where $\ell' = \tau(\ell)$ is called *positively oriented* if (20) holds and *negatively oriented* if (21) holds. Alternatively, we say that τ is *positive* (resp., *negative*) on T_ℓ.

Definition 4.7. Let L be the number of coherent pizza slices of Λ, same as the number of coherent pizza slices of Λ'. Since pizza slices of Λ and Λ' are ordered according to orientations of T and T', respectively, the characteristic correspondence τ induces a permutation υ of the set $[L] = \{1, \ldots, L\}$, such that $j = \upsilon(i)$ when T_ℓ is the i-th coherent pizza slice of Λ and $T'_{\tau(\ell)}$ is the j-th coherent pizza slice of Λ'. Since τ is a signed correspondence, it defines a sign function $s : [L] \to \{+, -\}$, where $s(l) = $ "$+$" (resp., $s(l) = $ "$-$") if τ is positive (resp., negative) on the l-th coherent pizza slice T_ℓ of Λ.

Proposition 4.8. (See [5, Proposition 4.10].) *Let (T, T') be a normal pair of Hölder triangles. Let Λ and Λ' be minimal pizzas on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$. Let T_ℓ and $T'_{\ell'}$, where $\ell' = \tau(\ell)$, be coherent pizza slices such that a pizza zone D of Λ adjacent to T_ℓ (say $D = D_\ell$, the case $D = D_{\ell-1}$ being similar) is a maximum zone M_i of Λ. Then the corresponding pizza zone D' of Λ' adjacent to $T'_{\ell'}$ (either $D' = D'_{\ell'}$ when τ is positive on T_ℓ or $D' = D'_{\ell'-1}$ when τ is negative) is a maximum zone M'_j of Λ', where $j = \sigma(i)$.*

Definition 4.9. Let T_ℓ be a coherent pizza slice of Λ such that $q_\ell \geq q_{\ell-1}$. A maximum zone M_i of Λ is called *right-adjacent* to T_ℓ if $tord(\lambda_\ell, M_i) \geq q_\ell$. If there are no maximum zones of Λ right-adjacent to T_ℓ, then T_ℓ is called *tied on the right*. Similarly, if T_ℓ is a coherent pizza slice of Λ such that $q_{\ell-1} \geq q_\ell$, then a maximum zone M_i of Λ is called *left-adjacent* to T_ℓ when $tord(\lambda_{\ell-1}, M_i) \geq q_{\ell-1}$. If there are no maximum zones left-adjacent to T_ℓ, then T_ℓ is called *tied on the left*. Note that T_ℓ may be tied on both sides, or have both right- and left-adjacent maximum zones, when $q_{\ell-1} = q_\ell$.

Remark 4.10. Let $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ be a coherent pizza slice of Λ such that $q_\ell \geq q_{\ell-1}$, and let M_i be a maximum zone of Λ right-adjacent to T_ℓ. Since the pizza zone D_ℓ of Λ has exponent $\nu_\ell \leq q_\ell$, either $\lambda_\ell \in M_i$ or $tord(\lambda_\ell, M_i) = q_\ell$. Similarly, if $q_{\ell-1} \geq q_\ell$ and M_i is a maximum zone of Λ left-adjacent to T_ℓ, then either $\lambda_{\ell-1} \in M_i$ or $tord(\lambda_{\ell-1}, M_i) = q_{\ell-1}$.

Definition 4.11. If a coherent pizza slice T_ℓ of Λ is tied on the right, it cannot be the last coherent pizza slice of Λ: if $\lambda_\ell = \gamma_2$ is a boundary arc of T, then it is a maximum zone adjacent to T_ℓ, and if $T_{\ell+1}$ is a transversal pizza slice of Λ with $q_{\ell+1} > q_\ell$, then either $\lambda_{\ell+1}$ belongs to a maximum zone adjacent to T_ℓ or $T_{\ell+2}$ is a coherent pizza slice of Λ, since two consecutive transversal pizza slices of a minimal pizza have either a maximal or a minimal common pizza zone. Thus, if T_ℓ is tied on the right, then either $T_{\ell+1}$ is a coherent pizza slice of Λ with $q_{\ell+1} > q_\ell$, or $T_{\ell+1}$ is a transversal pizza slice of Λ with $q_{\ell+1} > q_\ell$ and $T_{\ell+2}$ is a coherent pizza slice of Λ with $q_{\ell+2} > q_{\ell+1}$. The two coherent pizza slices, either T_ℓ and $T_{\ell+1}$ or T_ℓ and $T_{\ell+2}$, are called *right-tied*. Similarly, if a coherent pizza slice T_ℓ of Λ is tied on the left, then either $T_{\ell-1}$ is a coherent pizza slice of Λ with $q_{\ell-2} > q_{\ell-1}$, or $T_{\ell-1}$ is a transversal pizza slice of Λ with $q_{\ell-2} > q_{\ell-1}$ and $T_{\ell-2}$ is a coherent pizza slice of Λ with $q_{\ell-3} > q_{\ell-2}$. The two coherent pizza slices, either T_ℓ and $T_{\ell-1}$ or T_ℓ and $T_{\ell-2}$, are called *left-tied*.

Definition 4.12. A sequence $C = \{T_\ell, \ldots, T_{\ell+k}\}$ (resp., $C = \{T_\ell, \ldots, T_{\ell-k}\}$) of consecutive pizza slices of Λ, where T_ℓ and $T_{\ell+k}$ (resp., T_ℓ and $T_{\ell-k}$) are coherent pizza slices, is called a *rightward caravan* (resp., a *leftward caravan*) if each coherent pizza slice in C is tied on the right (resp., tied on the left), except the last pizza slice $T_{\ell+k}$ (resp., $T_{\ell-k}$) of C being not tied. The non-empty set $\mathcal{A}(C)$ of maximum zones of Λ right-adjacent to $T_{\ell+k}$ (resp., left-adjacent to $T_{\ell-k}$) is called the *adjacent set* of a caravan C.

If M_i is a maximum zone of Λ such that $M_i \prec T_\ell$ (resp., $M_i \prec T_{\ell-k}$), we say that $M_i \prec C$. If M_i is a maximum zone of Λ such that $T_{\ell+k} \prec M_i$ (resp., $T_\ell \prec M_i$), we say that $C \prec M_i$. In particular $C \prec M_i$ (resp., $M_i \prec C$) for any maximum zone $M_i \in \mathcal{A}(C)$.
A caravan C' of pizza slices of Λ' and its adjacent set $\mathcal{A}(C')$ are defined similarly, replacing the pizza slices and maximum zones of Λ with the pizza slices and maximum zones of Λ'.

Remark 4.13. It follows from Definitions 4.11 and 4.12 that each coherent pizza slice of Λ belongs to at least one caravan. If two rightwards (resp., leftwards) caravans are not disjoint, then one of them is a subset of another, and their adjacent sets of maximum zones are the same. A rightward (resp., leftward) caravan may consist of a single coherent pizza slice T_ℓ of Λ when T_ℓ is not tied on the right (resp., on the left).
If C is a caravan of pizza slices of Λ, then the set $V(C) = \bigcup_{k:T_k \in C} V(T_k)$ is a zone in $V(T)$. Partial order on zones in $V(T)$ (see Definition 2.38) induces total order on the set of all (rightward and leftward) caravans of pizza slices of Λ, such that $C_1 \prec C_2$ when $V(C_1) \prec V(C_2)$. Similarly, partial order on zones in $V(T')$ induces total order on the set of all caravans of pizza slices of Λ'.
If $C = \{T_\ell, \ldots, T_{\ell+k}\}$ is a rightward caravan of Λ and M_i is a maximum zone of Λ not right-adjacent to C, then either $M_i \prec T_\ell$ or $M_j \prec M_i$ for any maximum zone $M_j \in \mathcal{A}(C)$. Similarly, if $C = \{T_\ell, \ldots, T_{\ell-k}\}$ is a leftward caravan of Λ and M_i is a maximum zone of Λ not left-adjacent to C, then either $T_\ell \prec M_i$ or $M_i \prec M_j$ for any maximum zone $M_j \in \mathcal{A}(C)$.

Proposition 4.14. *Let (T, T') be a normal pair of Hölder triangles, and let Λ and Λ' be minimal pizzas on T and T', respectively, associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$. Let τ be the characteristic correspondence between coherent pizza slices of Λ and Λ'. If two coherent pizza slices of Λ (either T_ℓ and $T_{\ell+1}$ or T_ℓ and $T_{\ell+2}$) are tied, then τ is either positive on both of these pizza slices, with either $\tau(\ell+1) = \tau(\ell)+1$ or $\tau(\ell+2) = \tau(\ell)+2$, or negative on both of these pizza slices, with either $\tau(\ell+1) = \tau(\ell) - 1$ or $\tau(\ell+2) = \tau(\ell) - 2$. Moreover, coherent pizza slices of Λ' assigned by τ to tied pizza slices of Λ are also tied. If two tied pizza slices of Λ belong to a caravan C and the tied pizza slices of Λ' assigned to them by τ belong to a caravan $C' = \tau(C)$, then the set of maximum zones of Λ' adjacent to C' is $\mathcal{A}(C') = \{M'_{\sigma(i)} : M_i \in \mathcal{A}(C)\}$.*

Proof. Consider first the case of two consecutive pizza slices T_ℓ and $T_{\ell+1}$. We may assume that T_ℓ is tied on the right, τ is positive on T_ℓ, and $T_{\ell+1}$ is a coherent pizza slice with $q_{\ell+1} > q_\ell$, the other cases being similar. Let $\ell' = \tau(\ell)$, $\ell'' = \tau(\ell+1)$, and let $T'_{\ell'} = T(\lambda'_{\ell'-1}, \lambda'_{\ell'})$ and $T'_{\ell''} = T(\lambda'_{\ell''-1}, \lambda'_{\ell''})$ be coherent pizza slices of Λ' corresponding to T_ℓ and $T_{\ell+1}$, respectively.

Let us show first that τ is positive on $T_{\ell+1}$. If τ is negative on $T_{\ell+1}$, then [5, Proposition 3.9] implies that $\beta_{\ell+1} = q_\ell$, $tord(\lambda_\ell, \lambda'_{\ell'}) = tord(\lambda_\ell, \lambda'_{\ell''}) = q_\ell$ and $tord(\lambda_{\ell+1}, \lambda'_{\ell''-1}) = q_{\ell+1} > q_\ell$. Since $\beta_\ell < q_\ell$ and T' is normally embedded, we have $T'_{\ell'} \prec T'_{\ell''}$, thus $\lambda'_{\ell''-1} \subset T(\lambda'_{\ell'}, \lambda'_{\ell''})$ and $tord(\lambda'_{\ell'}, \lambda'_{\ell''}) = q_\ell$. As $q_{\ell+1} > q_\ell$, there is a maximum zone $M'_j \subset V(T(\lambda'_{\ell'}, \lambda'_{\ell''}))$ of Λ', such that $tord(M'_j, \lambda_\ell) \geq q_\ell$. If $j = \sigma(i)$, then $tord(M_i, \lambda_\ell) \geq q_\ell$, a contradiction.

We are going to show next that $\ell'' = \ell' + 1$. Since τ is positive on both T_ℓ and $T_{\ell+1}$ and $\beta_\ell < q_\ell = \beta_{\ell+1} < q_{\ell+1}$, [5, Proposition 3.9] implies that $T'_{\ell'} \prec T'_{\ell''}$, thus $\lambda'_{\ell''-1} \subset T(\lambda'_{\ell'}, \lambda'_{\ell''})$. Since $q'_{\ell'} = q_\ell = q'_{\ell''-1}$, we have $ord_\theta g = q_\ell$ for each arc $\theta \subset \tilde{T}' = T(\lambda'_{\ell'}, \lambda'_{\ell''-1})$, otherwise there would be a maximum zone M'_j of Λ' inside $V(\tilde{T}')$, an the corresponding maximum zone M_i of Λ, where $j=\sigma(i)$ would have $tord(M_i,\lambda_\ell)\ge q_\ell$, a contradiction.

The statement about the sets $\mathcal{A}(C)$ and $\mathcal{A}(C')$ of adjacent maximum zones follows from the isometry of the characteristic permutation σ (see Remark 3.5).

If T_ℓ and $T_{\ell+2}$ are two coherent pizza slices of Λ such that T_ℓ is tied on the right, $T_{\ell+1}$ is a transversal pizza slice, and τ is positive on T_ℓ, then the same arguments as above show that τ is positive on $T_{\ell+2}$, the pizza slices $T'_{\ell'}$ and $T'_{\ell''}$ of Λ' assigned by τ to T_ℓ and $T_{\ell+2}$ satisfy $T'_{\ell'}\prec T'_{\ell''}$, and $T'_{\ell'}$ is tied on the right. Since $q_{\ell+1}>q_\ell$, the pizza slices $T'_{\ell'}$ and $T'_{\ell''}$ are not consecutive pizza slices of Λ', thus $\ell''>\ell'+1$. Also, $T'_{\ell'+1}$ is not a coherent pizza slice of Λ', otherwise the same arguments as above applied to tied pizza slices $T'_{\ell'}$ and $T'_{\ell'+1}$ would imply that $T_{\ell+1}$ is a coherent pizza slice, a contradiction. Thus $T'_{\ell'+1}$ is a transversal pizza slice of Λ' and $T_{\ell'+2}$ is a coherent pizza slice of Λ'. Since $T'_{\ell''}$ and $T'_{\ell'+2}$ belong to the same rightward caravan C' as $T'_{\ell'}$, and the order of coherent pizza slices in the caravans C and C' is the same as the order of the exponents q and q' of their boundary arcs, we have $\ell''=\ell'+2$. This completes the proof of Proposition 4.14. □

Corollary 4.15. *Let (T,T') be a normal pair of Hölder triangles, with the minimal pizzas Λ and Λ' on T and T' associated with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$, and let C be a caravan of Λ. Then the following properties hold:*
(A) *The characteristic correspondence τ has the same sign, either positive or negative, on all coherent pizza slices of C. We say that τ is positive (resp., negative) on C.*
(B) *There is a caravan $C'=\tau(C)$ of pizza slices of Λ' such that τ defines a one-to-one correspondence between coherent pizza slices of C and C', preserving their order when τ is positive on C, and reversing their order when τ is negative on C. If C is a rightward (resp., leftward) caravan of Λ and τ is positive on C, then $C'=\tau(C)$ is a rightward (resp., leftward) caravan of Λ'. Similarly, if C is a rightward (resp., leftward) caravan of Λ and τ is negative on C, then $C'=\tau(C)$ is a leftward (resp., rightward) caravan of Λ'.*
(C) *If τ is positive on a rightward (resp., leftward) caravan C of Λ, then a maximum zone M_i of Λ is right-adjacent (resp., left-adjacent) to C if, and only if, the maximum zone $M'_{\sigma(i)}$ of Λ' is right-adjacent (resp., left-adjacent) to $C'=\tau(C)$. Similarly, if τ is negative on C, then a maximum zone M_i of Λ is right-adjacent (resp., left-adjacent) to C if, and only if, the maximum zone $M'_{\sigma(i)}$ of Λ' is left-adjacent (resp., right-adjacent) to $C'=\tau(C)$. Thus the adjacent set $\mathcal{A}(C)$ of a caravan C of pizza slices of Λ is mapped by σ to the adjacent set $\mathcal{A}(C')$ of the caravan $C'=\tau(C)$ of pizza slices of Λ'.*

Remark 4.16. If D_ℓ is a coherent pizza zone common to consecutive coherent pizza slices T_ℓ and $T_{\ell+1}$ of Λ, then [5, Proposition 3.9] implies that $T'_{\tau(\ell)}$ and $T'_{\tau(\ell+1)}$ are consecutive coherent pizza slices of Λ' with a common pizza zone $D'_{\ell'}$ of Λ' corresponding to D_ℓ. Thus τ must have the same sign on these two pizza slices, with either $\ell'=\tau(\ell)$ and $\ell'+1=\tau(\ell+1)$ when τ is positive, or $\ell'=\tau(\ell+1)$ and $\ell'+1=\tau(\ell)$ when τ is negative. Similarly, if $D_\ell=M_i$ is a maximum zone of Λ common to two coherent pizza slices T_ℓ and $T_{\ell+1}$, then Proposition 4.8 implies that $T'_{\tau(\ell)}$ and $T'_{\tau(\ell+1)}$ are consecutive coherent pizza slices of Λ' with a common pizza zone $D'_{\ell'}=M'_{\sigma(i)}$ corresponding to D_ℓ, thus τ must have the same sign on these two pizza slices.

Definition 4.17. If C is a rightward (resp., leftward) caravan of pizza slices of Λ and τ is positive (resp., negative) on C, let $j_+(C)=\min_{i:M_i\in\mathcal{A}(C)}\sigma(i)$ and $j_-(C)=j_+(C)-1$.

If C is a rightward (resp., leftward) caravan of pizza slices of Λ and τ is negative (resp., positive) on C, let $j_-(C) = \max_{i: M_i \in \mathcal{A}(C)} \sigma(i)$ and $j_+(C) = j_-(C) + 1$.

Proposition 4.18. *If C is a caravan of pizza slices of Λ and $C' = \tau(C)$ is the corresponding caravan of pizza slices of Λ', then the number of maximum zones M'_j of Λ' such that $M'_j \prec C'$ is $j_-(C)$. If C_1 and C_2 are two caravans of Λ such that $\tau(C_1) \prec \tau(C_2)$, then $j_-(C_1) \le j_-(C_2)$.*

Proof. We may assume that C is a rightward caravan and τ is positive on C, the other cases being similar. Then $C' = \tau(C)$ is a rightward caravan of Λ', according to item (B) of Corollary 4.15, and $C' \prec M'_j$ for all maximum zones $M'_j \in \mathcal{A}(C')$ of Λ'. Since $\mathcal{A}(C)$ is mapped to $\mathcal{A}(C')$ by σ, according to item (C) of Corollary 4.15, the maximum zone M'_{j+} has the minimal index $j = j+$ among all maximum zones $M'_j \in \mathcal{A}(C')$ of Λ'. Since all maximum zones M'_j of Λ' which are not in $\mathcal{A}(C')$ satisfy either $M'_j \prec C'$ or $M'_{j_+} \prec M'_j$, according to Remark 4.13, there are exactly $j_- = j_+ - 1$ maximum zones M'_j of Λ' such that $M'_j \prec C'$. □

Definition 4.19. Let $\mathcal{K} = \mathcal{M} \bigcup \mathcal{L}$ be the disjoint union of the set $\mathcal{M}$ of maximum zones of Λ and the set $\mathcal{L}$ of coherent pizza slices of Λ, and let $\mathcal{K}' = \mathcal{M}' \bigcup \mathcal{L}'$ be the corresponding set of maximum zones and coherent pizza slices of Λ'. Notice that the sets $\mathcal{K}$ and $\mathcal{K}'$ are totally ordered according to orientations of T and T', respectively (see Definition 2.38). Since the sets $\mathcal{K}$ and $\mathcal{K}'$ have the same number of elements $K = m + L$, the combination of permutations σ and υ of the sets $[m]$ and $[L]$, respectively, defines a permutation ω of the set $[K] = \{1, \ldots, K\}$, such that $k' = \omega(k)$ either when the k-th element of $\mathcal{K}$ corresponds to the i-th maximum zone M_i of Λ and the k'-th element of K corresponds to the $\sigma(i)$-th maximum zone $M'_{\sigma(i)}$ of Λ', or when the k-th element of $\mathcal{K}$ corresponds to the l-th coherent pizza slice T_ℓ of Λ and the k'-th element of K corresponds to the $\upsilon(l)$-th coherent pizza slice $T'_{\tau(\ell)}$ of Λ'. The permutation ω of $[K]$ is called the *combined characteristic permutation* of the pair (T, T').

Theorem 4.20. *Let (T, T') be a normal pair of Hölder triangles with the minimal pizzas Λ and Λ' on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$, and let (T, T'') be another normal pair of Hölder triangles with a minimal pizza on T associated with the distance function $dist(x, T'')$ combinatorially equivalent to Λ and a minimal pizza Λ'' on T'' associated with the distance function $dist(x'', T)$.*
Since the sets $\mathcal{M}'$ and $\mathcal{M}''$ of maximum zones of Λ' and Λ'' have the same number of elements m, and the sets $\mathcal{L}'$ and $\mathcal{L}''$ of coherent pizza slices of Λ' and Λ'' have the same number of elements L, the disjoint union $\mathcal{K}''$ of the sets $\mathcal{M}''$ and $\mathcal{L}''$ has the same number of elements $K = m + L$ as the set $\mathcal{K}'$. If the characteristic permutation σ of $[m]$ of the pair (T, T') is the same as the characteristic permutation of the pair (T, T''), the permutation υ of $[L]$ induced by the characteristic correspondence τ of the pair (T, T') is the same as the permutation of $[L]$ induced by the correspondence between coherent pizza slices of T and T'', and the sign function $s : [L] \to \{+, -\}$ of the pair (T, T') is the same as the sign function of the pair (T, T''), then the combined characteristic permutation ω of $[K]$ of the pair (T, T') is the same as the combined characteristic permutation of the pair (T, T'').

Proof. The permutation σ defines the same order on the sets $\mathcal{M}'$ and $\mathcal{M}''$ of maximum zones of Λ' and Λ'', respectively, consistent with orientations of T' and T'', since $\sigma(i) < \sigma(j)$ implies that $M'_{\sigma(i)} \prec M'_{\sigma(j)}$ and $M''_{\sigma(i)} \prec M''_{\sigma(j)}$. Similarly, the permutation υ defines the same order on the sets $\mathcal{L}'$ and $\mathcal{L}''$ of coherent pizza slices of Λ' and Λ'', respectively, consistent with orientations of T' and T''. To prove that the permutation ω is the same for (T,T') and (T,T''), it is enough to show that a maximum zone M'_j of Λ' precedes the k-th coherent pizza slice of Λ' if, and only if, a maximum zone M''_j of Λ'' precedes the k-th coherent pizza slice of Λ''. But this follows from Proposition 4.18: if $k = \upsilon(l)$ and C is the caravan of pizza slices of Λ containing the l-th coherent pizza slice of Λ, then the k-th coherent pizza slice of Λ' belongs to the caravan C' of Λ' and the k-th coherent pizza slice of Λ'' belongs to the caravan C'' of Λ', such that the number $j_-(C)$ of maximum zones of Λ' preceding C' is the same as the number of maximum zones of Λ'' preceding C'', since $j_-(C)$ is determined by the pizza Λ, permutation σ and the sign function s. □

Remark 4.21. If conditions of Theorem 4.20 are satisfied, we say that ω is *determined* by Λ, σ, υ and s, meaning that ω is the same for all normal pairs for which Λ, σ, υ and s are the same. In what follows, we are going to use this terminology for other invariants of normal pairs of Hölder triangles.

Definition 4.22. Let Λ be a minimal pizza on a normally embedded Hölder triangle T associated with a non-negative Lipschitz function f on T. Let m and L be the numbers of maximum zones and coherent pizza slices of Λ, respectively. Let σ and υ be permutations of the sets $[m] = \{1,\ldots,m\}$ and $[L] = \{1,\ldots,L\}$, respectively, and let $s : [L] \to \{+,-\}$ be a sign function on $[L]$. For any caravan C of Λ, let $\mathcal{A}(C)$ be the adjacent set of C (see Definition 4.12), and let $j_-(C)$ be the index defined by σ, υ and s (see Definition 4.17). A triple (σ,υ,s) is called *allowable* if it satisfies the following conditions:
(**A1**). *If the k-th and l-th coherent pizza slices of Λ belong to the same caravan C, then either $s(k) = s(l) = +$ and $\upsilon(l)-\upsilon(k) = l-k$, or $s(k) = s(l) = -$ and $\upsilon(l)-\upsilon(k) = k-l$.*
(**A2**) *If C_1 and C_2 are two caravans of Λ such that $\tau(C_1) \prec \tau(C_2)$, then $j_-(C_1) \le j_-(C_2)$.*
(**A3**) *If D_ℓ is either a coherent pizza zone or a maximum zone of Λ adjacent to the k-th and $(k+1)$-st coherent pizza slices of Λ, then $s(k) = s(k+1)$.*
Note that these conditions are satisfied when σ, υ and s are defined for a normal pair (T,T') such that $f(x) = dist(x,T')$ is the distance function on T (see Proposition 4.18 and Remark 4.16). In particular, these conditions are necessary to define the combined characteristic permutation ω of the pair (T,T') (see Definition 4.19).

Proposition 4.23. *Let (σ,υ,s) be an allowable triple, as in Definition 4.22, for a minimal pizza Λ on a normally embedded Hölder triangle T associated with a non-negative Lipschitz function f on T. Let $\mathcal{K}$ be the disjoint union of the set $\mathcal{M}$ of m maximum zones of Λ and the set $\mathcal{L}$ of L coherent pizza slices of Λ, ordered according to orientation of T. Then there exists a unique permutation ω of the set $[K] = \{1,\ldots,K\}$, where $K = m+L$, compatible with the permutations σ and υ on the subsets of $\mathcal{K}$ corresponding to $\mathcal{M}$ and $\mathcal{L}$, respectively (see Remark 4.24 below) and for any $k \in [K]$ such that the k-th element of $\mathcal{K}$ corresponds to a coherent pizza slice of Λ belonging to a caravan C, the number of indices $l \in [K]$ corresponding to maximum zones of Λ, such that $\omega(l) < \omega(k)$, is equal to $j_-(C)$. If the triple (σ,υ,s) is defined for a normal pair (T,T') such that Λ is a minimal pizza associated with the distance function $f(x) = dist(x,T')$ on T, then ω is the combined characteristic permutation of the pair (T,T') (see Definition 4.19).*

Remark 4.24. In Proposition 4.23, “compatible” means the following:
For any indices k and l of $[K]$ corresponding to the maximum zones M_i and M_j of Λ

(resp., to the i-th and j-th coherent pizza slices of Λ), we have $\omega(k) < \omega(l)$ if, and only if, $\sigma(i) < \sigma(j)$ (resp., $\upsilon(i) < \upsilon(j)$).

Proof of Proposition 4.23. Let K_m and K_L be the subsets of indices $k \in [K]$ corresponding to the maximum zones and coherent pizza slices of Λ, respectively. Since ω is compatible with σ and υ, the order of indices $\omega(k)$ in the subsets $\omega(K_m)$ and $\omega(K_L)$ of $[K]$ is determined by the permutations σ and υ, respectively. In particular, for each $k \in K_L$, the set $K_L(k)$ of indices $l \in K_L$, such that $\omega(l) < \omega(k)$, is known. Thus the permutation ω would be completely defined if, for each $k \in K_L$, the set $K_m(k)$ of indices $i \in K_m$, such that $\omega(i) < \omega(k)$, is known. If $k \in K_L$ corresponds to a j-th coherent pizza slice of Λ belonging to a caravan C of Λ, such that either C is a rightward caravan and $s(j) = +$ or C is a leftward caravan and $s(j) = -$, then, according to Definition 4.17, the set $K_m(k)$ consists of all indices of K_m corresponding to the maximum zones M_l of Λ such that $\sigma(l) < \min_{i\in\mathcal{A}(C)} \sigma(i)$. Condition (**A1**) of Definition 4.22 implies that the value $s(j)$ is the same for all coherent pizza slices of Λ belonging to the same caravan C. Similarly, if $k \in K_L$ corresponds to a j-th coherent pizza slice of Λ belonging to a caravan C of Λ, such that either C is a rightward caravan and $s(j) = -$ or C is a leftward caravan and $s(j) = +$, then the set $K_m(k)$ consists of all indices of K_m corresponding to the maximum zones M_l of Λ such that $\sigma(l) \le \max_{i\in\mathcal{A}(C)} \sigma(i)$. Condition (**A2**) of Definition 4.22 implies that $K_m(j) \subseteq K_m(j')$ when $\upsilon(j) < \upsilon(j')$. Thus ω is determined by Λ, σ, υ and s.
If (T, T') is a normal pair of Hölder triangles, Λ is a minimal pizza associated with the distance function $f(x) = dist(x, T')$ on T, and σ, υ and s are defined by the characteristic permutation σ and characteristic correspondence τ of the pair (T, T'), then the permutation ω is the combined characteristic permutation of the pair (T, T'), due to Definition 4.19 and Theorem 4.20. □

Remark 4.25. The characteristic correspondence τ can be extended to a correspondence between the pizza zones of Λ and Λ' adjacent to coherent pizza slices as follows: If T_ℓ is a coherent pizza slice of Λ and $T'_{\ell'} = \tau(T_\ell)$, we can define $\tau(D_{\ell-1}) = D'_{\ell'-1}$ and $\tau(D_\ell) = D'_{\ell'}$ (resp., $\tau(D_{\ell-1}) = D'_{\ell'}$ and $\tau(D_\ell) = D'_{\ell'-1}$) if τ is positive (resp., negative) on T_ℓ.
Proposition 4.5 implies that this correspondence preserves pizza toppings: $\mu(\tau(D_{\ell-1})) = \mu(D_{\ell-1})$ and $\mu(\tau(D_\ell)) = \mu(D_\ell)$, $tord(D_{\ell-1}, \tau(D_{\ell-1})) = q_{\ell-1}$ and $tord(D_\ell, \tau(D_\ell)) = q_\ell$.
If τ is positive on T_ℓ, then $q'_{\ell'-1} = q_{\ell-1},\ q'_{\ell'} = q_\ell,\ \mu'_{\ell'}(q'_{\ell'-1}) = \mu_\ell(q_{\ell-1})$ and $\mu'_{\ell'}(q'_{\ell'}) = \mu_\ell(q_\ell)$.
If τ is negative on T_ℓ, then $q'_{\ell'-1} = q_\ell,\ q'_{\ell'} = q_{\ell-1},\ \mu'_{\ell'}(q'_{\ell'-1}) = \mu_\ell(q_\ell)$ and $\mu'_{\ell'}(q'_{\ell'}) = \mu_\ell(q_{\ell-1})$.
This correspondence between the pizza zones of Λ and Λ' is not necessarily one-to-one: a pizza zone of Λ common to two coherent pizza slices may be "split," assigned by τ to two different pizza zones of Λ', and two pizza zones of Λ may be assigned to the same "split" pizza zone of Λ'. A boundary arc of T adjacent to a coherent pizza slice of Λ may be assigned by τ to an interior pizza zone of Λ', and an interior pizza zone of Λ may be assigned to a boundary arc of T' adjacent to a coherent pizza slice of Λ'. However, the correspondence between the pizza zones of Λ and Λ' defined by τ is one-to-one on coherent pizza zones and on the maximum zones (see Remark 4.16), and on the pizza zones common to tied coherent triangles, due to Proposition 4.14.
To recover one-to-one correspondence between *essential* pizza zones of Λ and Λ' (see Definition 7.1) compatible with σ on the maximum zones and assigning the boundary arcs of T' to the boundary arcs of T, we are going to define in Section 7 *pre-pizzas* $\tilde{\Lambda}$

and $\tilde{\Lambda}'$ (see Definition 7.3) removing *non-essential* arcs from pizzas Λ and Λ', and *twin pre-pizzas* $\check{\Lambda}$ and $\check{\Lambda}'$ (see Definition 7.6) expanding pre-pizzas $\tilde{\Lambda}$ and $\tilde{\Lambda}'$ by *twin arcs*.

5. The $\{\sigma\tau\}$-pizza invariant

Let (T,T') be a normal pair of β-Hölder triangles $T=T(\gamma_1,\gamma_2)$ and $T'=T(\gamma'_1,\gamma'_2)$. Let $\{D_\ell\}_{\ell=0}^{p}$ and $\{D'_{\ell'}\}_{\ell'=0}^{p'}$ be the sets of pizza zones for the minimal pizzas $\Lambda_T=\{T_\ell\}_{\ell=1}^{p}$ and $\Lambda'_{T'}=\{T'_{\ell'}\}_{\ell'=1}^{p'}$ associated with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$, where $T_\ell=T(\lambda_{\ell-1},\lambda_\ell)$, $T'_{\ell'}=T(\lambda'_{\ell'-1},\lambda'_{\ell'})$, and the arcs $\lambda_\ell\in D_\ell$ and $\lambda'_{\ell'}\in D_{\ell'}$ are selected so that the pizzas Λ_T and $\Lambda'_{T'}$ are compatible (see Definition 7.11). Let $\{M_i\}_{i=1}^{m}$ and $\{M'_j\}_{j=1}^{m}$ be the maximum zones in $V(T)$ and $V(T')$ for the functions f and g. Let $j=\sigma(i)$ be the characteristic permutation σ of the pair (T,T') (see Definition 3.4), and let $\ell'=\tau(\ell)$ be the characteristic correspondence between the sets of coherent pizza slices of Λ_T and $\Lambda'_{T'}$ (see Definition 4.6).

Definition 5.1. The *$\{\sigma\tau\}$-pizza* on $T\cup T'$ (see [5, Definition 4.12]) consists of the minimal pizzas Λ_T and $\Lambda'_{T'}$, characteristic permutation σ and characteristic correspondence τ.

Definition 5.2. Two $\{\sigma\tau\}$-pizzas $(\Lambda_T,\Lambda'_{T'},\sigma_T,\tau_T)$ on $T\cup T'$ and $(\Lambda_S,\Lambda'_{S'},\sigma_S,\tau_S)$ on $S\cup S'$ are *combinatorially equivalent* if pizzas Λ_T and Λ_S are combinatorially equivalent (see Definition 2.26), pizzas $\Lambda'_{T'}$ and $\Lambda'_{S'}$ are combinatorially equivalent, $\sigma_T=\sigma_S$ and $\tau_T=\tau_S$.

It was shown in [5, Theorem 4.13] that the $\{\sigma\tau\}$-pizza is an outer Lipschitz invariant of a pair of normally embedded Hölder triangles. The main result of this section is the following theorem (see [5, Conjecture 4.14]).

Theorem 5.3. *Let (T,T') and (S,S') be two normal pairs of Hölder triangles. If the $\{\sigma\tau\}$-pizza on $T\cup T'$ is combinatorially equivalent to the $\{\sigma\tau\}$-pizza on $S\cup S'$, then there is an orientation-preserving outer bi-Lipschitz homeomorphism $H:T\cup T'\to S\cup S'$ such that $H(T)=S$ and $H(T')=S'$.*

We prove it first (see Theorem 5.7 below) for totally transversal pairs of Hölder triangles.

Lemma 5.4. *Let (T,T') be a totally transversal pair of β-Hölder triangles (see Definition 4.1) with the pizza Λ_T on T associated with the distance function $f(x)=dist(x,T')$. Let $q_\ell=ord_{\lambda_\ell}f$, for $0\le\ell\le p$. Then, unless $tord(T,T')\le\beta$, one of the boundary arcs of each Hölder triangle T_ℓ belongs to a maximum zone M_i of Λ_T.*
If $\gamma\subset T_\ell$ is an arc such that $\gamma\notin M_i$, then $ord_\gamma f=tord(\gamma,M_i)$.

Proof. This follows immediately from Definition 4.1. □

Remark 5.5. For a totally transversal pair (T,T'), the $\{\sigma\tau\}$-pizza does not contain the correspondence τ, as the sets of coherent pizza slices of Λ_T and $\Lambda'_{T'}$ are empty. Also, Lemma 5.4 implies that the pizza Λ_T is completely determined by exponents β_ℓ of the pizza slices T_ℓ and exponents $q_\ell=ord_{\lambda_\ell}f$. Similarly, the pizza $\Lambda'_{T'}$ is completely determined by exponents $\beta'_{\ell'}$ of the pizza slices $T'_{\ell'}$ and exponents $q'_{\ell'}=ord_{\lambda'_{\ell'}}g$.

Lemma 5.6. *Let (T,T') be a totally transversal pair of β-Hölder triangles with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$. Let γ' be an arc of a pizza slice $T'_{\ell'}$ of $\Lambda'_{T'}$, and let $q=tord(\gamma',T)$. Let $Z_{\gamma'}$ be the set of arcs $\gamma\subset T$ such that*

$tord(\gamma,\gamma')=q$. If $q\le\beta$ then $Z_{\gamma'}=T$. Otherwise, there is a unique maximum zone $M_i\subset V(T)$ such that $M'_{\sigma(i)}$ is adjacent to $T'_{\ell'}$, $tord(\theta,M_i)=q$ and $Z_{\gamma'}$ is the set of all arcs $\gamma\in T$ such that $tord(\gamma,M_i)\ge q$. Moreover $tord(\gamma,M_i)=tord(\gamma,\gamma')$ for any $\gamma\in V(T)\setminus Z_{\gamma'}$.

Proof. The statement follows from the non-archimedean property of the tangency order, and from [4, Lemma 3.3]. □

Theorem 5.7. *Let (T,T') and (S,S') be two totally transversal pairs of Hölder triangles. If the $\{\sigma\tau\}$-pizzas of the pairs (T,T') and (S,S') are combinatorially equivalent, then there is an orientation-preserving outer bi-Lipschitz homeomorphism $H:T\cup T'\to S\cup S'$ such that $H(T)=S$ and $H(T')=S'$.*

Proof. Let $\Lambda_T=\{T_\ell\}_{\ell=1}^p$ and $\Lambda'_{T'}=\{T'_{\ell'}\}_{\ell'=1}^{p'}$ be minimal pizzas on T and T' associated with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$, where $T_\ell=T(\lambda_{\ell-1},\lambda_\ell)$ and $T'_{\ell'}=T(\lambda'_{\ell'-1},\lambda'_{\ell'})$. Let $\Lambda_S=\{S_\ell\}_{\ell=1}^p$ and $\Lambda'_{S'}=\{S'_{\ell'}\}_{\ell'=1}^{p'}$, where $S_\ell=T(\theta_{\ell-1},\theta_\ell)$ and $S'_{\ell'}=T(\theta'_{\ell'-1},\theta'_{\ell'})$, be minimal pizzas on S and S' associated with the distance functions $\phi(y)=dist(y,S')$ and $\psi(y')=dist(y',S)$, respectively.

Since the pizzas Λ_T and Λ_S are equivalent, the Hölder triangles T_ℓ and S_ℓ have the same exponent β_ℓ for each ℓ. Thus, for $1\le\ell\le p$, there exists a bi-Lipschitz homeomorphism $H_\ell:T_\ell\to S_\ell$ preserving the distance to the origin, such that $H_\ell(\lambda_{\ell-1})=\theta_{\ell-1}$ and $H_\ell(\lambda_\ell)=\theta_\ell$. Similarly, for $1\le\ell'\le p'$, there exists a bi-Lipschitz homeomorphism $H'_{\ell'}:T'_{\ell'}\to S'_{\ell'}$ preserving the distance to the origin, such that $H'_{\ell'}(\lambda'_{\ell'-1})=\theta'_{\ell'-1}$ and $H'_{\ell'}(\lambda'_{\ell'})=\theta'_{\ell'}$. Let us define a map $H:T\cup T'\to S\cup S'$ so that its restriction to each Hölder triangle T_ℓ coincides with H_ℓ, and its restriction to each Hölder triangle $T'_{\ell'}$ coincides with $H'_{\ell'}$. Since T, T', S and S' are normally embedded, H defines outer bi-Lipschitz homeomorphisms $T\to S$ and $T'\to S'$. In particular, for any two arcs γ_1 and γ_2 in T we have $tord(H(\gamma_1),H(\gamma_2))=tord(\gamma_1,\gamma_2)$, and for any two arcs γ'_1 and γ'_2 in T' we have $tord(H(\gamma'_1),H(\gamma'_2))=tord(\gamma'_1,\gamma'_2)$.

Since $H(\lambda_\ell)=\theta_\ell$ and the pizzas on T and S are equivalent, we have $tord(\lambda_\ell,T')=ord_{\lambda_\ell}f=ord_{\theta_\ell}\phi=tord(H(\lambda_\ell),S')$ for $0\le\ell\le p$. Similarly, $tord(\lambda'_{\ell'},T)=tord(H(\lambda'_{\ell'}),S)$ for $0\le\ell'\le p'$. This implies, in particular, that H maps the maximum zones M_i for f to the maximum zones N_i for ϕ, and the maximum zones M'_j for g to the maximum zones N'_j for ψ. Since $\sigma_S=\sigma_T$, we have $\bar q_i=tord(M_i,M'_j))=tord(N_i,N'_j)$, where $j=\sigma_T(i)=\sigma_S(i)$.

Let γ and γ' be any two arcs in T and T', respectively. We are going to show that Lemma 5.6 implies $tord(H(\gamma),H(\gamma'))=tord(\gamma,\gamma')$. Let $\gamma'\subset T'_{\ell'}$, a pizza slice of λ'_T, let $q=tord(\gamma',T)$, and let $Z_{\gamma'}$ be the set of all arcs in T having tangency order q with γ'. According to Lemma 5.6, $Z_{\gamma'}$ is the set of all arcs $\gamma\subset T$ such that $tord(\gamma,M_i)\ge q$. Thus $tord(H(\gamma),H(\gamma'))=tord(H(\gamma),N_i)=q$ for all arcs $\gamma\in Z_{\gamma'}$. If $\gamma'\in M'_{\sigma(i)}$, then $Z_{\gamma'}=M_i$, thus $H(Z_{\gamma'})=\check M_i$ is the set of arcs in S having tangency order q with $H(\gamma')$. If $\gamma'\notin M'_{\sigma(i)}$, then Lemma 5.6 applied to (S,S') implies that $tord(\gamma,\gamma')=tord(H(\gamma),H(\gamma'))$ for any two arcs $H(\gamma)\subset S$ and $H(\gamma')\subset S'$ such that $tord(H(\gamma),H(\gamma'))=q$. Thus $H(Z_{\gamma'})$ coincides with the set $Z_{H(\gamma')}$ of arcs in S having tangency order q with $H(\gamma')$. If $\gamma\notin Z_{\gamma'}$, then $H(\gamma)\notin Z_{H(\gamma')}$, and Lemma 5.6 implies that $tord(H(\gamma),H(\gamma'))=tord(H(\gamma),N_i)=tord(\gamma,M_i)=tord(\gamma,\gamma')$. This proves that $tord(H(\gamma),H(\gamma'))=tord(\gamma,\gamma')$ for all arcs $\gamma\subset T$ and $\gamma'\subset T'$.

Since H preserves the tangency orders of any two arcs in $T\cup T'$, it is outer bi-Lipschitz by Proposition 2.5. This completes the proof of Theorem 5.7. $\square$

Proof of Theorem 5.3. Let (T,T') and (S,S') be two normal pairs of β-Hölder triangles. Let $\Lambda_T=\{T_\ell\}_{\ell=1}^{p}$ and $\Lambda'_{T'}=\{T'_{\ell'}\}_{\ell'=1}^{p'}$ be minimal pizzas on T and T' associated with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$, where $T_\ell=T(\lambda_{\ell-1},\lambda_\ell)$ and $T'_{\ell'}=T(\lambda'_{\ell'-1},\lambda'_{\ell'})$. Let $\Lambda_S=\{S_\ell\}_{\ell=1}^{p}$ and $\Lambda'_{S'}=\{S'_{\ell'}\}_{\ell'=1}^{p'}$ be minimal pizzas on S and S' associated with the distance functions $\phi(y)=dist(y,S')$ and $\psi(y')=dist(y',S)$, where $\S_\ell=T(\theta_{\ell-1},\theta_\ell)$ and $S'_{\ell'}=T(\theta'_{\ell'-1},\theta'_{\ell'})$.

Let $\{D_\ell\}_{\ell=0}^{p}$ and $\{D'_{\ell'}\}_{\ell'=0}^{p'}$ be the pizza zones of Λ_T and $\Lambda'_{T'}$, respectively (see [5, Lemma 2.28]), such that $\lambda_\ell\in D_\ell$ and $\lambda'_{\ell'}\in D'_{\ell'}$. Let $\{\Delta_\ell\}_{\ell=0}^{p}$ and $\{\Delta'_{\ell'}\}_{\ell'=0}^{p'}$ be the pizza zones of Λ_S and $\Lambda'_{S'}$, respectively, such that $\theta_\ell\in\Delta_\ell$ and $\theta'_{\ell'}\in\Delta'_{\ell'}$.

Since the pizzas Λ_T and Λ_S are equivalent, we have $q_\ell=ord_{\lambda_\ell}f=ord_{\theta_\ell}\phi$. Since the pizzas $\Lambda'_{T'}$ and $\Lambda'_{S'}$ are equivalent, we have $q'_{\ell'}=ord_{\lambda'_{\ell'}}g=ord_{\theta'_{\ell'}}\psi$.

Let D_ℓ be a coherent pizza zone of Λ_T, i.e., $\mu(D_\ell)=\nu(\lambda_\ell)<q_\ell$. Since D_ℓ is a maximal perfect q_ℓ-order zone for f (see Lemma 2.35 and [5, Proposition 3.9]) there is a unique maximal perfect q_ℓ-order zone $\tau(D_\ell)\subset V(T')$ for g (see Remark 4.25) such that $\mu(\tau(D_\ell))=\mu(D_\ell)$ and $tord(D_\ell,\tau(D_\ell))=q_\ell$. Note that the correspondence τ is well defined on D_ℓ because D_ℓ is a coherent zone. Since D_ℓ is coherent, it is a common pizza zone for two coherent pizza slices T_ℓ and $T_{\ell+1}$ of Λ_T, thus $\tau(D_\ell)=D'_{\ell'}$ is a common pizza zone for the pizza slices $T'_{\tau(\ell)}$ and $T'_{\tau(\ell+1)}$. In particular, τ has the same sign (either positive or negative) on T_ℓ and $T_{\ell+1}$, $\ell'=\tau(\ell)$ and $\ell'+1=\tau(\ell+1)$ when τ is positive, $\ell'=\tau(\ell+1)$ and $\ell'+1=\tau(\ell)$ when τ is negative.

Conversely, any coherent pizza zone $D'_{\ell'}$ of $\Lambda'_{T'}$ coincides with $\tau(D_\ell)$ for some coherent pizza zone D_ℓ of Λ_T.

In each coherent pizza zone $D'_{\ell'}=\tau(D_\ell)$ we can choose an arc $\lambda'_{\ell'}$ so that $tord(\lambda_\ell,\lambda'_{\ell'})=q_\ell$. Selecting any arc $\lambda'_{\ell'}$ in each transversal pizza zone $D'_{\ell'}$, we obtain a minimal pizza $\Lambda'_{T'}$ on T' associated with g, such that any coherent pair of pizza slices $(T_\ell,T'_{\tau(\ell)})$ satisfies the condition (17). In particular, according to [5, Theorem 3.20, Proposition 3.2], there is an outer bi-Lipschitz homeomorphism $h_\ell:\Gamma_\ell\to T'_{\tau(\ell)}$, where Γ_ℓ is the graph of $f|_{T_\ell}$, such that $(id,h_\ell):T_\ell\cup\Gamma_\ell\to T_\ell\cup T'_{\tau(\ell)}$ is an outer bi-Lipschitz homeomorphism. Similarly, given a minimal pizza Λ_S on S associated with ϕ, we can define a minimal pizza $\Lambda'_{S'}$ on S' associated with ψ, such that, for any coherent pair $(S_\ell,S'_{\tau(\ell)})$ of pizza slices, there is an outer bi-Lipschitz homeomorphism $\eta_\ell:\Phi_\ell\to S'_{\tau(\ell)}$, where Φ_ℓ is the graph of $\phi|_{S_\ell}$, such that $(id,\eta_\ell):S_\ell\cup\Phi_\ell\to S_\ell\cup S'_{\tau(\ell)}$ is an outer bi-Lipschitz homeomorphism.

Let $H:T\cup T'\to S\cup S'$ be a bi-Lipschitz homeomorphism such that $H(\lambda_\ell)=\check\lambda_\ell$ for $\ell=0,\ldots,p$, and $H(\lambda'_{\ell'})=\theta'_{\ell'}$ for $\ell'=0,\ldots,p'$. Then H, restricted to any coherent pair of pizza slices $(T_\ell,T'_{\tau(\ell)})$, maps it to a coherent pair of pizza slices $(S_\ell,S'_{\tau(\ell)})$. Given a homeomorphism $H|_T$, we may choose a homeomorphism $H|_{T'}$ so that the following diagram is commutative.

$$\begin{array}{ccccc} S_\ell & \xrightarrow{(id,\phi)} & \Phi_\ell & \xrightarrow{\eta_\ell} & S'_{\tau(\ell)} \\ {\scriptstyle H|_T}\uparrow & & & & \uparrow{\scriptstyle H|_{T'}} \\ T_\ell & \xrightarrow{(id,f)} & \Gamma_\ell & \xrightarrow{h_\ell} & T'_{\tau(\ell)} \end{array} \tag{22}$$

We are going to prove that a homeomorphism $H:T\cup T'\to S\cup S'$, such that its restriction to coherent pairs of Hölder triangles satisfies (22), is outer bi-Lipschitz.

Remark 5.8. Note that, for a coherent pair (T,T') of Hölder triangles, such that T is a pizza slice associated with a Lipschitz function f and T' is a graph of f, a mapping $H: T\cup T' \to T\cup T'$ bi-Lipschitz on each of the two triangles may be not outer bi-Lipschitz on their union, even when H^*f is Lipschitz contact equivalent to f.

For example, let $T=\{x\ge 0,\ y\ge 0\}$ be the first quadrant in the xy-plane and T' the graph of a function $z=f(x,y)=y^2$ on T. Let $H(x,y,0)=(x,y,0)$ and $H(x,y,y^2)=(x,2y,4y^2)$. If $\gamma=\{x\ge 0,\ y=x^2,\ z\equiv 0\}$ and $\gamma'=\{x\ge 0,\ y=x^2,\ z=x^4\}$ are arcs in T and T', then $tord(\gamma,\gamma')=4$ but $tord(H(\gamma),H(\gamma'))=2$.

Lemma 5.9. *For any arc $\gamma'\subset T'$, we have $tord(H(\gamma'),S)=tord(\gamma',T)$.*

Proof. Let $q=tord(\gamma',T)$. If $\gamma'\subset T'_{\ell'}$, where $\ell'=\tau(\ell)$, belongs to a coherent pizza slice, the statement follows from (22), as H is compatible with the graph structure of the pair $(T_\ell,T'_{\ell'})$. If γ' belongs to a transversal pizza slice $T'_{\ell'}$, then either $\lambda'_{\ell'-1}$ or $\lambda'_{\ell'}$ is the supporting arc $\tilde\lambda'_{\ell'}$ of $T'_{\ell'}$ with respect to g (see Definition 2.16). We may assume that $\tilde\lambda'_{\ell'}=\lambda'_{\ell'}$, the case of $\tilde\lambda'_{\ell'}=\lambda'_{\ell'-1}$ being similar. Then, since $T'_{\ell'}$ is transversal, we have either $tord(\gamma',\lambda'_{\ell'})>tord(\lambda'_{\ell'},T)=q$ or $q=tord(\gamma',\lambda'_{\ell'})<tord(\lambda'_{\ell'},T)$. Since $H: T'_{\ell'}\to S'_{\ell'}$ is a bi-Lipschitz map such that $H(\lambda'_{\ell'})=\check\lambda'_{\ell'}$, this implies $tord(H(\gamma'),S)=tord(\gamma',T)$. □

Corollary 5.10. *The homeomorphism $H: T\cup T'\to S\cup S'$ preserves the minimal pizzas on T and T' associated with the distance functions f and g, respectively: $Q_\phi(H(T_\ell))=Q_f(T_\ell)$, $\mu_{H(T_\ell),\phi}\equiv\mu_{T_\ell,f}$ for $1\le\ell\le p$, and $Q_\psi(H(T'_{\ell'}))=Q_g(T'_{\ell'})$, $\mu_{H(T'_{\ell'}),\psi}\equiv\mu_{T'_{\ell'},g}$ for $1\le\ell'\le p'$. In particular, H maps coherent pizza slices of T and T' to coherent pizza slices of S and S', and transversal pizza slices of T and T' to transversal pizza slices of S and S', respectively.*

Let $q=tord(\gamma',T)$, where γ' is an arc in T'. We are going to choose an arc λ in T such that $tord(\gamma',\lambda)=tord(H(\gamma'),H(\lambda))=q$. If $\gamma'\subset T'_{\tau(\ell)}$ belongs to a coherent pizza slice of T' of a minimal pizza associated with g, then $\lambda\subset T_\ell$ can be chosen so that γ' is its image under composition of the maps (id,f) and h_ℓ in the bottom row of the commutative diagram (22). Otherwise, if γ' belongs to a transversal pizza slice $T'_{\ell'}$ of the minimal pizza associated with g, then $q=tord(\gamma',\lambda')$, where $\lambda'\subset T'$ is the supporting arc of $T'_{\ell'}$. Then $q'=tord(\lambda',T)\ge q$, and λ' is either a boundary arc of a coherent pizza slice of T' or belongs to a maximum pizza zone M'_j. In any case, there is an arc $\lambda\subset T$ which is either a boundary arc of a coherent pizza slice of T or belongs to a maximum zone M_i, where $j=\sigma(i)$, such that $tord(\lambda,\lambda')=tord(H(\lambda),H(\lambda'))=q'$. Thus $tord(\theta,\lambda)=\min(q,q')=q$. Since $H|_{T'}$ is a bi-Lipschitz homeomorphism, we have $tord(H(\gamma'),H(\lambda'))=q$, thus $tord(H(\gamma'),H(\lambda))=\min(tord(H(\gamma'),H(\lambda')),tord(H(\lambda),H(\lambda')))=\min(q,q')=q$.

If γ is any arc in T, then either $tord(\gamma,\lambda)\ge q$ and $tord(\gamma',\gamma)=tord(H(\gamma'),H(\gamma))=q$ or $tord(\gamma,\lambda)<q$ and $tord(\gamma',\gamma)=tord(H(\gamma'),H(\gamma))<q$ by the non-archimedean property of the tangency order. This implies that H preserves the tangency orders of any two arcs in $T\cup T'$, thus H is an outer bi-Lipschitz homeomorphism by Proposition 2.5.

This completes the proof of Theorem 5.3 □

6. Realization Theorem for Totally Transversal Pairs: Blocks

In this section we formulate the necessary and sufficient conditions for the existence of a totally transversal normal pair (T,T') of Hölder triangles with the given $\{\sigma\tau\}$-pizza invariant. Since there are no coherent pizza slices in a totally transversal pizza, the correspondence τ is trivial in that case. The permutation σ must be consistent with the metric data defined by the pizzas on T and on T'. In fact, we are going to show that consistency conditions can be defined for only one of the two pizzas, say Λ_T, so that the second pizza $\Lambda'_{T'}$ exists and is unique up to combinatorial equivalence.

Let (T,T'), where $T=T(\gamma_1,\gamma_2)$ and $T'=T(\gamma'_1,\gamma'_2)$, be a normal pair of β-Hölder triangles. Let $M=\{M_i\}_{i=1}^m$ and $M'=\{M'_j\}_{j=1}^m$ be the sets of maximum zones for the minimal pizzas on T and T' associated with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$, ordered according to the orientations of T and T'. Assume that $m>0$, thus at least one maximum zone exists. Let $\bar{q}_i=\bar{q}'_{\sigma(i)}=tord(M_i,M'_{\sigma(i)})$, where σ is a characteristic permutation of the pair T and T'. Let $\bar\beta_i=tord(M_i,M_{i+1})$ for $i=1,\ldots,m-1$. If $M_1\neq\{\gamma_1\}$, let $\bar\beta_0=tord(\gamma_1,M_1)$, otherwise let $\bar\beta_0=\beta$. If $M_m\neq\{\gamma_2\}$, let $\bar\beta_m=tord(M_m,\gamma_2)$, otherwise $\bar\beta_m=\beta$.

The following statement follows from the definition of the maximum zone for a minimal pizza (see Definition 3.2).

Lemma 6.1. *For each $i=1,\ldots,m$ the following inequality hold:*

$$\bar{q}_i>\max(\bar\beta_{i-1},\bar\beta_i). \tag{23}$$

Proof. If $M_i=D_\ell$ is an interior maximum zone of a minimal pizza on T associated with f, then M_i is a q_ℓ-order zone for f, thus $\bar q_i=q_\ell$, and either $q_\ell>q_{\ell+1}\geq tord(D_\ell,D_{\ell+1})\geq\bar\beta_i$ or $q_\ell=q_{\ell+1}>tord(D_\ell,D_{\ell+1})\geq\bar\beta_i$. In both cases $\bar q_i>\bar\beta_i$. Similarly, $\bar q_i>\bar\beta_{i-1}$. If $M_1=\{\gamma_1\}$, then $\bar q_1>\beta$. If $M_m=\{\gamma_2\}$, then $\bar q_m>\beta$. □

Definition 6.2. Let (T,T'), where $T=T(\gamma_1,\gamma_2)$ and $T'=T(\gamma'_1,\gamma'_2)$, be a totally transversal normal pair of Hölder triangles, and let Λ and Λ' be minimal pizzas on T and T' associated with the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$, respectively. Let $\check\lambda_0,\ldots,\check\lambda_{n-1}$ be a set of n arcs in T, ordered according to the orientation of T, such that $\check\lambda_0=\gamma_1$, $\check\lambda_{n-1}=\gamma_2$, and each maximum zone M_i of Λ contains exactly one of these arcs. Here $n=m$ if $M_1=\{\gamma_1\}$ and $M_m=\{\gamma_2\}$, $n=m+2$ if $M_1\neq\{\gamma_1\}$ and $M_m\neq\{\gamma_2\}$, $n=m+1$ otherwise. Note that, if $\check\lambda_j\in M_i$, then $i=j$ when $M_1\neq\{\gamma_1\}$, otherwise $i=j+1$. Let $\check q_j=ord_{\check\lambda_j}f$, in particular, $\check q_j=\bar q_i$ when $\check\lambda_j\in M_i$. Let π be a permutation of the set $[n]=\{0,\ldots,n-1\}$ such that $\pi(0)=0$, $\pi(n-1)=n-1$, and π is compatible with the permutation σ of the maximum zones: $\pi(j)=\sigma(j)$ when $\check\lambda_j\in M_j$, $\pi(j)+1=\sigma(j+1)$ when $\check\lambda_j\in M_{j+1}$. Let $\check\lambda'_0=\gamma'_1$, $\check\lambda'_{n-1}=\gamma'_2$. For $i=1,\ldots,m$ and $\check\lambda_j\in M_i$, let $\check\lambda'_{\pi(j)}\in M'_{\sigma(i)}$ be any arc such that $\check q_j=tord(\check\lambda_j,\check\lambda'_{\pi(j)})=\bar q_i$. This defines a set of n arcs $\check\lambda'_0,\ldots,\check\lambda'_{n-1}$ in T', ordered according to the orientation of T'.

Definition 6.3. Let π be a permutation of a set $[n]=\{0,\ldots,n-1\}$. A segment (a non-empty set of consecutive indices) $\{j,\ldots,k\}\subseteq[n]$ is called a *block* of π (see [1]) if the set $\{\pi(j),\ldots,\pi(k)\}$ is also a set of consecutive indices (not necessarily in increasing order). A block is *trivial* if it contains a single element, or if it is equal $[n]$.

Lemma 6.4. The following properties of blocks of a permutation π of $[n]$ hold:

- **(i)** If a segment B of $[n]$ is a block of π then the set $\pi(B)$ is a block of π^{-1}, thus $B\mapsto\pi(B)$ defines one-to-one correspondence between the blocks of π and π^{-1}.
- **(ii)** If two blocks B and B' of π have non-empty intersection, then $B\cap B'$ and $B\cup B'$ are also blocks of π.
- **(iii)** Each non-empty subset J of $[n]$ is contained in a unique *minimal block* $B_\pi(J)$.

(iv) Let $S(J)$ be the minimal segment of $[n]$ containing a non-empty subset J of $[n]$. Then $|S(J)| = |J|$ if, and only if, J is a segment.

(v) Let us define $\mathfrak{S}(J) = \pi^{-1}(S(\pi(S(J)))$, set $\mathfrak{S}^1(J) = \mathfrak{S}(J)$ and $\mathfrak{S}^k(J) = \mathfrak{S}(\mathfrak{S}^{k-1}(J))$ for $k > 1$. Then $\mathfrak{S}^k(J) \subseteq \mathfrak{S}^{k+1}(J)$, $\mathfrak{S}^k(J) \subseteq B_\pi(J)$ and $\mathfrak{S}^k(J) = \mathfrak{S}^{k+1}(J)$ if and only if $\mathfrak{S}^k(J) = B_\pi(J)$.

Proof. (i) From the definition of a block, $\pi(B)$ is a segment of $[n]$ and $\pi^{-1}(\pi(B)) = B$.
(ii) Since $\pi(B)$ and $\pi(B')$ are segments of $[n]$, $\pi(B \cup B')$ is not a segment only when $\pi(B) \cap \pi(B') = \pi(B \cap B')$ is empty, thus $B \cap B'$ is empty, a contradiction.
(iii) Since $[n]$ is a block of $[n]$, the set J is contained in a minimal block of $[n]$, which is unique by (ii).
(iv) We have $J \subseteq S(J)$, and equality holds only when J is a segment.
(v) Since $Y \subseteq S(I) \subseteq B_\pi(I)$ and $\pi(S(I)) \subseteq S(\pi(S(I)) \subseteq \pi(B_\pi(I))$ for any subset Y of $[n]$, we have $Y \subset \mathfrak{S}(I) \subseteq B_\pi(I)$, thus $|\mathfrak{S}(I)| \geq |I|$, and equality holds only if $I = S(I)$ is a segment of $[n]$ and $\pi(S(I)) = \pi(I)$ is a segment of $[n]$, in which case $I = B_\pi(I)$ is a block of π. $\square$

Definition 6.5. For $i \neq j$, $0 \leq i, j \leq n-1$, let $B_{ij} = B_\pi(\{i,j\})$ be the minimal block of π containing $\{i,j\}$ (see Lemma 6.4) and let $B'_{ij} = B_{\pi^{-1}}(\{i,j\})$ be the minimal block of π^{-1} containing $\{i,j\}$.

Theorem 6.6. *Let (T, T') be a normal pair of Hölder triangles. Let $\{\check\lambda_0, \ldots, \check\lambda_{n-1}\}$ and $\{\check\lambda'_0, \ldots, \check\lambda'_{n-1}\}$ be set of arcs in T and T', respectively, and let $\pi : [n] \to [n]$ be a permutation, as in Definition 6.2. Then*

$$tord(\check\lambda_i, \check\lambda_j) \leq tord(\check\lambda_k, \check\lambda_l) \text{ for } \{k,l\} \subset B_{ij}, \tag{24}$$

$$tord(\check\lambda'_i, \check\lambda'_j) \leq tord(\check\lambda'_k, \check\lambda'_l) \text{ for } \{k,l\} \subset B'_{ij}. \tag{25}$$

Proof. For each subset $I \subset N = 2^{[n]}$, where $N = 2^{[n]} \setminus \emptyset$ is the set of all non-empty subsets of $[n]$, we define $T_I = T(\check\lambda_j, \check\lambda_k) \subset T$, where j and k are the minimal and maximal values of $i \in I$. Let $T'_I = T(\check\lambda'_j, \check\lambda'_k) \subset T'$, where j and k are the minimal and maximal values of $i \in I$. For $I = \{i,j\}$, Lemma 6.1 implies that exponents of the Hölder triangles $T'_{\pi(S(I))}$ and $T_{\mathfrak{S}^1(I)}$ are equal to the exponent β of T_I. It follows that $T_{B_\pi(I)}$ is also a β-Hölder triangle. $\square$

Corollary 6.7. *If $B_{ij} = B_{kl}$ then $tord(\check\lambda_i, \check\lambda_j) = tord(\check\lambda_k, \check\lambda_l)$.*

Remark 6.8. Since $\pi(0) = 0$ and $\pi(n-1) = n-1$, the segments $\{0, \ldots, n-2\}$ and $\{1, \ldots, n-1\}$ are blocks of π. This implies that, for $i \neq j$ and $\{i,j\} \neq \{0, n-1\}$, each block B_{ij} is non-trivial. We have $B_{ij} = \{i,j\}$ if, and only if, $j = i \pm 1$ and $\pi(j) = \pi(i) \pm 1$.

Example 6.9. The permutation $\pi = (0,3,1,4,2,5)$ for normal pair of Hölder triangles in Figure 1 has three non-trivial blocks: $B_{01} = \{0, \ldots, 4\}$, $B_{45} = \{1, \ldots, 5\}$, $B_{12} = B_{23} = B_{34} = \{1, \ldots, 4\}$. Accordingly, $tord(\check\lambda_0, \check\lambda_1) = tord(\check\lambda'_0, \check\lambda'_1) \leq tord(\check\lambda_1, \check\lambda_2) = tord(\check\lambda'_1, \check\lambda'_2) = tord(\check\lambda_2, \check\lambda_3) = tord(\check\lambda'_2, \check\lambda'_3) = tord(\check\lambda_3, \check\lambda_4) = tord(\check\lambda'_3, \check\lambda'_4) \geq tord(\check\lambda_4, \check\lambda_5) = tord(\check\lambda'_4, \check\lambda'_5)$ in Theorem 6.6.

Definition 6.10. Let $T = T(\gamma_1, \gamma_2)$ be a normally embedded Hölder triangle oriented from γ_1 to γ_2, and let $f(x)$ be a totally transversal function on T (see Definition 4.1). Let

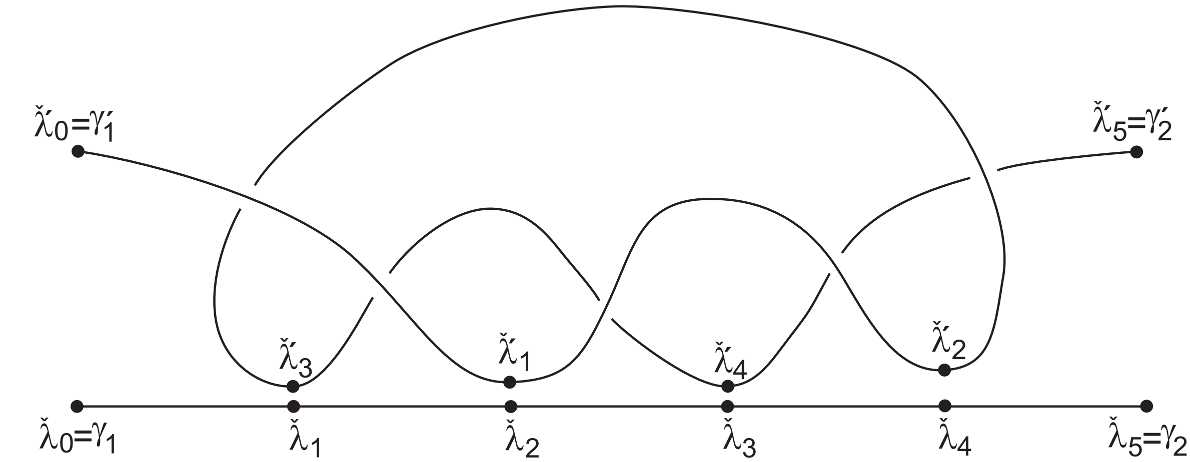

FIGURE 1. Normally embedded triangles in Example 6.9.

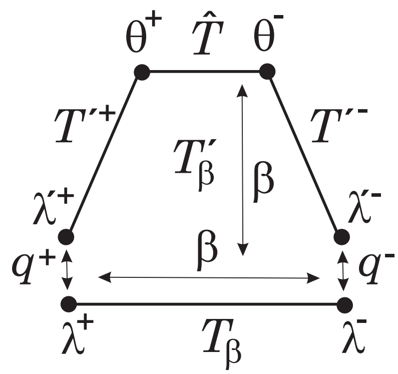

FIGURE 2. Model pair of triangles in Definition 6.12.

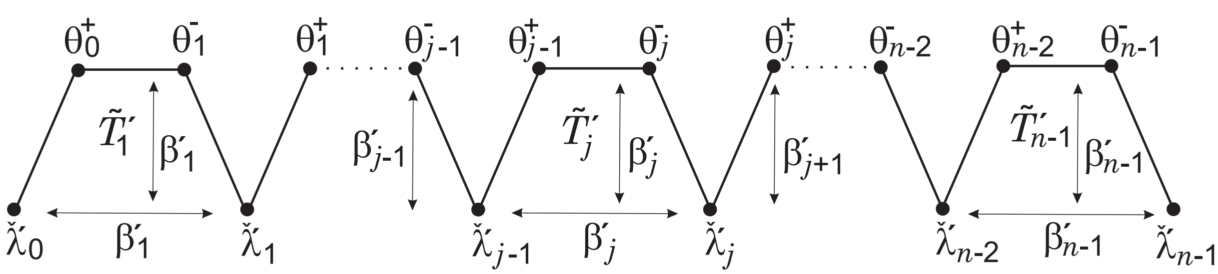

FIGURE 3. Arcs $\check\lambda'_j$ and $\theta^\pm_j$ in the proof of Theorem 6.14.

Λ be a minimal pizza on T associated with f. A family $\{\check\lambda_j\}_{j=0}^{n-1}$ of n arcs in T ordered according to the orientation of T, such that $\check\lambda_0 = \gamma_1$ and $\check\lambda_{n-1} = \gamma_2$, is called a *supporting family* associated with f if one of the following conditions is satisfied.

(A) Both arcs $\check\lambda_0 = \gamma_1$ and $\check\lambda_{n-1} = \gamma_2$ are maximum zones, and each arc $\check\lambda_j$ belongs to a maximum zone M_{j+1} of Λ.

(B) The arc $\check\lambda_0 = \gamma_1$ is a maximum zone, the arc $\check\lambda_{n-1} = \gamma_2$ is a minimum zone, and an arc $\check\lambda_j$ belongs to a maximum zone M_{j+1} of Λ when $j < n-1$.

(C) The arc $\check\lambda_0 = \gamma_1$ is a minimum zone, the arc $\check\lambda_{n-1} = \gamma_2$ is a maximum zone, and an arc $\check\lambda_j$ belongs to a maximum zone M_j of Λ when $j > 0$.

(D) Both arcs $\check\lambda_0 = \gamma_1$ and $\check\lambda_{n-1} = \gamma_2$ are minimum zones, and an arc $\check\lambda_j$ belongs to a maximum zone M_j of Λ when $0 < j < n-1$.

Definition 6.11. A permutation π of $[n] = \{0, \dots, n-1\}$ is called *admissible* with respect to a Lipschitz function f if the supporting family $\{\check\lambda_j\}_{j=0}^{n-1}$ satisfies inequalities (24).

Definition 6.12. Let q^+, q^-, β be three exponents, such that $\beta \le \min(q^+, q^-)$. Let $T_\beta \subset \mathbb{R}^2_{u,v}$, be the standard β-Hölder triangle (see Definition 2.6) bounded by the arcs $\lambda^+ = \{u \ge 0,\, v \equiv 0\}$ and $\lambda^- = \{u \ge 0,\, v = u^\beta\}$. Let $\lambda'^+ = \{(u,v) \in \lambda^+,\, z = u^{q^+}\}$ and $\lambda'^- = \{(u,v) \in \lambda^-,\, z = u^{q^-}\}$ be two arcs in the space $\mathbb{R}^3_{u,v,z}$.
Let $\theta^+ = \{(u,v,z) \in \lambda'^+,\, w = u^\beta\}$ and $\theta^- = \{(u,v,z) \in \lambda'^-,\, w = u^\beta\}$ be two arcs in the space $\mathbb{R}^4_{u,v,z,w}$.

Let $T'^+ = T(\lambda'^+, \theta^+)$ be the β-Hölder triangle in $\mathbb{R}^4_{u,v,z,w}$ obtained as the union of straight line segments with endpoints $\lambda'^+ \cap \{u = u_0\}$ and $\theta^+ \cap \{u = u_0\}$, parallel to the w-axis, over small non-negative u_0. Similarly, let $T'^- = T(\lambda'^-, \theta^-)$ be the β-Hölder triangle in $\mathbb{R}^4_{u,v,z,w}$ obtained as the union of straight line segments with endpoints $\lambda'^- \cap \{u = u_0\}$ and $\theta^- \cap \{u = u_0\}$, parallel to the w-axis. Let $\hat T' = T(\theta^+, \theta^-)$ be the β-Hölder triangle in $\mathbb{R}^4_{u,v,z,w}$ obtained as the union of straight line segments with endpoints $\theta^+ \cap \{u = u_0\}$ and $\theta^- \cap \{u = u_0\}$. Note that T'^+ and $\hat T'$ have a common boundary arc θ^+, while $\hat T'$ and T'^- have a common boundary arc θ^-. Thus $T'_\beta = T'^+ \cup \hat T' \cup T'_-$ (see Fig. 2) is a β-Hölder triangle. The pair (T_β, T'_β) of β-Hölder triangles is called a (q^+, q^-, β)-*model*.

Remark 6.13. The (q^+, q^-, β)-model (T_β, T'_β) is a totally transversal normal pair of β-Hölder triangles, with the given tangency orders q^+ and q^- of their boundary arcs, where $\beta \le \min(q^+, q^-)$. Let $f(x) = dist(x, T')$ be the distance function on T. If $\beta = q^+ = q^-$, then T_β is a single pizza slice of a minimal pizza associated with f, and both boundary arcs of T_β are minimum zones for f. If $\max(q^+, q^-) > \beta = \min(q^+, q^-)$, then T_β is a single pizza slice of a minimal pizza associated with f, one of its boundary arcs is a maximum zone for f, another one is a minimum zone. If $\beta < \min(q^+, q^-)$, then T_β has two pizza slices of a minimal pizza associated with f, both boundary arcs of T_β are maximum zones for f, and the set $G(T_\beta)$ of its generic arcs is a minimum zone.

The following theorem implies that the necessary conditions (23) and (24) on the exponents $\bar q_i$ and $tord(\check\lambda_i, \check\lambda_j)$ are sufficient for the existence and uniqueness, up to outer Lipschitz equivalence, of a totally transversal normal pair (T, T') of Hölder triangles with the given characteristic permutation σ of a minimal pizza associated with the totally transversal distance function f on T. Given a normally embedded Hölder triangle T and a supporting family $\{\check\lambda_i\}_{i=0}^{n-1}$ of arcs in T for the minimal pizza Λ on T associated with f, a Hölder triangle T' is constructed as follows. Adding an extra variable z, we define a family of arcs $\{\check\lambda'_j\}_{j=0}^{n-1}$ that will become a supporting family for a minimal pizza Λ' on T' associated with the distance function $g(x') = dist(x', T)$. Each arc $\check\lambda'_j$ is the graph of a function $z = u^{\bar q_i}$ over the arc $\check\lambda_i$, where $j = \pi(i)$. Next, any two consecutive arcs $\check\lambda'_{j-1}$ and $\check\lambda'_j$ are "connected" by a Hölder triangle T'_j based on the model Hölder triangle $T'_{\beta'_j}$ (see Definition 6.12 and Figure 2 above) where $\beta'_j = tord(\check\lambda'_{j-1}, \check\lambda'_j)$. The Hölder triangle T' is defined as the union of Hölder triangles T'_j. To show that T' is normally embedded, we use conditions (23) and (24) to prove that triangles T'_j are normally embedded and pairwise transversal, thus T' is combinatorially normally embedded (see Definition 2.9 and Proposition 2.10).

Theorem 6.14. *Let $T = T(\gamma_1, \gamma_2)$ be a normally embedded Hölder triangle oriented from γ_1 to γ_2, and let $\{\check\lambda_j\}_{j=0}^{n-1}$ be a supporting family of arcs in T associated with a non-negative totally transversal Lipschitz function f on T (see Definition 6.10). Then, for any admissible with respect to f permutation π of $[n] = \{0, \ldots, n-1\}$, there exists a unique, up to outer Lipschitz equivalence, totally transversal normal pair (T, T') of Hölder triangles, such that the function $dist(x, T')$ on T is contact equivalent to f and the permutation π is associated with the characteristic permutation σ of the pair (T, T'), as in Definition 6.2.*

Proof. We may assume that $T = T_\beta$ is a standard β-Hölder triangle (5) in $\mathbb{R}^2_{uv}$, the arcs $\check\lambda_i$ are germs at the origin of the graphs $\{u \geq 0,\ v = \check\lambda_i(u)\}$ of Lipschitz functions $\check\lambda_i(u) \geq 0$, and each $T_i = T(\check\lambda_{i-1}, \check\lambda_i)$ is a β_i-Hölder triangle. From the non-archimedean property of the tangency order, we have $tord(\check\lambda_i, \check\lambda_l) = \min\big(tord(\check\lambda_i, \check\lambda_k), tord(\check\lambda_k, \check\lambda_l)\big)$ for $i < k < l$. It will be convenient in this proof to parameterize all arcs by u instead of the distance to the origin.

To define a totally transversal pair (T, T'), we are going to construct a normally embedded Hölder triangle T' in several steps. In Step 1, we define a set of n arcs $\check\lambda'_j$ that will be a supporting family for T'. In Step 2, the triangle T' is constructed as the union of $n-1$ subtriangles $T'_j = T(\check\lambda'_{j-1}, \check\lambda'_j)$ based on the models from Definition 6.12. In Step 3, we prove that T' is normally embedded. In step 4, we prove that the distance function $f(x) = dist(x, T')$ on T is totally transversal and satisfies conditions of Theorem 6.14, thus (T, T') is a totally transversal normal pair.

Step 1. For $j = 0, \ldots, n-1$, let $\check\lambda'_j \subset \mathbb{R}^3_{u,v,z}$ be the arc $\{(u, v) \in \check\lambda_i,\ z = u^{\check q_i}\}$, where $j = \pi(i)$ and $\check q_i = ord_{\check\lambda_i} f$. Let $\check q'_j = \check q_i = tord(\check\lambda'_j, T)$. The set of arcs $\{\check\lambda'_j\}_{j=0}^{n-1}$ will be a supporting family for the Hölder triangle T' associated with the distance function $g(x') = dist(x', T)$ on T'. In particular, $\gamma'_1 = \check\lambda'_0$ and $\gamma'_2 = \check\lambda'_{n-1}$ will be the boundary arcs of T'. Let $\beta'_j = tord(\check\lambda'_{j-1}, \check\lambda'_j)$ for $j = 1, \ldots, n-1$. From the non-archimedean property of the tangency order we have, for $j = \pi(i_1)$ and $k = \pi(i_2)$, isometry

$$tord(\check\lambda'_j, \check\lambda'_k) = tord(\check\lambda_{i_1}, \check\lambda_{i_2}). \tag{26}$$

Since T is normally embedded and the arcs $\check\lambda_i$ satisfy the block inequalities (24), the arcs $\check\lambda'_j$ satisfy combinatorial normal embedding inequalities (see Definition 2.9) necessary for the Hölder triangle T' with a supporting family $\{\check\lambda'_j\}_{j=0}^{n-1}$ to be normally embedded:

$$tord(\check\lambda'_j, \check\lambda'_l) = \min\big(tord(\check\lambda'_j, \check\lambda'_k), tord(\check\lambda'_k, \check\lambda'_l)\big) \text{ for } j < k < l. \tag{27}$$

This follows from (26) when $j = \pi(i_1)$, $k = \pi(i_2)$ and $l = \pi(i_3)$, where $i_2 \in [i_1, i_3]$, as $tord(\check\lambda_{i_1}, \check\lambda_{i_3}) = \min\big(tord(\check\lambda_{i_1}, \check\lambda_{i_2}), tord(\check\lambda_{i_2}, \check\lambda_{i_3})\big)$ in that case. If, e.g., $i_1 < i_3 < i_2$ then $i_2 \in B_\pi(\{i_1, i_3\})$, thus $tord(\check\lambda_{i_1}, \check\lambda_{i_2}) = tord(\check\lambda_{i_1}, \check\lambda_{i_3}) \leq tord(\check\lambda_{i_2}, \check\lambda_{i_3})$ due to (24). The other cases are similar.

Since the arcs $\check\lambda_i$ satisfy inequalities (24), the arcs $\check\lambda'_j$ satisfy the block inequalities (25).

Step 2. Consider the product $\mathbb{R}^{n+2}_{u,v,z,\mathbf{w}}$ of $\mathbb{R}^3_{u,v,z}$ and $\mathbb{R}^{n-1}_{\mathbf{w}}$,where $\mathbf{w} = (w_1, \ldots, w_{n-1})$. We define $2n-2$ arcs $\theta^\pm_j \subset \mathbb{R}^{n+2}_{u,v,z,\mathbf{w}}$ as follows:
For $0 < j \leq n-1$, let $\theta^-_j = \{(u, v, z) \in \check\lambda'_j,\ w_j = u^{\beta'_j}\} \subset \mathbb{R}^4_{u,v,z,w_j}$.
For $0 \leq j < n-1$, let $\theta^+_j = \{(u, v, z) \in \check\lambda'_j,\ w_{j+1} = u^{\beta'_{j+1}}\} \subset \mathbb{R}^4_{u,v,z,w_{j+1}}$.

From the inequalities (23) and (24), we get $\beta'_j \leq \min(\check q'_{j-1}, \check q'_j)$ for each $j = 1, \ldots, n-1$.

For $1 \le j \le n-1$, consider a $(\check{q}'_{j-1}, \check{q}'_j, \beta'_j)$-model $(T_{\beta'_j}, T'_{\beta'_j})$ (see Definition 6.12), where $T_{\beta'_j} = T(\lambda^+, \lambda^-)$ is the standard β'_j-Hölder triangle and $T'_{\beta'_j} = T'^+ \cup \hat{T}' \cup T'^-$ is the union of three β'_j-Hölder triangles $T'^+ = T(\lambda'^+, \theta^+)$, $\hat{T}' = T(\theta^+, \theta^-)$ and $T'^- = T(\lambda'^-, \theta^-)$.

Let $h_j : T'_{\beta'_j} \to \mathbb{R}^{n+2}_{u,v,z,\mathbf{w}}$ be a map preserving the variable u, such that $h_j(\lambda'^+) = \lambda'_{j-1}$, $h_j(\lambda'^-) = \lambda'_j$, $h_j(\theta^+) = \theta^+_{j-1}$, $h_j(\theta^-) = \theta^-_j$ and, for any small positive u_0, the map h_j is linear on each of the three straight line segments of $T'_{\beta'_j} \cap \{u = u_0\}$. This defines h_j completely. We'll show in Step 3 that h_j defines an outer bi-Lipschitz map $T'_{\beta'_j} \to \tilde{T}'_j$, where $\tilde{T}'_j = T'^+_{j-1} \cup \hat{T}'_j \cup T'^-_j$, $T'^+_{j-1} = h_j(T'^+)$, $\hat{T}'_j = h_j(\hat{T}')$, $T'^-_j = h_j(T'^-)$ (see Figure 3).

The arcs $\theta^\pm_j$ will belong to minimum zones of a minimal pizza on T' associated with g. These arcs will be located on $T' = T(\gamma'_1, \gamma'_2)$ in the following order (see Figure 3):

$$\check{\lambda}'_0, \theta^+_0, \theta^-_1, \check{\lambda}'_1, \theta^+_1, \ldots, \theta^-_{j-1}, \check{\lambda}'_{j-1}, \theta^+_{j-1}, \theta^-_j, \check{\lambda}'_j, \theta^+_j, \ldots, \theta^-_{n-2}, \check{\lambda}'_{n-2}, \theta^+_{n-2}, \theta^-_{n-1}, \check{\lambda}'_{n-1}. \tag{28}$$

Note that each variable w_j has the order β'_j on the arcs θ^-_{j-1} and θ^+_j, and is identically zero on all arcs θ^-_k for $k \ne j-1$, on all arcs θ^+_k for $k \ne j$, and on all arcs $\check{\lambda}'_k$.

Step 3. Let us show first that each Hölder triangle $\tilde{T}'_j$ is normally embedded. It is the union of three normally embedded β'_j-Hölder triangles $T(\check{\lambda}'_{j-1}, \theta^+_{j-1})$, $T(\theta^-_j, \check{\lambda}'_j)$ and $T(\theta^+_{j-1}, \theta^-_j)$, each of them being a family of straight line segments. The first two of them are families of straight line segments of length $u^{\beta'_j}$ parallel to the axis w_j, the third one connects them by a family of the straight line segments in $\{w_j = u^{\beta'_j},\ w_k \equiv 0 \text{ for } k \ne j\}$. Since $tord(\check{\lambda}'_{j-1}, \check{\lambda}'_j) = \beta'_j$, the tangency order between $T(\check{\lambda}'_{j-1}, \theta^+_{j-1})$ and $T(\theta^-_j, \check{\lambda}'_j)$ is also β'_j. This implies that $\tilde{T}'_j$ is normally embedded.

To show that T' is combinatorially normally embedded (see Definition 2.9) we need to prove that any two Hölder triangles $\tilde{T}'_j$ and $\tilde{T}'_k$ are transversal. Let $\eta \subset \tilde{T}'_j$ and $\eta' \subset \tilde{T}'_k$ be any two arcs, and let $\mathrm{proj}_T\eta$ and $\mathrm{proj}_T\eta'$ be their projections to T. Note that the variable w_j, which is non-zero on $\tilde{T}'_j$, vanish on $\tilde{T}'_k$, and the variable w_k, which is non-zero on $\tilde{T}'_k$, vanish on $\tilde{T}'_j$. Thus $tord(\eta, \eta') \le tord(\eta, \mathrm{proj}_T\eta')$ and $tord(\eta, \eta') \le tord(\mathrm{proj}_T\eta, \eta')$

If $k = j+1$, then $\tilde{T}'_j$ and $\tilde{T}'_{j+1}$ are consecutive Hölder triangles in T', and $\tilde{T}'_j \cap \tilde{T}'_{j+1} = \check{\lambda}'_j$.

When $tord(\eta, \check{\lambda}'_{j-1}) = tord(\eta, \check{\lambda}'_j) = \beta'_j$, then $tord(\eta, T) = ord_\eta w_j = \beta'_j$ and $tord(\eta, \eta') \le tord(\eta, \mathrm{proj}_T\eta') \le \beta'_j = tord(\eta, \check{\lambda}'_j)$. Similarly, when $tord(\eta', \check{\lambda}'_j) = tord(\eta', \check{\lambda}'_{j+1}) = \beta'_{j+1}$, then $tord(\eta, \eta') \le \beta'_{j+1} = tord(\eta', \check{\lambda}'_{j+1})$.

If $tord(\eta, \check{\lambda}'_j) > \beta'_j$, then $\eta \subset T(\check{\lambda}'_j, \theta^-_j)$ and $tord(\eta, \check{\lambda}'_j) = ord_\eta w_j \le tord(\eta, \eta')$, as $w_j \equiv 0$ on $\tilde{T}'_{j+1}$. Similarly, if $tord(\eta', \check{\lambda}'_j) > \beta'_{j+1}$, then $\eta' \subset T(\check{\lambda}'_j, \theta^+_j)$ and $tord(\eta', \check{\lambda}'_j) = ord_{\eta'} w_{j+1} \le tord(\eta, \eta')$.

If $tord(\eta, \check{\lambda}'_{j-1}) > \beta'_j$ and $tord(\eta', \check{\lambda}'_{j+1}) > \beta'_{j+1}$, then $tord(\eta, \lambda'_j) = \beta'_j \le tord(\eta, \eta')$ by the non-archimedean property of the tangency order, as $tord(\lambda'_{j-1}, \lambda'_{j+1}) \ge \min(\beta'_j, \beta'_{j+1})$ by (27). Thus the Hölder triangles $\tilde{T}'_j$ and $\tilde{T}'_{j+1}$ are transversal, with $\tilde{\gamma} = \tilde{\gamma}' = \check{\lambda}'_j$ in Definition 2.8.

Let $j < k-1, j-1 = \sigma(i_-), j = \sigma(i_+), k-1 = \sigma(l_-), k = \sigma(l_+)$.

If $tord(\check{\lambda}'_j, \check{\lambda}'_{k-1}) < \max(\beta'_j, \beta'_k)$, then $\tilde{T}'_j$ and $\tilde{T}'_k$ are transversal by the non-archimedean property of the tangency order, thus we may assume that $tord(\check{\lambda}'_j, \check{\lambda}'_{k-1}) \ge \max(\beta'_j, \beta'_k)$.

The same arguments as in the case $k = j+1$ above show that $tord(\eta, \eta') \le \beta'_j = tord(\eta, \check{\lambda}'_j) = tord(\eta, \check{\lambda}'_{k-1})$ when $tord(\eta, \check{\lambda}'_{j-1}) = tord(\eta, \check{\lambda}'_j) = \beta'_j$ and $tord(\eta, \eta') \le \beta'_k = tord(\eta', \check{\lambda}'_{k-1}) = tord(\eta', \check{\lambda}'_j)$ when $tord(\eta', \check{\lambda}'_{k-1}) = tord(\eta', \check{\lambda}'_k) = \beta'_k$.

If $\kappa = tord(\eta, \check\lambda'_j) > \beta'_j$, then $\eta \subset T(\check\lambda'_j, \theta^-_j)$ and $\kappa = ord_\eta w_j$. If $\kappa \le tord(\check\lambda'_j, \check\lambda'_{k-1})$, then $tord(\eta, \check\lambda_{k-1}) = \kappa$, and the same arguments as in the case $k = j+1$ show that $tord(\eta, \eta') \le tord(\eta, \lambda'_{k-1})$, as $w_j \equiv 0$ on $\tilde T'_k$.

Otherwise, if $\kappa > \alpha$, then $tord(\eta, \check\lambda_{k-1}) = \alpha = tord(\eta, \eta')$ when $tord(\eta', \check\lambda'_{k-1}) \ge \alpha$ and $tord(\eta, \check\lambda_{k-1}) < \alpha = tord(\eta, \eta')$ when $tord(\eta', \check\lambda'_{k-1}) < \alpha$. Similarly, if $tord(\eta', \check\lambda'_{k-1}) > \beta'_k$, then $\eta' \subset T(\check\lambda'_{k-1}, \theta^+_k)$ and $tord(\eta', \check\lambda'_j) \le tord(\eta, \eta')$.

If $tord(\eta, \check\lambda'_{j-1}) > \beta'_j$, then $tord(\eta, \eta') \le tord(\eta, \check\lambda'_{k-1})$ by the non-archimedean property of the tangency order, by the same argument as in the case $k = j+1$. Similarly, if $tord(\eta', \check\lambda'_k) > \beta'_k$, then $tord(\eta, \eta') \le tord(\eta', \check\lambda'_j)$ by the non-archimedean property of the tangency order. Thus the Hölder triangles $\tilde T'_j$ and $\tilde T'_k$ are transversal, with $\tilde\gamma = \check\lambda'_j$ and $\tilde\gamma' = \check\lambda'_{k-1}$ in Definition 2.8.

By Proposition 2.10, the Hölder triangle T' is normally embedded.

Step 4. We show first that the distance function $g(x') = dist(x', T)$ on T' is totally transversal. This would imply that $f(x) = dist(x, T')$ is a totally transversal function on T', thus its contact equivalence class is completely determined by the exponents β_j of T_j and the orders $\check q_i$ of $f|_{\check\lambda_i}$.

Let η be any arc in T'_j. If $tord(\eta, \check\lambda'_{j-1}) > \beta'_j$ then $\eta \subset T(\check\lambda'_{j-1}, \theta^+_{j-1})$ and $tord(\eta, T) = ord_\eta w_j$. If $tord(\eta, \check\lambda'_j) > \beta'_j$, then $\eta \subset T(\check\lambda'_j, \theta^-_j)$ and $tord(\eta, T) = ord_\eta w_j$. Otherwise, if $\eta \in G(T'_j)$, then $tord(\eta, T) = \beta'_j$. This implies that the distance function $g|_{T'_j}$ is totally transversal. Since $T' = \bigcup_j T'_j$ is normally embedded, the distance function $g(x) = dist(x, T)$ on T' is totally transversal. Proposition 4.5 implies that the distance function $f(x) = dist(x, T')$ on T is also totally transversal: a coherent slice T_ℓ for a minimal pizza on T associated with f would correspond to a coherent slice for a minimal pizza on T' associated with g, a contradiction. Lemma 4.3 implies that the function f is determined, up to contact Lipschitz equivalence, by the exponents β_j and $\check q_i$. The permutation π is associated with the characteristic permutation σ of the pair (T, T') by the construction of arcs λ'_j. Thus the distance function f satisfies conditions of Theorem 6.14. □

7. Realization Theorem for General Pairs

In this section we formulate the necessary and sufficient conditions for the existence of a general normal pair (T, T') of Hölder triangles with the given $\{\sigma\tau\}$-pizza invariant. For this purpose we define the notions of *pre-pizza* and *twin pre-pizza* associated with a non-negative Lipschitz function f on a normally embedded Hölder triangle T. Note that the permutation σ acts on maximum zones of the minimal pizzas Λ and Λ' associated with the distance functions on T and T', and the correspondence τ acts (although not one-to-one, see Remark 4.25) on the boundary arcs of coherent pizza slices of Λ and Λ'. Additionally, the boundary arcs γ_1 and γ_2 of T naturally correspond to the boundary arcs γ'_1 and γ'_2 of T'. All these arcs are called *essential* (see Definition 7.1), and all other arcs of Λ and Λ' are called *non-essential*. Each non-essential arc of Λ is a common boundary arc of two transversal pizza slices.

A pre-pizza $\tilde\Lambda$ is obtained from the minimal pizza Λ on T associated with f by deleting non-essential arcs and replacing two transversal pizza slices adjacent to each non-essential arc by a single Hölder triangle of $\tilde\Lambda$. The pizza Λ can be recovered from a pre-pizza $\tilde\Lambda$ by restoring non-essential arcs (see Remark 7.5).

A twin pre-pizza $\check{\Lambda}$ is obtained from a pre-pizza $\tilde{\Lambda}$ by adding new arcs, called *twin arcs*. The twin arcs are defined so that the correspondence τ becomes a one-two-one correspondence on the set of pizza zones of $\check{\Lambda}$. The pre-pizza $\tilde{\Lambda}$ can be recovered from a twin pre-pizza $\check{\Lambda}$ by replacing each pair of twin arcs by a single arc.

We define a permutation ϖ of the set of indices of $\check{\Lambda}$, using the permutations σ of the maximum zones of Λ, the permutation υ of coherent pizza slices of Λ defined by the correspondence τ, and the sign function s on coherent pizza slices of Λ defined by τ. It is an analog of the permutation π defined for the totally transversal case in the previous section. Propositions 7.19 and 7.21 describe the combinatorial and metric properties of ϖ for a given normal pair (T, T') of Hölder triangles. These properties become *admissibility conditions* (see Definition 7.24) for the permutation ϖ, necessary and sufficient for the existence and uniqueness of a normal pair (T, T'), when only a pizza Λ on T, the permutations σ and υ, and the sign function s are given.

Given a minimal pizza Λ associated with a non-negative Lipschitz function f on a normally embedded Hölder triangle T and an admissible permutation ϖ, the construction of a normal pair (T, T') is similar to that in the previous section, although the admissibility conditions are more complicated. For a given pizza Λ, permutations σ and υ, and a sign function s, an admissible permutation ϖ, if exists, is unique, and the existence conditions for ϖ are explicitly formulated.

Definition 7.1. Let $T = T(\gamma_1, \gamma_2)$ be a normally embedded β-Hölder triangle oriented from γ_1 to γ_2. Let $\Lambda = \{T_\ell\}_{\ell=1}^p$, where $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$, be a minimal pizza on T associated with a non-negative Lipschitz function f on T, and let $\{D_\ell\}_{\ell=0}^p$ be the corresponding pizza zones in $V(T)$ (see Lemma 2.35) such that $\lambda_\ell \in D_\ell$. If T_ℓ is a coherent pizza slice, then the pizza zones $D_{\ell-1}$ and D_ℓ are called *primary pizza zones*, the boundary arcs $\lambda_{\ell-1}$ and λ_ℓ of T_ℓ are called *primary arcs*, and their indices are called *primary indices* of Λ.

The pair $(D_{\ell-1}, D_\ell)$ is called a *primary pair* of pizza zones, the pair $(\lambda_{\ell-1}, \lambda_\ell)$ is called a *primary pair* of arcs, and the pair $(\ell-1, \ell)$ is called a *primary pair* of indices of Λ.

A pizza zone D_ℓ, and an arc $\lambda_\ell \in D_\ell$, is called *essential* if D_ℓ is either a boundary arc of T, or a maximum zone of Λ, or a primary pizza zone. Otherwise, D_ℓ and λ_ℓ are called *non-essential.*

Lemma 7.2. *If D_ℓ is a non-essential pizza zone of a minimal pizza Λ associated with a non-negative Lipschitz function f on a normally embedded Hölder triangle T, then both $D_{\ell-1}$ and $D_{\ell+1}$ are essential pizza zones of Λ, and D_ℓ is a common pizza zone of two transversal pizza slices T_ℓ and $T_{\ell+1}$. Moreover, D_ℓ is a transversal minimum zone of Λ:*

$$\mu(D_\ell) = q_\ell = \beta_{\ell-1} = \beta_\ell < \min(q_{\ell-1}, q_{\ell+1}). \tag{29}$$

Proof. Since Λ is a minimal pizza on T associated with f, a common pizza zone D_ℓ of two transversal pizza slices of Λ is either a maximum or a minimum zone. As D_ℓ is non-essential, it is a minimum zone. This implies that each of the zones $D_{\ell-1}$ and $D_{\ell+1}$ is either a maximum zone or a boundary zone of a coherent pizza slice, thus both $D_{\ell-1}$ and $D_{\ell+1}$ are essential pizza zones of Λ. Since D_ℓ is adjacent to two transversal pizza slices, it is a transversal zone. $\square$

Definition 7.3. Let $\{D_\ell\}_{\ell=0}^p$ be the pizza zones of a minimal pizza $\Lambda = \{T_\ell\}_{\ell=1}^p$, where $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$, associated with a non-negative Lipschitz function f on a normally embedded oriented Hölder triangle T, and let $\{M_i\}_{i=1}^m$, be the maximum zones of Λ. Let

$\{\tilde{D}_j\}_{j=0}^{N-1}$ be essential pizza zones of Λ (see Definition 7.1) ordered according to orientation of T, and let $\tilde{\lambda}_j \in \tilde{D}_j$ be essential arcs of Λ. Let $\tilde{T}_j = T(\tilde{\lambda}_{j-1}, \tilde{\lambda}_j)$ for $0 < j < N$. Then $\tilde{T}_j$ is either a coherent pizza slice of Λ, or a transversal pizza slice T_ℓ of Λ such that both zones $D_{\ell-1}$ and D_ℓ are essential, or the union of two adjacent transversal pizza slices T_ℓ and $T_{\ell+1}$ of Λ such that D_ℓ is a non-essential pizza zone of Λ.
A *pre-pizza* $\tilde{\Lambda} = \{\tilde{T}_j\}_{j=1}^{N-1}$ on T associated with f is a decomposition of T into Hölder triangles $\tilde{T}_j$ with the following *toppings* $\{\tilde{q}_j, \tilde{\beta}_j, \tilde{Q}_j, \tilde{\mu}_j, \tilde{\nu}_j\}$:
1) If $\tilde{\lambda}_j = \lambda_\ell$, then $\tilde{q}_j = ord_{\tilde{\lambda}_j} f = q_\ell$ and $\tilde{\nu}_j = \nu_T(\tilde{\lambda}_j, f) = \nu_\ell$.
2) If $\tilde{T}_j = T_\ell$ is a coherent pizza slice of Λ, then $\tilde{\beta}_j = \beta_\ell$, $\tilde{Q}_j = [\tilde{q}_{j-1}, \tilde{q}_j] = Q_\ell$, and either $\tilde{\mu}_j(q) = \mu_\ell(q)$ is an affine function on $\tilde{Q}_j$, or $\tilde{\mu}_j(\tilde{q}_j) = \tilde{\beta}_j$ is a single exponent when $\tilde{Q}_j = \{\tilde{q}_j\}$ is a point.
3) If $\tilde{D}_j = D_\ell = M_i$ is a maximum zone of Λ, then $\tilde{q}_j = q_\ell = \bar{q}_i$, $\tilde{\beta}_j = \beta_\ell$ when $j > 0$ and $\tilde{\beta}_{j+1} = \beta_{\ell+1}$ when $j < N-1$. In that case $\tilde{D}_j$ is called a *maximum zone* of $\tilde{\Lambda}$.
4) If $\tilde{T}_j = T_\ell \cup T_{\ell+1}$ is the union of two pizza slices of Λ, then $\tilde{q}_{j-1} = q_{\ell-1}$, $\tilde{q}_j = q_{\ell+1}$ and $\tilde{\beta}_j = \beta_{\ell-1} = \beta_\ell < \min(\tilde{q}_{j-1}, \tilde{q}_j)$. In that case, $(j-1, j)$ is called a *gap pair* of indices of $\tilde{\Lambda}$.
A pair $(j-1, j)$ of consecutive indices of $\tilde{\Lambda}$ is called *primary* if $\tilde{T}_j$ is a coherent pizza slice and *secondary* otherwise. If Λ is a totally transversal pizza, then all pairs $(j-1, j)$ of indices of $\tilde{\Lambda}$ are secondary, and $\tilde{\Lambda}$ is called a *totally transversal* pre-pizza.

Lemma 7.4. *Let $\tilde{\Lambda} = \{\tilde{T}_j\}_{j=1}^{N-1}$ be a pre-pizza associated with a non-negative Lipschitz function f on a normally embedded Hölder triangle T corresponding to a minimal pizza Λ on T associated with f. We may assume that at least one maximum zone of Λ exists, since only in the trivial case $tord(T, T') \le \beta$ there are no maximum zones. A pizza zone $\tilde{D}_j$ of $\tilde{\Lambda}$ is a maximum zone if, and only if, the following conditions are satisfied:*

$$\text{If } j > 0, \text{ then either } (j-1, j) \text{ is a gap pair or } \tilde{q}_j \ge \tilde{q}_{j-1}. \tag{30}$$

$$\text{If } j < N-1, \text{ then either } (j, j+1) \text{ is a gap pair or } \tilde{q}_j \ge \tilde{q}_{j+1}. \tag{31}$$

Proof. If $j > 0$ and $\tilde{D}_j$ is a maximum zone of $\tilde{\Lambda}$, then $\tilde{T}_j$ is either a coherent pizza slice T_ℓ of Λ or a transversal pizza slice T_ℓ of Λ or the union of two transversal pizza slices $T_{\ell-1}$ and T_ℓ of Λ with a common minimum pizza zone D_ℓ. In the first case, $\tilde{q}_j = q_\ell \ge \tilde{q}_{j-1} = q_{\ell-1}$. In the second case, $\tilde{q}_j = q_\ell > q_{\ell-1} = \tilde{q}_{j-1}$. In the third case, $(j-1, j)$ is a gap pair of indices of $\tilde{\Lambda}$. Thus condition (30) is satisfied for $\tilde{D}_j$.
Similarly, if $j < N-1$ and $\tilde{D}_j$ is a maximum zone, then condition (31) is satisfied for $\tilde{D}_j$.
If $\tilde{D}_j$ is not a maximum zone of $\tilde{\Lambda}$, then either $(j-1, j)$ or $(j, j+1)$ is not a gap pair. If $(j, j+1)$ is a gap pair and $(j-1, j)$ is not a gap pair, then $\tilde{T}_j = T_\ell$ is a pizza slice of Λ and $\tilde{q}_{j-1} = q_{\ell-1} > q_\ell = \tilde{q}_j$, thus condition (30) is not satisfied.
Similarly, if $(j-1, j)$ is a gap pair and $(j, j+1)$ is not a gap pair, then $\tilde{T}_{j+1} = T_\ell$ is a pizza slice of Λ and $\tilde{q}_{j+1} = q_{\ell+1} > q_\ell = \tilde{q}_j$, thus condition (31) is not satisfied.
If both $(j-1, j)$ and $(j, j+1)$ are not gap pairs, then either $j = 0$ and $\tilde{T}_1 = T_1$ is a pizza slice of Λ such that $\tilde{q}_1 = q_1 > q_0 = \tilde{q}_0$ or $j = N-1$ and $\tilde{T}_{N-1} = T_p$ is a pizza slice of Λ such that $\tilde{q}_{N-2} = q_{p-1} > q_p = \tilde{q}_{N-1}$ or $0 < j < N-1$, $\tilde{T}_j = T_\ell$ and $\tilde{T}_{j+1} = T_{\ell+1}$ are pizza slices of Λ, and either $\tilde{q}_{j-1} = q_{\ell-1} > q_\ell = \tilde{q}_j$ or $\tilde{q}_{j+1} = q_{\ell+1} > q_\ell = \tilde{q}_j$, or both $\tilde{q}_{j-1} > \tilde{q}_j$ and $\tilde{q}_{j+1} > \tilde{q}_j$, thus at least one of conditions (30) and (31) is not satisfied. $\square$

Remark 7.5. A minimal pizza Λ on T associated with f can be recovered as a refinement of a pre-pizza $\tilde{\Lambda}$, adding a generic arc in each Hölder triangle $\tilde{T}_j$ such that $(j-1, j)$ is a gap pair. If f is totally transversal, then all pizza zones of Λ are either maximum or minimum zones, with all interior minimum zones being non-essential. Thus a totally transversal pre-pizza $\tilde\Lambda$ on T is a decomposition of T by the arcs λ_ℓ in interior maximum zones of Λ.

Definition 7.6. Let $\{\tilde\lambda_j\}_{j=0}^{N-1}$ be the arcs of a pre-pizza $\tilde\Lambda = \{\tilde T_j\}_{j=1}^{N-1}$ on T associated with a non-negative Lipschitz function f. A *twin pre-pizza* $\check\Lambda$ on T associated with f is a decomposition of T into Hölder triangles $\check T_k$ obtained by the following operations:
1. Each arc $\tilde\lambda_j$ of $\tilde\Lambda$ common to consecutive coherent pizza slices $\tilde T_j$ and $\tilde T_{j+1}$, such that $\tilde D_j$ is a transversal pizza zone of $\tilde\Lambda$ (i.e., $\tilde\nu_j = \mu(\tilde D_j) = \tilde q_j$, see Definition 4.1) and $\tilde D_j$ is not a maximum zone, is replaced by *twin arcs* $\tilde\lambda_j^- \in \tilde D_j$ and $\tilde\lambda_j^+ \in \tilde D_j$, such that $\tilde\lambda_j^- \prec \tilde\lambda_j^+$ and $tord(\tilde\lambda_j^-, \tilde\lambda_j^+) = \tilde q_j$.
2. If the boundary arc $\gamma_1 = \tilde\lambda_0$ of T is a transversal minimum zone (see Definition 4.1) and $\tilde T_1$ is a coherent pizza slice of $\tilde\Lambda$, then we add an interior arc $\tilde\lambda_0^+ \subset \tilde T_1$, such that $tord(\tilde\lambda_0^+, \tilde\lambda_0) = \tilde q_0$. Since $\mu_T(\tilde\lambda_0, f) = \tilde q_0$, we have $ord_{\tilde\lambda_0^+} f = \tilde q_0$.
3. If the boundary arc $\gamma_2 = \tilde\lambda_{N-1}$ of T is a transversal minimum zone and $\tilde T_{N-1}$ is a coherent pizza slice of $\tilde\Lambda$, then we add an interior arc $\tilde\lambda_{N-1}^- \subset \tilde T_{N-1}$, such that $tord(\tilde\lambda_{N-1}^-, \tilde\lambda_{N-1}) = \tilde q_{N-1}$. Since $\mu_T(\tilde\lambda_{N-1}, f) = \tilde q_{N-1}$, we have $ord_{\tilde\lambda_{N-1}^-} f = \tilde q_{N-1}$.

Applying these operations to $\tilde\Lambda$ and ordering all arcs according to orientation of T, we obtain a *twin pre-pizza* $\check\Lambda = \{\check T_k\}_{k=1}^{\mathcal N-1}$ on T associated with f, where $\check T_k = T(\check\lambda_{k-1}, \check\lambda_k)$ is a $\check\beta_k$-Hölder triangle, with the set $\{\check\lambda_k\}_{k=0}^{\mathcal N-1}$ of $\mathcal N \ge N$ arcs. If $\check\lambda_k \in \tilde D_j$, then we define $\check q_k = \tilde q_j$ and $\check\nu_k = \tilde\nu_j$. If $\check\lambda_1 = \tilde\lambda_0^+$, then $\check q_1 = \check\nu_1 = \tilde q_0$. If $\check\lambda_{\mathcal N-2} = \tilde\lambda_{N-1}^-$, then $\check q_{\mathcal N-2} = \check\nu_{\mathcal N-2} = \tilde q_{N-1}$. A pair of arcs $(\check\lambda_{k-1}, \check\lambda_k)$ such that $\check T_k \subset \tilde T_j$ is a coherent pizza slice for f, and the pair of indices $(k-1, k)$ of $\check\Lambda$, is called *primary.* In that case, $\check\beta_k = \tilde\beta_j$, $\check q_{k-1} = \tilde q_{j-1}$, $\check q_k = \tilde q_j$, $\check Q_k = \tilde Q_j$, and $\check\mu_k(q) = \tilde\mu_j(q)$ is an affine function on $\check Q_k$ or $\check\mu_k(\check q_k) = \check\beta_k$ when $\check Q_k = \{\check q_k\}$ is a point. If a pair $(k-1, k)$ of consecutive indices of $\check\Lambda$ is not primary, it is called a *secondary* pair of indices.
If $\check\lambda_k \subset \tilde D_j$, where $\tilde D_j$ is a maximum zone of $\tilde\Lambda$, then $\check\lambda_k$ is called a *maximum arc* of $\check\Lambda$ and k is called a *maximum index* of $\check\Lambda$. In that case, $\check q_k = \tilde q_j = \bar q_i$, $\check\beta_k = \tilde\beta_j$ and $\check\beta_{k+1} = \tilde\beta_{j+1}$.
If $\check T_k = \tilde T_j$, where $(j-1, j)$ is a gap pair of indices of $\tilde\Lambda$, then $(k-1, k)$ is called a *gap pair* of indices of $\check\Lambda$, $\check q_{k-1} = \tilde q_{j-1}$, $\check q_k = \tilde q_j$ and $\check\beta_k = \tilde\beta_j < \min(\check q_{k-1}, \check q_k)$.
Twin pairs of arcs of $\check\Lambda$ are defined as either $(\check\lambda_{k-1}, \check\lambda_k) = (\tilde\lambda_j^-, \tilde\lambda_j^+)$, where $\tilde\lambda_j^-$ and $\tilde\lambda_j^+$ are twin arcs in $\tilde D_j$, or $(\check\lambda_0, \check\lambda_1)$ when $\check\lambda_1 = \tilde\lambda_0^+$, or $(\check\lambda_{\mathcal N-2}, \check\lambda_{\mathcal N-1})$ when $\check\lambda_{\mathcal N-2} = \tilde\lambda_{N-1}^-$, and a pair $(k-1, k)$ of indices of twin arcs is called a *twin pair* of indices of $\check\Lambda$.
The number $\mathcal N$ of arcs $\check\lambda_k$ of $\check\Lambda$ is equal to the number N of arcs $\tilde\lambda_j$ of $\tilde\Lambda$ plus the number os twin pairs of arcs of $\check\Lambda$.

Remark 7.7. Let $\{\check\lambda_k\}_{k=0}^{\mathcal N-1}$ be the arcs of a twin pre-pizza $\check\Lambda$ on a Hölder triangle T corresponding to a pre-pizza $\tilde\Lambda$ on T associated with a Lipschitz function f. If $(\check\lambda_{k-1}, \check\lambda_k)$ is a secondary pair of arcs of $\check\Lambda$, then exactly one of the following three properties holds:
(A) $\check\lambda_{k-1}$ and $\check\lambda_k$ are twin arcs and $\check q_{k-1} = \check q_k = \check\beta_k$.
If $1 < k < \mathcal N - 1$, then $\check\lambda_{k-1}$ and $\check\lambda_k$ belong to a transversal pizza zone $\tilde D_j$ of $\tilde\Lambda$ adjacent to two coherent pizza slices, such that $\tilde D_j$ is not a maximum zone of $\tilde\Lambda$, thus both $(k-2, k-1)$ and $(k, k+1)$ are primary pairs of indices of $\check\Lambda$ corresponding to primary pairs $(j-1, j)$ and $(j, j+1)$ of indices of $\tilde\Lambda$, such that the following holds (see Lemma 7.4):

$$\max(\check\mu_{k-1}(\check q_{k-1}), \check\mu_{k+1}(\check q_k)) = \check\beta_k, \quad \check q_{k-1} = \check q_k < \max(\check q_{k-2}, \check q_{k+1}) = \max(\tilde q_{j-1}, \tilde q_{j+1}). \tag{32}$$

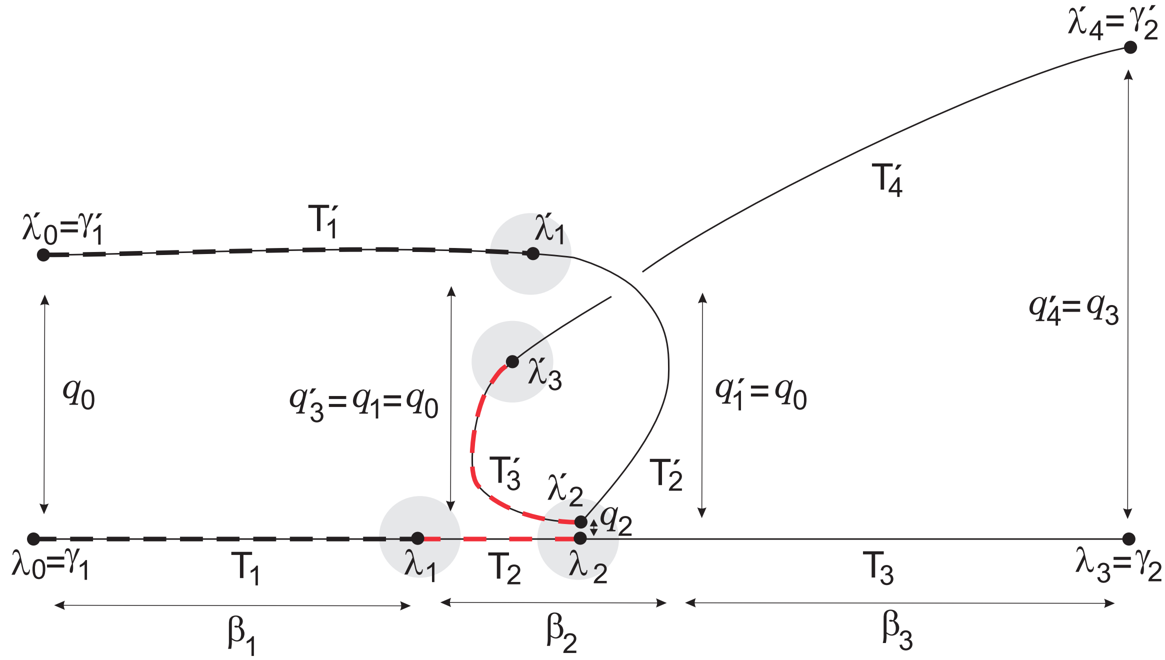

Figure 4. A normal pair of Hölder triangles in Example 7.8. Coherent pizza slices are shown in dashed lines.

If $k = 1$, then $\check\lambda_0 = \gamma_1$ is a transversal boundary arc of T which is not a maximum arc of $\tilde\Lambda$, thus $(1, 2)$ is a primary pair of indices of $\check\Lambda$ corresponding to a primary pair $(0, 1)$ of indices of $\tilde\Lambda$, such that

$$\check\mu_2(\check q_1) = \check\beta_1, \quad \check q_0 = \check q_1 < \check q_2 = \tilde q_1 \quad \text{(see (31) in Lemma 7.4).} \tag{33}$$

If $k = \mathcal{N} - 1$, then γ_2 is a transversal boundary arc of T which is not a maximum arc of $\tilde\Lambda$, thus $(\mathcal{N} - 3, \mathcal{N} - 2)$ is a primary pair of indices of $\check\Lambda$, corresponding to a primary pair $(N - 2, N - 1)$ of indices of $\tilde\Lambda$, such that

$$\check\mu_{\mathcal{N}-2}(\check q_{\mathcal{N}-2}) = \check\beta_{\mathcal{N}-1}, \quad \check q_{\mathcal{N}-2} = \check q_{\mathcal{N}-1} < \check q_{\mathcal{N}-3} = \tilde q_{N-2} \quad \text{(see (30) in Lemma 7.4).} \tag{34}$$

(B) $T(\check\lambda_{k-1}, \check\lambda_k)$ is a transversal pizza slice of a minimal pizza on T associated with f, thus $\check q_{k-1} = \check\nu_{k-1} \neq \check q_k = \check\nu_k$ and $\check\beta_k = \min(\check q_{k-1}, \check q_k)$.
(C) $T(\check\lambda_{k-1}, \check\lambda_k)$ is the union of two transversal pizza slices of a minimal pizza on T associated with f, thus $\check\beta_k < \min(\check q_{k-1}, \check q_k)$, and $(k - 1, k)$ is a gap pair of indices of $\check\Lambda$.

Example 7.8. A normal pair (T, T') of Hölder triangles in Figure 4 has different numbers of pizza slices in T and T'. This is related to the presence of twin arcs in T and the absence of them in T'. The minimal pizza Λ on T associated with the distance function f has three pizza slices T_1, T_2 and T_3, with exponents β_1, β_2 and β_3, such that $\beta_2 > \beta_1$ and $\beta_2 > \beta_3$. The minimal pizza Λ' on T' associated with the distance function g has four pizza slices T'_1, T'_2, T'_3 and T'_4, with exponents $\beta'_1 = \beta_1$, $\beta'_2 = \beta'_3 = \beta_2$ and $\beta'_4 = \beta_3$. The pizza slices T_1 and T_2 of Λ, and the pizza slices T'_1 and T'_3 of Λ', are coherent. The orders q_ℓ of f on the pizza zones D_ℓ of Λ satisfy inequalities $q_2 > q_0 = q_1 > q_3$, and the orders $q'_{\ell'}$ of g on the pizza zones $D'_{\ell'}$ of Λ' satisfy equalities $q'_0 = q'_1 = q'_3 = q_0$, $q'_2 = q_2$ and $q'_4 = q_3$. The pizza zones $D_0 = \{\gamma_1\} = M_1$ and $D_2 = M_2$ are the only maximum zones of Λ, and the permutation σ is trivial: $\sigma(1) = 1$ and $\sigma(2) = 2$. The correspondence τ, where $\tau(1) = 1$ and $\tau(2) = 3$, is positive on T_1 and negative on T_2.

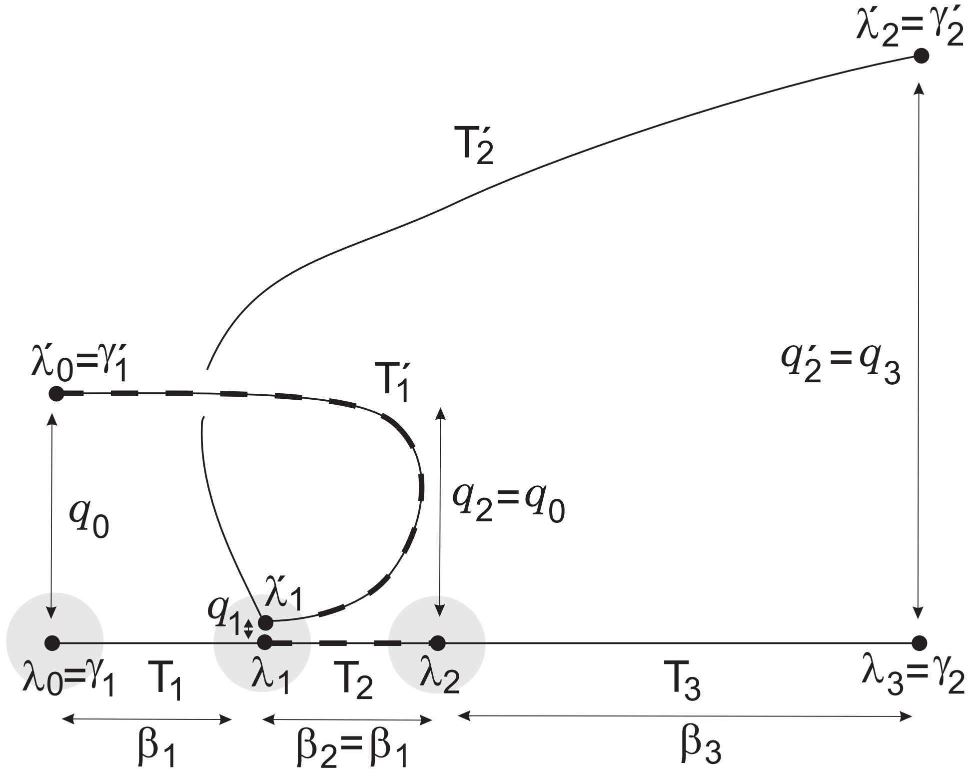

FIGURE 5. A normal pair of Hölder triangles in Example 7.9. Coherent pizza slices are shown in dashed lines.

There are no non-essential pizza zones in T, thus pre-pizza $\tilde{\Lambda}$ is the same as pizza: $N = n = 4$ and $\tilde{\lambda}_j = \lambda_j$ for $j = 0, \dots, 3$. The zones $\tilde{D}_1 = D_1$ and $\tilde{D}_2 = D_2$ of $\tilde{\Lambda}$ are transversal: $\mu(D_1) = q_0$ and $\mu(D_2) = q_2$. The arc $\tilde{\lambda}_1 = \lambda_1$, a common boundary arc of two coherent pizza slices, belongs to a transversal pizza zone D_1 of Λ which is not a maximum zone. By Definition 7.6, it should be replaced by twin arcs $\tilde{\lambda}_1^-$ and $\tilde{\lambda}_1^+$ in the twin pre-pizza $\check{\Lambda}$. Thus $\mathcal{N} = 5$, $\check{\lambda}_0 = \gamma_1$, $\check{\lambda}_1 = \tilde{\lambda}_1^-$, $\check{\lambda}_2 = \tilde{\lambda}_1^+$, $\check{\lambda}_3 = \tilde{\lambda}_2 = \lambda_2$, $\check{\lambda}_4 = \gamma_2$. There are no non-essential pizza zones in T', thus pre-pizza $\tilde{\Lambda}'$ is the same as pizza: $N' = n' = 5$ and $\tilde{\lambda}'_j = \lambda'_j$ for $j = 0, \dots, 4$. The zones $\tilde{D}'_1 = D'_1$, $\tilde{D}'_2 = D'_2$ and $\tilde{D}'_3 = D'_3$ are transversal: $\mu(D'_1) = \mu(D'_3) = q_0$ and $\mu(D'_2) = q_2$. Since each of the arcs λ'_1, λ'_2 and λ'_3 in these zones is not a common boundary arc of two coherent pizza slices, the twin pre-pizza $\check{\Lambda}'$ is the same as pre-pizza $\tilde{\Lambda}'$: $\mathcal{N}' = 5 = \mathcal{N}$ and $\check{\lambda}'_k = \tilde{\lambda}'_k = \lambda'_k$ for $k = 0, \dots, 4$.

Example 7.9. A normal pair (T, T') of Hölder triangles in Figure 5 has different numbers of pizza slices in T and T'. There are no twin arcs in T, but there is a twin arc in T' corresponding to its boundary arc γ'_1. The minimal pizza Λ on T associated with the distance function f has three pizza slices T_1, T_2 and T_3, with exponents β_1, β_2 and β_3, such that $\beta_1 = \beta_2 > \beta_3$. The minimal pizza Λ' on T' associated with the distance function g has two pizza slices T'_1 and T'_2, with exponents $\beta'_1 = \beta_1$ and $\beta'_2 = \beta_3$. The pizza slices T_2 of Λ and T'_1 of Λ' are coherent. The orders q_ℓ of f on the pizza zones D_ℓ of Λ satisfy inequalities $q_1 > q_0 > q_3$, and the orders $q'_{\ell'}$ of g on the pizza zones $D'_{\ell'}$ of Λ' satisfy equalities $q'_0 = q_0$, $q'_1 = q_1$ and $q'_2 = q_3$. The pizza zones $D_0 = \{\gamma_1\}$ and D_3 are minimum zones of Λ, the pizza zone D_1 is the only maximum zone of Λ, and the permutation σ is trivial: $\sigma(1) = 1$. The correspondence τ, where $\tau(2) = 1$, is negative on T_2.

There are no non-essential pizza zones of Λ, thus $\tilde\Lambda=\Lambda$, $N=n=4$ and $\tilde\lambda_j=\lambda_j$ for $j=0,\ldots,3$. The zones $\tilde D_1=D_1$ and $\tilde D_2=D_2$ are transversal: $\mu(D_1)=q_0$ and $\mu(D_2)=q_2$. Since there is a single coherent pizza slice T_2 of Λ, and both boundary arcs of T_2 are interior arcs of T, there are no twin arcs in T, thus $\check\Lambda=\tilde\Lambda$, $\mathcal{N}=N=4$ and $\check\lambda_k=\tilde\lambda_k$ for $k=0,\ldots,3$. There are no non-essential pizza zones in T', thus $\tilde\Lambda'=\Lambda'$, $N'=n'=3$ and $\tilde\lambda'_j=\lambda'_j$ for $j=0,1,2$. Since $\tilde\lambda'_0=\lambda'_0$ is a transversal boundary arc of T' adjacent to a coherent pizza slice $\tilde T'_1=T'_1$ of Λ', and $\{\lambda'_0\}$ is a minimum zone of Λ', the twin pre-pizza $\check\Lambda'$ contains a twin arc $\check\lambda'_1$ of the boundary arc $\check\lambda'_0=\tilde\lambda'_0=\gamma'_1$ of T', such that $tord(\check\lambda'_0,\check\lambda'_1)=q'_0$, $\mathcal{N}'=4$, $\check\lambda'_2=\tilde\lambda'_1$ and $\check\lambda'_3=\tilde\lambda'_2$.

Remark 7.10. In Examples 7.8 and 7.9 we have $\mathcal{N}'=\mathcal{N}$. We are going to prove (see Lemma 7.13) that this equality holds for any normal pair of Hölder triangles.

Definition 7.11. Let (T,T') be a normal pair of Hölder triangles, and let $\Lambda=\{T_\ell\}_{\ell=1}^{p}$ and $\Lambda'=\{T'_{\ell'}\}_{\ell=1}^{p'}$, where $T_\ell=T(\lambda_{\ell-1},\lambda_\ell)$ and $T'_{\ell'}=T(\lambda'_{\ell'-1},\lambda'_{\ell'})$, be minimal pizzas on T and T' associated with the distance functions f and g. Then Λ and Λ' are called *compatible* if, for any coherent pizza zones D_ℓ and $D'_{\ell'}=\tau(D_\ell)$ (see Remark 4.25) the pair of arcs $(\lambda_\ell,\lambda'_{\ell'})$, where $\lambda_\ell\in D_\ell$ and $\lambda'_{\ell'}\in D'_{\ell'}$, is normal:

$$tord(\lambda_\ell,\lambda'_{\ell'})=q_\ell=q'_{\ell'}. \tag{35}$$

It follows from [5, Proposition 3.9] that, for any normal pair (T,T') of Hölder triangles, there exist compatible minimal pizzas Λ and Λ' associated with the distance functions f and g. Since the sets of coherent pizza zones in pizzas, pre-pizzas and twin pre-pizzas are the same, pairs of pre-pizzas $\tilde\Lambda$ and $\tilde\Lambda'$, and pairs of twin pre-pizzas $\check\Lambda$ and $\check\Lambda'$, corresponding to compatible pairs of pizzas Λ and Λ', are also compatible.
In what follows, all pairs of pizzas, pre-pizzas and twin pre-pizzas associated with the distance functions on normal pairs of Hölder triangles are assumed to be compatible.

Remark 7.12. If $D_\ell=M_i$ is a transversal maximum pizza zone of Λ and $D_{\ell'}=M'_{\sigma(i)}$, or if D_ℓ is a transversal pizza zone of Λ adjacent to a coherent pizza slice and $D_{\ell'}=\tau(D_\ell)$ (see Remark 4.25) then $(\lambda_\ell,\lambda'_{\ell'})$ is a normal pair of arcs for any $\lambda_\ell\in D_\ell$ and $\lambda'_{\ell'}\in D_{\ell'}$.

Lemma 7.13. *Let (T,T') be a normal pair of Hölder triangles, with the distance functions f and g on T and T', respectively. Then the number $\mathcal{N}$ of arcs $\check\lambda_k$ in a twin pre-pizza $\check\Lambda$ on T associated with f (see Definition 7.6) is the same as the number of arcs $\check\lambda'_{k'}$ in a twin pre-pizza $\check\Lambda'$ on T' associated with g.*

Proof. The characteristic permutation σ defines a bijection between the sets of maximum zones of $\check\Lambda$ and $\check\Lambda'$. Thus the two sets have the same cardinality m. Similarly, the characteristic correspondence τ (see Definition 4.6) defines a bijection between the sets of coherent pizza slices of $\check\Lambda$ and $\check\Lambda'$. Thus the two sets have the same cardinality L. It follows from [5, Proposition 3.9] that there is a one-to-one correspondence between the sets of coherent interior pizza zones of $\check\Lambda$ and $\check\Lambda'$. Thus the two sets have the same cardinality n_2. It follows from [5, Proposition 4.10] that there is a one-to-one correspondence between the set of transversal maximum zones of $\check\Lambda$ adjacent to two coherent pizza slices and the set of transversal maximum zones of $\check\Lambda'$ adjacent to two coherent pizza slices. Thus the two sets have the same cardinality m_2. It follows from [5, Proposition 4.10] that there is a one-to-one correspondence between the set of transversal maximum zones of $\check\Lambda$ not adjacent to any coherent pizza slices and the set of transversal maximum zones of $\check\Lambda'$ not adjacent to any coherent pizza slices. Thus the two sets have the same cardinality m_0. The number δ of transversal boundary arcs of T which are minimum zones of $\check\Lambda$ is the same as the number of transversal boundary arcs of T' which are minimum zones of $\check\Lambda'$.

We are going to prove that $\mathcal{N} = 2L - n_2 - m_2 + m_0 + \delta$. Note that the number in the right-hand side is the same for $\check\Lambda$ and $\check\Lambda'$.

The number $2L$ counts the boundary arcs of coherent pizza slices of Λ, with the common boundary arcs of two coherent pizza slices being counted twice. If such a boundary arc belongs to a coherent pizza zone, then it must be counted once, and the number n_2 should be subtracted from $2L$. If such an arc belongs to a transversal maximum zone, then it must be counted once, and the number m_2 should be subtracted.
To each of the remaining common boundary arcs of two coherent slices correspond two twin arc of $\check\Lambda$, thus such arcs must be counted twice, as they are counted in $2L$. If a boundary arc $\check\gamma$ of T is adjacent to a coherent pizza slice, it is counted once in $2L$. However, if $\check\gamma$ is transversal and $\{\check\gamma\}$ is a minimum zone, then it corresponds to twin arcs in $\check\Lambda$, thus it should be counted twice. Accordingly, the number δ should be added.
This takes care of all boundary arcs of coherent pizza slices in $\check\Lambda$.
The remaining arcs in $\check\Lambda$ correspond to maximum zones not adjacent to any coherent pizza slices, thus the number m_0 should be added. □

Definition 7.14. Let (T, T') be a normal pair of Hölder triangles with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$ defined on T and T', respectively. If $\check\Lambda = \{\check\lambda_k\}_{k=0}^{\mathcal{N}-1}$ is a twin pre-pizza on T associated with f, and $\check\Lambda' = \{\check\lambda'_{k'}\}_{k'=0}^{\mathcal{N}-1}$ is a twin pre-pizza on T' associated with g, then the combination of the characteristic permutation σ and characteristic correspondence τ of the pair (T, T'), together with assignments $\check\lambda_0 \mapsto \check\lambda'_0$ and $\check\lambda_{\mathcal{N}-1} \mapsto \check\lambda'_{\mathcal{N}-1}$ for the boundary arcs of T and T', defines a one-to-one correspondence $\check\tau$ between the sets of arcs $\{\check\lambda_k\}$ and $\{\check\lambda'_{k'}\}$: if $(\check\lambda_{k-1}, \check\lambda_k)$ is a primary pair of arcs of $\check\Lambda$ corresponding to a coherent pizza slice T_ℓ of Λ, then the coherent pizza slice of $\check\Lambda'$ corresponding to the coherent pizza slice $T'_{\tau(\ell)}$ of Λ' is bounded by the arcs $\check\tau(\lambda_{\ell-1})$ and $\check\tau(\lambda_\ell)$ of $\check\Lambda'$, in the same (resp., opposite) order when τ is positive (resp., negative) on T_ℓ, and if $\check\lambda_k$ is the arc of $\check\Lambda$ contained in a maximum zone M_i of Λ, then the maximum zone $M'_{\sigma(i)}$ of Λ' contains the arc $\check\tau(\check\lambda_k)$ of $\check\Lambda'$. Remark 4.16 implies that the definition of $\check\tau$ is consistent when an arc of $\check\Lambda$ is a boundary arcs of a coherent pizza slice of Λ and belongs either to a coherent pizza zone or to a maximum zone of Λ. Since the sets of arcs of $\check\Lambda$ and $\check\Lambda'$ are ordered according to orientations of T and T' and have the same number of elements $\mathcal{N}$, this defines a permutation ϖ of the set $[\mathcal{N}] = \{0, \ldots, \mathcal{N}-1\}$.

Remark 7.15. The permutation ϖ in Example 7.8 is $(0, 1, 3, 2, 4)$. Note that the zone $\check D_1$ is "split" (see Remark 4.25): the arc $\check\lambda_1 = \tilde\lambda_1^-$ is mapped to $\check\lambda'_1 = \lambda'_1$, but the arc $\check\lambda_2 = \tilde\lambda_1^+$ is mapped to $\check\lambda'_3 = \lambda'_3$.

Definition 7.16. Let Λ be a minimal pizza on a normally embedded Hölder triangle T with m maximum zones and L coherent pizza slices, associated with a non-negative Lipschitz function f on T, and let (σ, υ, s) be an allowable triple of a permutation σ of the set $[m] = \{1, \ldots, m\}$, a permutation υ of the set $[L] = \{1, \ldots, L\}$ and a sign function $s : [L] \to \{+, -\}$ (see Definition 4.22). Let $\check\Lambda = \{\check T_k\}_{k=1}^{\mathcal{N}}$ be the corresponding twin pre-pizza associated with f. Then there is a unique permutation ϖ of the set $[\mathcal{N}] = \{0, \ldots, \mathcal{N}-1\}$ with the following properties:

1) $\varpi(0) = 0$ and $\varpi(\mathcal{N}-1) = \mathcal{N}-1$.
2) ϖ is compatible with the permutation σ: if the arcs $\check\lambda_k$ and $\check\lambda_l$ of $\check\Lambda$ belong to maximum zones M_i and M_j of Λ, then $\varpi(k) < \varpi(l)$ if, and only if, $\sigma(i) < \sigma(j)$.
3) ϖ is compatible with the permutation υ: if the arcs $\check\lambda_k$ and $\check\lambda_l$ of $\check\Lambda$, where $k \neq l$, are boundary arcs of the i-th and j-th coherent pizza slice of $\check\Lambda$, then $\varpi(i) < \varpi(j)$ if, and only if, $\upsilon(i) < \upsilon(j)$.
4) ϖ is compatible with the sign function s in the following sense: if $\check\lambda_{k-1}$ and $\check\lambda_k$ are boundary arcs of the j-th coherent pizza slice $\check T_k$ of $\check\Lambda$, then $\varpi(k-1) = \varpi(k) - 1$ when $s(j) = +$ and $\varpi(k-1) = \varpi(k) + 1$ when $s(j) = -$.
5) ϖ is compatible with the permutation ω of the set $[K] = \{1, \ldots, K\}$, where $K = m+L$ (see Definition 4.19 and Proposition 4.23) in the following sense: if an arc $\check\lambda_l$ of $\check\Lambda$ belongs to a maximum zone M_i, and if $\check T_k$ is a coherent pizza slice of $\check\Lambda$, then $\varpi(l)$ is less than at least one of the indices $\varpi(k-1)$ and $\varpi(k)$ if, and only if, the image by ω of the index in $[K]$ corresponding to M_i is less than the image by ω of the index in $[K]$ corresponding to $\check T_k$. Since ω is determined by Λ, σ, υ and s (see Remark 4.21) this condition may be reformulated as follows:
5′) If an arc $\check\lambda_l$ of $\check\Lambda$ belongs to a maximum zone M_i of Λ, and if $\check T_k = T(\check\lambda_{k-1}, \check\lambda_k)$ is a coherent pizza slice of $\check\Lambda$ which belongs to a caravan C, then $\varpi(l) < \max(\varpi(k-1), \varpi(k))$ if, and only if, $\sigma(i) \leq j_-(C)$ (see Definition 4.17 and Proposition 4.18).
A permutation ϖ with these properties is called *determined* by Λ, σ, υ and s.

Proposition 7.17. *Let (T, T'), be a normal pair of Hölder triangles with the minimal pizzas Λ and Λ' on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$, respectively. Let m be the number of maximum zones of Λ. Let a permutation σ of the set $[m] = \{1, \ldots, m\}$ be the characteristic permutation of the pair (T, T') (see Definition 3.4). Let L be the number of coherent pizza slices of Λ, and let the permutation υ of the set $[L] = \{1, \ldots, L\}$ and the sign function $s : [L] \to \{+, -\}$ be defined by the characteristic correspondence τ of the pair (T, T') (see Definition 4.7). Let $\mathcal{N}$ be the number of arcs of the twin pre-pizza $\check\Lambda$ on T, and let ϖ of the permutation of the set $[\mathcal{N}] = \{0, \ldots, \mathcal{N}-1\}$ (see Definition 7.14). Then the permutation ϖ is determined by Λ, σ, υ and s (see Definition 7.16). if (T, T'') is another normal pair with the same minimal pizza Λ on T, the same characteristic permutation σ, the same permutation υ and the same sign function s, then the pairs (T, T') and (T, T'') have the same permutation ϖ (see Remark 4.21).*

Proof. According to Theorem 4.20, the combined characteristic permutation ω of the set $[K] = \{1. \ldots, K\}$, where $K = m + L$, is determined by the pizza Λ, permutations σ and υ, and the sign function s. The order of coherent pizza slices in $\check\Lambda'$ is the same as their order in Λ', which is determined by the permutation υ. The sign function s defines the order of boundary arcs of each coherent pizza slice $\check T'$ of $\check\Lambda'$: Let $\check T'$ be the k'-th coherent pizza slice of $\check\Lambda'$, where $k' = \upsilon(k)$, and let $\check T$ be the k-th coherent pizza slice of $\check\Lambda$. If $s(k) = +$ (resp., $s(k) = -$) then the boundary arcs of $\check T'$ have the same (resp., opposite) order as the corresponding boundary arcs of $\check T$. If $\check T' \prec \check T''$ are two coherent pizza slices of $\check\Lambda'$, then either they do not have common boundary arcs and the boundary arcs of $\check T'$ precede the boundary arcs of $\check T''$, or they have a common boundary arc λ' (which belongs either to a coherent pizza zone or to a maximum zone of $\check\Lambda'$) and the other boundary arc of $\check T'$ precedes λ', while λ' precedes the other boundary arc of $\check T''$.

To define ϖ, it is enough to know the number of maximum zones of $\check\Lambda'$ preceding each of its coherent pizza slice, but it is the same as the number of maximum zones of Λ' preceding each of its coherent pizza slice, which was defined in Proposition 4.18. □

Remark 7.18. The properties of blocks in Lemma 6.4 hold if we replace the permutation π of the set $[n]$ by the permutation ϖ of the set $[\mathcal{N}]$.

Proposition 7.19. *Let (T, T') be a normal pair of Hölder triangles with the minimal pizzas Λ and Λ' on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$, and let $\check\Lambda$ and $\check\Lambda'$ be the corresponding twin pre-pizzas. If $\mathcal{N}$ is the number of arcs in $\check\Lambda$, same as the number of arcs in $\check\Lambda'$, then the permutation ϖ of the set $[\mathcal{N}] = \{0, \ldots, \mathcal{N}-1\}$ in Definition 7.14 satisfies the following properties:*
1. *$\varpi(0) = 0$ and $\varpi(\mathcal{N}-1) = \mathcal{N}-1$.*
2. *An arc $\check\lambda_k$ of $\check\Lambda$ belongs to a maximum zone M_i if, and only if, an arc $\check\lambda'_{\varpi(k)}$ of $\check\Lambda'$ belongs to the maximum zone $M'_{\sigma(i)}$, thus ϖ is compatible with σ on the maximum zones (see Remark 4.24).*
3. *The permutation ϖ is compatible with the permutation υ, the sign function s, and the combined characteristic permutation ω of the pair (T, T').*
4. *If C is a caravan of Λ, then the indices in $\check\Lambda$ of the boundary arcs of coherent pizza slices of C are mapped by ϖ to the indices in $\check\Lambda'$ of the boundary arcs of the corresponding coherent pizza slices of the caravan $C' = \tau(C)$ of Λ', in the same (resp., opposite) order when $s(C) = +$ (resp., when $s(C) = -$). In particular, the set of indices in $\check\Lambda$ of the boundary arcs of all pizza slices of C is a block of ϖ, on which ϖ acts either preserving or reversing the order.*
5. *The set of indices in $\check\Lambda$ of the maximum zones in $\mathcal{A}(C)$ adjacent to a caravan C of Λ is mapped by ϖ to the set of indices in $\check\Lambda'$ of the maximum zones in $\mathcal{A}(C')$ adjacent to the caravan $C' = \tau(C)$ of Λ'.*

Proof. Properties **1** and **2** are part of Definition 7.14 of the permutation ϖ.
Property **3** follows from Definition 7.14, since the action of ϖ on the primary pairs of indices of $\check\Lambda$ is compatible with the action of τ on the boundary arcs of coherent pizza slices of Λ, while the permutation υ, sign function s and combined characteristic permutation ω are defined by that action of τ.
Property **4** follows from Proposition 4.18, where this property is stated for the permutation υ, and from compatibility of ϖ with τ, υ and ω.
Property **5** follows from the normal embedding property of T and T'. □

Remark 7.20. Let (T, T') be a normal pair of Hölder triangles with the minimal pizzas Λ and Λ' on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$, and let $\check\Lambda$ and $\check\Lambda'$ be the corresponding twin pre-pizzas. Then the map $\check\lambda_k \mapsto \check\lambda'_{\varpi(k)}$ from the set $\{\check\lambda_k\}$ of arcs of $\check\Lambda$ to the set $\{\check\lambda'_{k'}\}$ of arcs of $\check\Lambda'$ induced by ϖ is an isometry with respect to the metric ξ (see Definition 2.4) defined by the tangency order: $tord(\check\lambda'_{\varpi(i)}, \check\lambda'_{\varpi(j)}) = tord(\check\lambda_i, \check\lambda_j)$.

Proposition 7.21. *Let (T, T') be a normal pair of Hölder triangles with the minimal pizzas Λ and Λ' on T and T' associated with the distance functions $f(x) = dist(x, T')$ and $g(x') = dist(x', T)$, and let $\check\Lambda$ and $\check\Lambda'$ be the corresponding twin pre-pizzas.*
If $(k'-1, k')$ is a secondary pair of indices of $\check\Lambda'$, where $k'-1 = \varpi(i)$ and $k' = \varpi(j)$, then $\check\beta'_{k'} = tord(\check\lambda_i, \check\lambda_j)$, and one of the following properties holds:

(**A**) $(k'-1,k')$ *is a twin pair of indices of* $\check\Lambda'$ *and* $\check q_i=\check q_j=tord(\check\lambda_i,\check\lambda_j)$.
There are three subcases of (**A**)*:*
($\mathbf{A_1}$) $1<k'<\mathcal{N}-1,\ (k'-2,k'-1)$ *and* $(k',k'+1)$ *are primary pairs of indices of* $\check\Lambda'$,

$$\max(\check\mu'_{k'-1}(\check q'_{k'-1}),\check\mu'_{k'+1}(\check q'_{k'}))=\check\beta'_{k'},\quad \check q'_{k'-1}=\check q'_{k'}<\max(\check q'_{k'-2},\check q'_{k'+1}),\tag{36}$$

the arcs $\check\lambda_i$ *and* $\check\lambda_j$ *belong either to two transversal pizza zones of* Λ *which are not maximum zones or to the same transversal pizza zone of* Λ *which is not a maximum zone.*
($\mathbf{A_2}$) $k'=1,\ i=0,\ (1,2)$ *is a primary pair of indices of* $\check\Lambda'$,

$$\check\mu'_2(\check q'_1)=\check\beta'_1,\quad \check q'_0=\check q'_1<\check q'_2,\tag{37}$$

the arc γ_1 *is a transversal boundary arc of* T *which is not a maximum zone, the arc* $\check\lambda_j$ *belongs either to a transversal pizza zone of* Λ *which is not a maximum zone or to the* $\check q_0$*-order zone of* f *containing* γ_1*.*
($\mathbf{A_3}$) $k'=\mathcal{N}-1,\ j=\mathcal{N}-1,\ (\mathcal{N}-3,\mathcal{N}-2)$ *is a primary pair of indices of* $\check\Lambda'$,

$$\check\mu'_{\mathcal{N}-2}(\check q'_{\mathcal{N}-2})=\check\beta'_{\mathcal{N}-1},\quad \check q'_{\mathcal{N}-2}=\check q'_{\mathcal{N}-1}<\check q'_{\mathcal{N}-3},\tag{38}$$

the arc γ_2 *is a transversal boundary arc of* T *which is not a maximum zone, the arc* $\check\lambda_i$ *belongs either to a transversal pizza zone of* Λ *which is not a maximum zone or to the* $\check q_{\mathcal{N}-1}$*-order zone of* f *containing* γ_2*.*
(**B**) $T(\check\lambda'_{k'-1},\check\lambda'_{k'})$ *is a transversal pizza slice of* Λ', $\check q_i\neq\check q_j$ *and* $\min(\check q_i,\check q_j)=tord(\check\lambda_i,\check\lambda_j)$.
(**C**) $(k'-1,k')$ *is a gap pair of indices of* $\check\Lambda'$ *and* $\min(\check q_i,\check q_j)>tord(\check\lambda_i,\check\lambda_j)$.

Proof. The statement follows from Remark 7.7 applied to $\check\Lambda'$ instead of $\check\Lambda$, and from Remark 7.20. In particular, (36), (37) and (38) follow from (32), (33) and (34), respectively. Properties (**B**) and (**C**) follow from the corresponding properties in Remark 7.7. □

The following theorem is an analog of Theorem 6.6.

Theorem 7.22. *Let* $\mathcal{B}_{ij}=B_\varpi(\{\check\lambda_i,\check\lambda_j\})$ *be the minimal block of* ϖ *containing* $\{i,j\}$*, and let* $\mathcal{B}'_{ij}=B_{\varpi^{-1}}(\{\check\lambda'_i,\check\lambda'_j\})$ *be the minimal block of* ϖ^{-1} *containing* $\{i,j\}$*. Then*

$$tord(\check\lambda_i,\check\lambda_j)\le tord(\check\lambda_k,\check\lambda_l)\ \textit{for}\ \{k,l\}\subset\mathcal{B}_{ij},\tag{39}$$

$$tord(\check\lambda'_i,\check\lambda'_j)\le tord(\check\lambda'_k,\check\lambda'_l)\ \textit{for}\ \{k,l\}\subset\mathcal{B}'_{ij}.\tag{40}$$

Remark 7.23. If a pair (T,T') is totally transversal, then $\mathcal{N}=n$, $\varpi=\pi$, the sets of arcs $\{\check\lambda_j\}$ in Definitions 7.14 and 6.10 are the same, and the inequalities (39) and (40) are the same as the inequalities (24) and (25).

Let Λ be a minimal pizza on a normally embedded Hölder triangle T associated with a non-negative Lipschitz function $f(x)$ on T, and let $\check\Lambda=\{\check T_k\}_{k=0}^{\mathcal{N}-1}$, where $\check T_k=T(\check\lambda_{k-1},\check\lambda_k)$, be the corresponding twin pre-pizza on T. Let $\varpi:[\mathcal{N}]\to[\mathcal{N}]$, where $[\mathcal{N}]=\{0,\dots,\mathcal{N}-1\}$, be a permutation, such that $\varpi(0)=0,\ \varpi(\mathcal{N}-1)=\mathcal{N}-1$, and each primary pair $(k-1,k)$ of indices of $\check\Lambda$, corresponding to a coherent pizza slice $\check T_k$, is mapped by ϖ to a pair of consecutive indices $(\varpi(k-1),\varpi(k))$, either $\varpi(k-1)=\varpi(k)-1$ or $\varpi(k-1)=\varpi(k)+1$. We are going to formulate *admissibility conditions* for the permutation ϖ, necessary and sufficient for the existence of a normal pair (T,T') of Hölder triangles such that f is contact equivalent to the distance function $dist(x,T')$ on T and the permutation ϖ is compatible with the permutations σ and υ, sign function s and combined characteristic permutation ω of the pair (T,T'). If such a pair (T,T') exists, then it is unique up to outer Lipschitz equivalence, due to Theorem 5.3.

Definition 7.24. Let Λ be a minimal pizza on a normally embedded Hölder triangle T, associated with a non-negative Lipschitz function f on T. Let m and L be the number of maximum zones and coherent pizza slices of Λ, respectively. Let σ and υ be allowable permutations of $[m]=\{1,\ldots,m\}$ and $[L]=\{1,\ldots,L\}$, respectively, and let $s:[L]\to\{+,-\}$ be an allowable sign function (see Definition 4.22). Let ω be the unique permutation of the set $[K]=\{1,\ldots,K\}$, where $K=m+L$, compatible with (σ,υ,s) (see Proposition 4.23). Let $\check\Lambda=\{\check T_k\}_{k=1}^{\mathcal N}$, where $\check T_k=T(\check\lambda_{k-1},\check\lambda_k)$, be the twin pre-pizza on T corresponding to Λ. A permutation ϖ of the set $[\mathcal N]=\{0,\ldots,\mathcal N-1\}$, such that $\varpi(0)=0$ and $\varpi(\mathcal N-1)=\mathcal N-1$, is called *admissible* with respect to Λ, σ, υ and s, if the following *admissibility conditions* are satisfied.
1. The permutation ϖ is determined by Λ σ, υ and s (See Definition 7.16).
2. The block inequalities (39) for ϖ are satisfied.

Theorem 7.25. *Let Λ be a minimal pizza on a normally embedded Hölder triangle $T=T(\gamma_1,\gamma_2)$ associated with a non-negative Lipschitz function $f(x)$ on T. Let m and L be the numbers of maximum zones and coherent pizza slices of Λ, respectively, and let $(\sigma,\ \upsilon,\ s)$, where σ is a permutation of the set $[m]=\{1,\ldots,m\}$, υ is a permutation of the set $[L]=\{1,\ldots,L\}$ and $s:[L]\to\{+,-\}$ is a sign function on $[L]$, be an allowable triple (see Definition 4.22). If $\check\Lambda=\{\check T_k\}_{k=1}^{\mathcal N}$, where $\check T_k=T(\check\lambda_{k-1},\check\lambda_k)$, is a twin pre-pizza on T associated with f and ϖ is an admissible permutation of the set $[\mathcal N]=\{0,\ldots,\mathcal N-1\}$, then there exists a unique, up to outer Lipschitz equivalence, normal pair (T,T') of Hölder triangles, such that the distance function $dist(x,T')$ on T is Lipschitz contact equivalent to f, the permutation σ is the characteristic permutation of the pair (T,T'), the permutation υ and sign function s are defined by the characteristic correspondence τ of the pair (T,T') (see Definition 4.7) and ϖ is associated with σ and τ as in Definition 7.14.*

Proof. The proof is similar to the proof of Theorem 6.14, except Step 2 where coherent pizza slices of $\check\Lambda'$ are defined. We may assume that $T=T_\beta$ is a standard β-Hölder triangle (5) in $\mathbb R^2_{uv}$. In particular, the arcs $\check\lambda_k$ in T are germs at the origin of the graphs $\{u\ge 0,\, v=\check\lambda_k(u)\}$, where $\check\lambda_k(u)\ge 0$ are Lipschitz functions, and each Hölder triangle $\check T_k=T(\check\lambda_{k-1},\check\lambda_k)$, for $k=1,\ldots,\mathcal N-1$, is a $\check\beta_k$-Hölder triangle. It follows from the non-archimedean property of the tangency order that $tord(\check\lambda_k,\check\lambda_l)=\min_{i:k<i<l}(tord(\check\lambda_k,\check\lambda_i),tord(\check\lambda_i,\check\lambda_l))$. We are going to construct a normally embedded β-Hölder triangle T' with a twin pre-pizza $\check\Lambda'$ on T' associated with the distance function $g(x')=dist(x',T)$.

Step 1. We define a family of arcs $\{\check\lambda'_{k'}\subset\mathbb R^3_{u,v,z}\}$ for $k'=\varpi(k)$ as the germs $\{(u,v)\in\check\lambda_k,\ z=u^{\check q_k}\}$, where $\check q_k=ord_{\check\lambda_k}f$. They will be the arcs of a twin pre-pizza $\check\Lambda'$ on T' associated with g. Let $\check q'_{\varpi(k)}=\check q_k$ and $\check\nu'_{\varpi(k)}=\check\nu_k$ for $k=0,\ldots,\mathcal N-1$. Note that $\check\nu_k=\check q_k$ when an arc $\check\lambda_k$ belongs to a transversal pizza zone of $\check\Lambda$. This implies that $\check\nu'_{k'}=\check q'_{k'}$ for $k'=\varpi(k)$ in that case. The arcs $\gamma'_1=\check\lambda'_0$ and $\gamma'_2=\check\lambda'_{\mathcal N-1}$ will be the boundary arcs of T'. Let $\check\beta'_{k'}=tord(\check\lambda'_{k'-1},\check\lambda'_{k'})$ for $k'=1,\ldots,\mathcal N-1$. Pizza slices $\check T'_{k'}$ of $\check\Lambda'$ will be $\check\beta'_{k'}$-Hölder triangles.

From the non-archimedean property of the tangency order we have, for $j=\varpi(i_1)$ and $k=\varpi(i_2)$, isometry

$$tord(\check\lambda'_j,\check\lambda'_k)=tord(\check\lambda_{i_1},\check\lambda_{i_2}),\tag{41}$$

same as (26) in Step 1 of the proof of Theorem 6.14. Since T is normally embedded and the arcs $\check\lambda_i$ satisfy the block inequalities (39), the arcs $\check\lambda'_j$ satisfy combinatorial normal embedding inequalities (see Definition 2.9) necessary for the Hölder triangle T' to be normally embedded:

$$tord(\check\lambda'_j,\check\lambda'_l)=\min_k\big(tord(\check\lambda'_j,\check\lambda'_k),tord(\check\lambda'_k,\check\lambda'_l)\big)\ \text{ for } j<k<l, \tag{42}$$

same as (27) in Step 1 of the proof of Theorem 6.14.

Step 2. For each primary pair of indices $(k-1,k)$ of $\check\Lambda$ corresponding to a coherent pizza slice $\check T_k$ of $\check\Lambda$, we define a $\check\beta_k$-Hölder triangle as follows.
Consider the standard pizza slice

$$z=\psi_{\check\beta_k,\check q_{k-1},\check q_k,\check\mu_k}(u,v) \tag{43}$$

(see Definition 2.23) on the standard $\check\beta_k$-Hölder triangle $T_{\check\beta_k}$, then translate its graph in $\mathbb{R}^3_{u,v,z}$ to a graph $T(\check\lambda'_{\varpi(k-1)},\check\lambda'_{\varpi(k)})$ of a Lipschitz function over $\check T_k$ by a bi-Lipschitz homeomorphism H_k in $\mathbb{R}^2_{u,v}$ preserving the u-coordinate, such that $H_k(\{v\equiv 0\})=\check\lambda_{k-1}$ and $H_k(\{v=u^{\check\beta_k}\})=\check\lambda_k$. It follows from Proposition 7.19 that $\varpi(k-1)$ and $\varpi(k)$ are consecutive indices, either $\varpi(k-1)=\varpi(k)-1$ or $\varpi(k-1)=\varpi(k)+1$. The Hölder triangle $T(\check\lambda'_{k'-1},\check\lambda'_{k'})$, where $k'=\max(\varpi(k-1),\varpi(k))$, will be a coherent pizza slice $\check T'_{k'}$ of $\check\Lambda'$, with the same width function $\mu(q)$ as the standard pizza slice (43), and $(k'-1,k')$ will be a primary pair of indices of $\check\Lambda'$.

Step 3. If $k'-1$ and k' are consecutive indices in $[\mathcal{N}]$ such that $(k'-1,k')$ is not a primary pair of indices of $\check\Lambda'$ defined in Step 2, then $(k'-1,k')$ will be a secondary pair of indices of $\check\Lambda'$. The arcs $\check\lambda'_{k'-1}$ and $\check\lambda'_{k'}$ are already defined in Step 1. We define two arcs $\theta^+_{k'-1}=\{(u,v,z)\in\check\lambda'_{k'-1},\ w_{k'}=u^{\check\beta'_{k'}}\}$ and $\theta^-_{k'}=\{(u,v,z)\in\check\lambda'_{k'},\ w_{k'}=u^{\check\beta'_{k'}}\}$ in $\mathbb{R}^4_{u,v,z,w_{k'}}$, then consider the $(\check q'_{k'-1},\check q'_{k'},\beta'_{k'})$-model $(T_{\beta'_{k'}},T'_{\beta'_{k'}})$ (see Definition 6.12) and define a map $h_{k'}:T'_{\beta'_{k'}}\to\mathbb{R}^4_{u,v,z,w_{k'}}\subset\mathbb{R}^{\mathcal{N}+2}_{u,v,z,\mathbf{w}}$, where $\mathbf{w}=(w_1,\ldots,w_{\mathcal{N}-1})$, as in Step 2 of the proof of Theorem 6.14.

Let $\check T'_{k'}=T(\check\lambda'_{k'-1},\check\lambda'_{k'})=h_{k'}(T'_{\beta'_{k'}})\subset\mathbb{R}^{\mathcal{N}+2}_{u,v,z,\mathbf{w}}$. There are three possible cases (see Proposition 7.21).

(A) If $\check q'_{k'-1}=\check q'_{k'}=\beta'_{k'}$, then the arcs $\check\lambda'_{k'-1}$ and $\check\lambda'_{k'}$ will be twin arcs of $\check\Lambda'$.

(B) If $\check q'_{k'-1}\neq\check q'_{k'}$ and $\min(\check q'_{k'-1},\check q'_{k'})=\beta'_{k'}$, then $\check T'_{k'}$ will be a transversal pizza slice of $\check\Lambda'$.

(C) If $\min(\check q'_{k'-1},\check q'_{k'})>\beta'_{k'}$, then $\check T'_{k'}$ will be the union of two transversal pizza slices, and $(k'-1,k')$ will be a gap pair of indices of $\check\Lambda'$.

Step 4. Let $T'=\bigcup_{k'=1}^{\mathcal{N}}\check T'_{k'}$ be the union of all Hölder triangles defined in Steps 2 and 3. To prove that T' is normally embedded, we are going to show that it is combinatorially normally embedded (see Definition 2.9).

As in Step 3 of the proof of Theorem 6.14, we show first that each partial Hölder triangle $\check T'_{k'}=T(\check\lambda'_{k'-1},\check\lambda'_{k'})$ is normally embedded. For a primary pair of indices $(k'-1,k')$ of $\check\Lambda'$, the triangle $\check T'_{k'}$ is normally embedded as the graph of a Lipschitz function. For a secondary pair of indices $(k'-1,k')$ of $\check\Lambda'$, the triangle $\check T'_{k'}$ is constructed from the $(\check q'_{k'-1},\check q'_{k'},\beta'_{k'})$-model $(T_{\beta'_{k'}},T'_{\beta'_{k'}})$ (see Definition 6.12), and the same arguments as in Step 3 of the proof of Theorem 6.14 show that it is normally embedded.

Next, we have to show that any two Hölder triangles $\check T'_{k'}$ and $\check T'_{l'}$, where $k'\neq l'$, are transversal. If both pairs of indices $(k'-1,k')$ and $(l'-1,l')$ are primary, then the two

Hölder triangles are transversal as the graphs of Lipschitz functions on two subtriangles $\check{T}_k$ and $\check{T}_l$ of T, where $k \neq l$.
If one of the two pairs of indices, say $(k'-1, k')$, is secondary and another one is primary, the proof is similar to the argument in Step 3 of the proof of Theorem 6.14, as the variable $w_{k'}$, which is non-zero on $\check{T}'_{k'}$, vanishes on the graph of a Lipschitz function $\check{T}'_{l'}$.
Finally, if both pairs of indices are secondary, the proof is exactly the same as in Step 3 of Theorem 6.14.

Step 5. We are going to show that the twin pre-pizza on T associated with the distance function $\tilde{f}(x) = dist(x, T')$ is combinatorially equivalent to the twin pre-pizza $\check{\Lambda}$ on T associated with f.

If $\check{T}_k$ is a coherent pizza slice of $\check{\Lambda}$, then, by construction in Step 2 of this proof, the corresponding sub-triangle $\check{T}'_{k'}$ of λ', where $k' = \max(\varpi(k-1), \varpi(k))$, is a graph of a Lipschitz function on $\check{T}_k$ Lipschitz contact equivalent to $f|_{\check{T}_k}$. Thus the pair $(\check{T}_k, \check{T}'_{k'})$ is outer Lipschitz equivalent to the pair $(\check{T}_k, graph(f|_{\check{T}_k})$. Let us show that $\tilde{f}|_{\check{T}_k}$ is Lipschitz contact equivalent to $f|_{\check{T}_k}$.

For any arc η' of a coherent pizza slice $\check{T}'_{l'}$ of $\check{\Lambda}'$, where $l' = \max(\varpi(l-1), \varpi(l))$ and $l \neq k$, the order of tangency of η' with any arc η of $\check{T}_k$ cannot exceed $ord_\eta f$, as $\check{T}'_{k'}$ and $\check{T}'_{l'}$ are graphs of functions contact equivalent to restrictions of the Lipschitz function f on T to distinct pizza slices $\check{T}_k$ and $\check{T}_l$ of $\check{\Lambda}$. The same arguments as in Step 4 of the proof of Theorem 6.14 show that the order of tangency of an arc η of $\check{T}_k$ with any arc η' of a non-coherent triangle $T'_{\ell'}$ of $\check{\Lambda}'$ cannot exceed $ord_\eta f$, as the variable $w_{l'}$ involved in construction of $\check{T}'_{l'}$ in Step 3 of this proof vanishes on $\check{T}'_{k'}$.
Thus $f|_{\check{T}_k}$ is contact equivalent to restriction of $\tilde{f}$ to T_k.

The same arguments as in Step 4 of the proof of Theorem 6.14 show that any non-coherent triangle $\check{T}_k$ of $\check{\Lambda}$ is either a transversal pizza slice for the distance function $\tilde{f}$ or the union of two transversal pizza slices with a common minimum zone. In both cases, $\tilde{f}$ restricted to $\check{T}_k$ is completely determined by the exponent $\check{\beta}_k$ of $\check{T}_k$ and the orders $\check{q}_{k-1}$ and $\check{q}_k$ of f on its boundary arcs. This implies that $\tilde{f}$ restricted to $\check{T}_k$ is contact equivalent to f on all triangles $\check{T}_k$ of $\check{\Lambda}$.

This completes the proof of Theorem 7.25. □

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