DESCRIPTION OF DEFORMATIONS WITH CONSTANT MILNOR NUMBER FOR HOMOGENEOUS FUNCTIONS

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In [1], V. I. Arnol'd showed that the multiplicity* $\mu(f)$ of quasi-homogeneous function f with isolated singularities is not changed upon the addition of small terms of the same degree as in f, or of any terms of higher degree. V. I. Arnol'd assumed that the small deformations of quasi-homogeneous function f which do not alter $\mu(f)$ are exhausted by the aforementioned deformations to within the action of the group of changes of variables, and he proved this hypothesis for all 0- and 1-modal quasi-homogeneous functions ([1, 2], see [3] as well). We shall prove this hypothesis for homogeneous functions.

Definition 1. Let f: $C^n \to C$ be an analytic function with the isolated singularity f(0) = 0 at the origin of coordinates. Let $F(x, \lambda)$ be a deformation of function f(x). We call the stratum $\mu \equiv$ const of deformation $F(x, \lambda)$ the sprout at 0 of the set of those values of parameter λ for which $F(x, \lambda)$ has a singular point close to 0 of the same multiplicity as f(x) with zero critical value.

THEOREM 2. Let f: $C^n \to C$ be a homogeneous function of degree d and let $\mu(f) < \infty$. Let

$$F(x, \lambda) = f(x) + \sum_{i=1}^{\mu-1} \lambda_i \varphi_i(x) \tag{1}$$

be a transversal to the orbit of function f in the space of those functions g such that dg(0) = 0 and g(0) = 0. We assume that the $\varphi_i(x)$ are homogeneous and that $\deg \varphi_i(x) \geqslant d \Leftrightarrow i = 1, \ldots, m$. Then, stratum $\mu \equiv$ const of deformation $F(x, \lambda)$ is given by the equations $\lambda_{m+1} = \ldots = \lambda_{\mu-1} = 0$.

COROLLARY 3. The modality of a homogeneous function equals its internal modality (see [1], definition 8.6).

For the proof of Theorem 2 we use the following result of Le Dung Trang, Saito [4] and Teissier [5].

THEOREM 4. Let G: $C^n \times C \to C$ be a one-parameter deformation of analytic function f: $C^n \to C$ such that, for sufficiently small λ, $G(0, \lambda) = 0$, $d_x G(0, \lambda) = 0$, and $\mu(G(\cdot, \lambda)) = \mu(f)$. Then, as $|x| \to 0$,

$$\left| \frac{\partial G(x, \lambda)}{\partial \lambda} \right| = o\,|d_x G(x, \lambda)| \tag{2}$$

uniformly in λ for sufficiently small λ.

Assertion 5. Let f be a homogeneous function of degree d, $\mu(f) < \infty$, and let $\varphi_2, \ldots, \varphi_{d-1}$ be homogeneous functions, $\deg \varphi_i = i$, $\sum_{i=2}^{d-1} \varphi_i \not\equiv 0$. Then,

$$\mu\left(f + \sum_{i=2}^{d-1} \varphi_i\right) < \mu(f).$$

Proof. We set $G(x, \lambda) = f(x) + \sum_{i=2}^{d-1} \lambda^{d-i} \varphi_i(x)$. Then, $G(tx, \lambda) \equiv t^d G(x, t\lambda)$, so that $\mu(G(\cdot, \lambda)) = \mu(G(\cdot, 1))$ when $\lambda \neq 0$. We assume that $\mu(G(\cdot, 1)) = \mu(f)$. Then the conditions of Theorem 4 hold for G. At the same time,

*Here and henceforth, we dub the Milnor number of singularities the multiplicity.

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$$d_x G(\lambda x, \lambda) = \lambda^{d-1}\left(df(x) + \sum_{i=2}^{d-1} d\varphi_i(x)\right), \quad \frac{\partial G}{\partial \lambda}(\lambda x, \lambda) = \lambda^{d-1}\left(\sum_{i=2}^{d-1}(d-i)\varphi_i(x)\right),$$

and, if we choose a point x_0 at which $\sum_{i=2}^{d-1}(d-i)\varphi_i(x_0) = c \neq 0$ then, on the curve $\gamma(t) = (tx_0, t)$, as $t \to 0$

$$|d_x G(\gamma(t))| \leqslant C|t|^{d-1} \quad \text{and} \quad \left|\frac{\partial G}{\partial \lambda}(\gamma(t))\right| \sim ct^{d-1};$$

consequently, relationship (2) does not hold, and this proves assertion 5.

Assertion 6. Let f: $C^n \to C$ be a quasi-homogeneous function of weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ and degree d. Let

$$F(x, t) = f(x) + \sum \lambda_i(t)\varphi_i(x), \tag{3}$$

where the φ_i are quasi-homogeneous functions of weight α and degree d_i, while the $\lambda_i(t)$ are analytic functions, $\lambda_i(0) = 0$ if $d_i \leq d$. Let $\lambda_i(t) = c_i t^{k_i} + o(|t|^{k_i})$. We set $\nu = \max_{i:d_i<d} \frac{d-d_i}{k_i}$. Let

$$G(x, t) = f(x) + \sum_{i:d_i<d} \tau_i(t)\varphi_i(x),$$

where

$$\tau_i(t) = \begin{cases} c_i t^{d-d_i}, & \text{if } \dfrac{d-d_i}{k_i} = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for sufficiently small t, we have $\mu(G(\cdot, t)) \geq \mu(F(\cdot, t))$.

Proof. We set $H(x, t, \varepsilon) = \varepsilon^{-d} F(\varepsilon^{d_1} x_1, \ldots, \varepsilon^{d_n} x_n, \varepsilon^{\nu} t)$. Obviously, $H(x, t, \varepsilon) \to G(x, t)$ as $\varepsilon \to 0$. At the same time, $\mu(H(\cdot, t, \varepsilon)) = \mu(F(\cdot, \varepsilon^{\nu} t))$ when $\varepsilon \neq 0$. Assertion 6 now follows from the semi-continuity of μ in the Zariski topology.

Proof of Theorem 2. We assume that stratum $\mu \equiv$ const of deformation (1) is not contained in the set $\lambda_{m+1} = \ldots = \lambda_{\mu-1} = 0$. Then, in this stratum there lies a curve of the form of (3) (where $\alpha_1 = \ldots = \alpha_n = 1$), with $\sum_{i:d_i<d} \lambda_i(t)\varphi_i(x) \not\equiv 0$. From assertions 5 and 6, for sufficiently small $t \neq 0$, we obtain $\mu(F(\cdot, t)) \leq \mu(G(\cdot, t)) < \mu(f)$, which contradicts our assumption. Conversely, as shown by V. I. Arnol'd [1], the stratum $\mu \equiv$ const contains set $\lambda_{m+1} = \ldots = \lambda_{\mu-1} = 0$.

Remark 1. Let f: $C^n \to C$ be a quasi-homogeneous function of weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ and degree d, and let $\mu(f) < \infty$. Let $F(x, \lambda) = f(x) + \sum_{i=1}^{\mu-1} \lambda_i\varphi_i(x)$ be transversal to the orbit of f in the space of functions having at 0 a critical point with critical value 0, where each function φ_i is quasi-homogeneous of weight α and degree d_i. Following Arnol'd, we set

$$T_t(\lambda_1, \ldots, \lambda_{\mu-1}) = (e^{t(d-d_1)}\lambda_1, \ldots, e^{t(d-d_{\mu-1})}\lambda_{\mu-1}), \quad T_t'(x_1, \ldots, x_n) = (e^{\alpha_1 t}x_1, \ldots, e^{\alpha_n t}x_n).$$

Then, $F(T_t' x, T_t\lambda) = e^{td} F(x, \lambda)$ for any $t \in C$.

It follows from the last formulas that stratum $\mu \equiv$ const in space $C^{\mu-1}$ is invariant with respect to the action of flow $\{T_t : t \in C\}$ (just as for all stratifications by μ). In correspondence with the three possibilities: $d_i \lesseqgtr d$, space $C^{\mu-1}$ can be decomposed into the direct sum of the three subspaces $C^{\mu-1} = \Lambda^+ \oplus \Lambda^0 \oplus \Lambda$ such that, as $t \to +\infty$, flow $\{T_t\}$ is stretched on Λ^+, is fixed on Λ^0, and is contracted on Λ^-. It is geometrically obvious (and is proven in assertion 6) that if the nonzero analytic curve γ passes through $0 \in C^{\mu-1}$ and is not contained in $\Lambda^0 \oplus \Lambda^-$, then, in the closure of set $\bigcup_{t\in C} T_t\gamma$, there is contained a nonzero curve lying completely in Λ^+.

LITERATURE CITED

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