COMBINATORIAL FORMULAS FOR PONTRYAGIN CLASSES AND GL-INVARIANT CHAINS

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In this paper methods are given for constructing combinatorial formulas for Pontryagin classes, which for the case of the first Pontryagin class reduce to the formula obtained in the paper of Gel'fand, Losik, and myself [3]. The definition of Pontryagin classes in terms of the degeneration of systems of sections of fiber bundles (see [1, 2]) serves as the starting point for the construction of such formulas. Here this definition is reformulated in terms of GL-invariant chains in the space Hom $(\mathbf{R}^m, \mathbf{R}^n)$, i.e., chains that are invariant with respect to the action of the group GL(n, R) on this space (Sec. 4). To each cycle in the complex of such chains and collection of m sections of the bundle $E \to X$ with fiber R^n, corresponds a characteristic cycle in X. Here for each k there exists a GL-invariant cycle P_k such that the corresponding characteristic cycle represents the k-th Pontryagin class of the bundle E.

Thus, GL-invariant chains are equivalent here with characteristic differential forms used in [3]. Replacing forms by chains allowed one to avoid complicated analytic computations in the formulations and proofs and clarified the connection between the combinatorial formulas obtained and the classically defined Pontryagin classes.

In order to see how a characteristic cycle changes upon passage from one collection of sections to another, in Sec. 4, with each GL-invariant cycle ξ is associated a collection of GL-invariant chains $(\xi^{k,l})$, called a hypersimplicial GL-invariant cycle. The definition of this collection is closely connected with the concept of hypersimplex introduced in [3]. From such a collection one can, under certain conditions on the combinatorial manifold X which satisfies condition (A) of [3], construct a combinatorial cycle in X, whose homology class is determined by the original cycle ξ (Secs. 2 and 5). The basic result of the paper (Theorem 5.2) is that for the cycle P_k, corresponding to the k-th Pontryagin class, the combinatorial cycle constructed represents the k-th Pontryagin class of the manifold X.

In Secs. 1, 2, and 3 of this paper there is a reformulation in a convenient form for what follows of the theory of cooriented chains, spaces of configurations and hypersimplices from [3] (see also [4, 5]) with a series of changes which turned out to be necessary for the generalization of the formula of [3] to the higher classes.

In Sec. 4 the connection of GL-invariant chains with characteristic classes is shown and the existence of hypersimplicial GL-invariant characteristic cycles is proved (Proposition 4.6), which plays a key role in all the constructions.

In Sec. 5, with the aid of the constructions given in the previous points, combinatorial characteristic cycles are defined.

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1. Cooriented Chains

Let X be a smooth manifold and Y be a submanifold with corners in X. We denote by $N_X Y$ the normal bundle of Y in X.

Definition 1.1. By a coorientation of Y is meant an orientation of $N_X Y$.

Let Y and Y' be cooriented submanifolds such that $Y' \subset \partial Y$, $\dim Y' = \dim Y - 1$. Then $N_X Y' = N_X Y|_{Y'} \oplus N_Y Y'$, so the coorientation of Y determines a coorientation of Y' if one considers the bundle $N_Y Y'$ to be oriented by the exterior normal to Y' in Y. We set $\varepsilon(Y, Y') = 1$, if this coorientation coincides with the original coorientation of Y', and $\varepsilon(Y, Y') = -1$ in the opposite case. The number $\varepsilon(Y, Y')$ is called the incidence coefficient of the cooriented manifolds Y and Y'.

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Definition 1.2. By (rational, closed) cooriented chains of codimension p we mean locally finite linear combinations $\Sigma c_i Y_i$, where c_i are rational numbers and Y_i are cooriented submanifolds with corners of codimension p in X, where change of coorientation corresponds to multiplication by -1. The space of such chains is denoted by $\hat{C}^p(X)$. The boundary operator $\hat{\partial}\colon \hat{C}^p(X) \to \hat{C}^{p+1}(X)$, defined by the incidence coefficients $\varepsilon(Y, Y')$, turns $\bigoplus_{p\geqslant 0} \hat{C}^p(X)$ into a complex, which is denoted by $\hat{C}^*(X)$.

Remark 1.3. A coorientation of the manifold $Y \subset X$ allows one to define at each point $y \in Y$ corresponding orientations of Y and X such that $TX_y \cong TY_y \oplus N_X Y_y$. If Y' is a cooriented submanifold of $X, Y' \subset \partial Y$, $\dim Y' = \dim Y - 1$, and the orientations of Y and Y' are chosen in correspondence with some orientation of X, then $\varepsilon(Y, Y')$ coincides with the usual incidence coefficient of oriented manifolds. Corresponding orientations of Y and X can also be taken as a definition of coorientation (see [4]). Such a definition also makes sense in the case when X and Y have no smooth structures. We shall make use of this in what follows, speaking of coorientations of simplices in a combinatorial manifold.

Let $f\colon X' \to X$ be a smooth map, transverse to the cooriented submanifold $Y \subset X$. Then the manifold $Y' = f^{-1}(Y)$ in X' is equipped with a natural coorientation, since $N_{X'}Y' \cong f^*N_X Y$.

Definition 1.4. Let $Z = \Sigma c_i Y_i$ be a cooriented chain in X and $f\colon X' \to X$ be a smooth map which is transverse to Z (i.e., to all Y_i). By f^*Z is denoted the cooriented chain $\Sigma c_i f^{-1}(Y_i)$ in X'.

If f is also transverse to $\hat{\partial}Z$, then $\hat{\partial}f^*Z = f^*\hat{\partial}Z$. In particular, if f is a submersion, then there is defined a homomorphism of complexes $f^*\colon \hat{C}^*(X) \to \hat{C}^*(X')$.

If f is an arbitrary smooth map and Z is a cycle from $\hat{C}^p(X)$ such that f is transverse to Z, then f^*Z is a cycle from $\hat{C}^p(X')$ whose homology class depends only on the homology class of Z. Hence there is defined a map $f^*\colon H_p(\hat{C}^*(X)) \to H_p(\hat{C}^*(X'))$.

Let Y be a cooriented submanifold of codimension p in X and S be an oriented p-dimensional submanifold of X which is transverse to Y. At each point $y \in S \cap Y$ one can identify TS_y and $N_X Y_y$. We set $\varepsilon_y = 1$ if the orientations of these spaces coincide, $\varepsilon_y = -1$ if not. We define the intersection index (S, Y) as the sum of the numbers ε_y over all points y of $S \cap Y$.

Definition 1.5. Let $Z = \Sigma c_i Y_i$ be a cooriented chain and $\sigma = \Sigma d_j S_j$ be an oriented chain transverse to Z, codim Z = dim σ. The number $(\sigma, Z) = \sum_{i,j} c_i d_j (S_j, Y_i)$ is called the intersection index of the chains σ and Z.

Proposition 1.6. If Z is a cooriented chain of codimension $p - 1$ and σ is an oriented chain of dimension p, such that $\partial\sigma$ and Z, σ and $\hat{\partial}Z$ are transverse, then $(\partial\sigma, Z) = (-1)^p (\sigma, \hat{\partial}Z)$.

COROLLARY 1.7. If Z is a cycle from $\hat{C}^p(X)$ and σ is a cycle from $C_p(X, Q)$ transverse Z, then (σ, Z) depends only on the homology classes of Z and σ in $H_p(\hat{C}^*(X))$ and $H_p(X, Q)$ respectively.

Thus, there is defined a map $\iota\colon H_p(\hat{C}^*(X)) \to H^p(X, Q)$.

Proposition 1.8. The map ι is an isomorphism and for any smooth map $f\colon X' \to X$ there is a commutative diagram

$$\begin{array}{ccc} H_p(\hat{C}^*(X)) & \xrightarrow{\iota} & H^p(X, Q) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H_p(\hat{C}^*(X')) & \xrightarrow{\iota} & H^p(X', Q). \end{array}$$

Remark 1.9. If X is an algebraic manifold, then one can define semialgebraic cooriented chains in X. To define such chains, instead of submanifolds with corners it is necessary to take semialgebraic sets. All the assertions about cooriented chains carry over to semialgebraic cooriented chains (with the replacement of smooth maps by algebraic ones).

2. Combinatorial Manifolds and Configuration Spaces

Let X be a simplicial complex. For each simplex $X_i \subset X$, one denotes by SX_i the star of X_i, i.e., the subcomplex of X formed by simplices containing X_i.

Definition 2.1. The complex X is called an M-dimensional combinatorial manifold if for each simplex $X_i \subset X$ there exists a continuous map $SX_i \to \mathbf{R}^M$ linear on each simplex of SX_i and carrying a neighborhood (in SX_i) of any interior point of the simplex X_i into an open set in $\mathbf{R}^M$. Each such map is called a smoothing. Two smoothings are called equivalent if one of them is obtained from the other by an affine transformation in $\mathbf{R}^M$. The set Σ_i of equivalence classes of smoothings is called the configuration space† of the simplex X_i. This space has a natural structure as a smooth algebraic manifold.

If $X_i \supset X_j$, then $SX_i \subset SX_j$, so there is defined a map $\psi_{ij}: \Sigma_j \to \Sigma_i$, which is a submersion.

If $X_i \supset X_j \supset X_k$, then obviously $\psi_{ik} = \psi_{ij} \circ \psi_{jk}$.

Definition 2.2. We say that the combinatorial manifold X satisfies condition A_p if for each simplex $X_i \subset X$

$$H_0(\Sigma_i, \mathbf{Q}) = \mathbf{Q} \quad \text{for} \quad \operatorname{codim} X_i \leqslant p,$$
$$H_q(\Sigma_i, \mathbf{Q}) = 0 \quad \text{for} \quad 1 \leqslant q \leqslant p - \operatorname{codim} X_i.$$

Let us assume now that all simplices X_i of the combinatorial manifold X are cooriented. For each simplex X_j we set $s(j) = \{i : X_i \supset X_j, \dim X_i = \dim X_j + 1\}$. If $i \in s(j)$, then by ε_{ij} is denoted the incidence coefficient $\varepsilon(X_i, X_j)$.‡

We denote by Σ the system of spaces Σ_i and maps ψ_{ij}.

Definition 2.3. By a cooriented chain Z of codimension p of the system of spaces Σ is meant a collection $\{Z_i\}$, where Z_i is a semialgebraic cooriented chain in Σ_i of codimension $p - \operatorname{codim} X_i$. The space of such chains is denoted by $\hat{C}^p(\Sigma)$. The boundary operator $\hat{\partial}: \hat{C}^p(\Sigma) \to \hat{C}^{p+1}(\Sigma)$ is defined by the formula

$$(\hat{\partial}Z)_j = \partial Z_j + (-1)^{\operatorname{codim} Z_j + 1} \sum_{i \in s(j)} \varepsilon_{ij} \psi_{ij}^* Z_i.$$

THEOREM 2.4. Let X be a combinatorial manifold satisfying the condition A_p and Z be a cycle from $\hat{C}^p(\Sigma)$.

a) To each flag $X_{i_0} \supset X_{i_1} \supset \ldots \supset X_{i_l}$ ($\operatorname{codim} X_{i_l} \leqslant p$) of simplices of X one can assign an oriented rational l-chain $\sigma_{i_0 \ldots i_l}$ in Σ_{i_0} such that σ_{i_0} is a point and

$$\partial \sigma_{i_0 \ldots i_l} = \psi_{i_0 i_1} \sigma_{i_1 \ldots i_l} \sigma_{i_1 \ldots i_l} + \sum_{\nu=1}^{l} (-1)^{\nu} \sigma_{i_0 \ldots \hat{i}_\nu \ldots i_l}.$$

b) Let $\Gamma = \sum_{l=0}^{p} \sum'_{i_0 \ldots i_l} \varepsilon_{i_0 i_1} \ldots \varepsilon_{i_{l-1} i_l} (\sigma_{i_0 \ldots i_l}, Z_{i_0}) X_{i_l}$, where Σ' signifies that the summation is taken only over those collections $i_0 \ldots i_l$, for which $i_{\nu-1} \in s(i_\nu)$, $\operatorname{codim} X_{i_l} = p$. Then Γ is a cycle in $\hat{C}^p(\Sigma)$.

c) The homology class of the cycle Γ is independent of the choice of the chain $\sigma_{i_0 \ldots i_l}$ and is unchanged if one replaces the cycle Z by one which is homologous in $\hat{C}^p(\Sigma)$.

The proof of Paragraph a) is trivial. The proof of Paragraph b) is done by direct computation using Proposition 1.6.

3. Hypersimplices

Let V be a finite set, $q \geqslant 0$ be an integer. We denote by $I_p(V, q)$ the Abelian group with generators $\Delta_q^{k,l}(v_1, \ldots, v_{k+q})$, where $\{v_1, \ldots, v_{k+q}\}$ is any ordered collection of k + q dis-

†This space is homotopically equivalent with the configuration space from [3].

‡This definition differs from the definition of [3] by the sign $(-1)^{\operatorname{codim} X_j}$.

tinct elements of the set V, and k and l can assume the values $k = l = 0$ for $p=0$, $k \geqslant 1$, $1 \leqslant l \leqslant q$, $k+l=p+1$ for $p \geqslant 1$. These generators are connected by the relations of skew-symmetry with respect to permutations of the first $k + l$ elements of the collection $\{v_1, \ldots, v_{k+q}\}$ and symmetry with respect to permutations of the last $q - l$ elements. The differential $\partial: I_p(V, q) \to I_{p-1}(V, q)$ is defined by the formula

$$\partial\Delta_q^{1,1}(v_1, \ldots, v_{q+1}) = \Delta_q^{0,0}(v_2, \ldots, v_{q+1}) - \Delta_q^{0,0}(v_1, v_3, \ldots, v_{q+1}) \text{ for } p=1,$$

$$\partial\Delta_q^{k,l}(v_1, \ldots, v_{k+q}) = \sum_{\mu=1}^{k+l} (-1)^{\mu-1}\Delta_q^{k-1,l}(v_1, \ldots \hat{v}_\mu \ldots, v_{k+q}) + $$
$$+ \sum_{\mu=1}^{k+l} (-1)^{\mu-1}\Delta_q^{k,l-1}(v_1, \ldots \hat{v}_\mu \ldots, v_{k+q}, v_\mu) \text{ for } p>1,$$

while for $k = 1$ (respectively, $l = 1$) the first (respectively, second) sum in the last expression is omitted. It is easy to verify that $\partial^2 = 0$. The complex $I_*(V, q) = \bigoplus_{p \geqslant 0} I_p(V, q)$ is called a hypersimplicial complex and its generators $\Delta_q^{k,l}$ are called hypersimplices.

If $V' \subset V$, we denote by $I_*(V, V', q)$ the subcomplex of $I_*(V, q)$ generated by the hypersimplices $\Delta_q^{k,l}(v_1, \ldots, v_{k+q})$, in which $\{v_{k+l+1}, \ldots, v_{k+q}\} \supset V'$.

Proposition 3.1. $H_k(I_*(V, V', q)) = 0$ for $k > 0$, $H_0(I_*(V, V', q)) = \mathbf{Z}$.

If X is a combinatorial manifold and X_i is a simplex of it, we denote by VX_i and VSX_i the sets of vertices of the simplex X_i and of its star SX_i. We set $I_*(i) = I_*(VSX_i, VX_i, M + 1)$, where $M = \dim X$. It is obvious that if $X_j \supset X_i$, then $I_*(i) \supset I_*(j)$.

Definition 3.2. By a hypersimplicial filament of the manifold X is meant a collection $\{b_i\}$, where $b_i \in I_p(i)$, $p = \operatorname{codim} X_i$, and $\partial b_i = \sum_{j \in s(i)} \varepsilon_{ij} b_j$.

The existence of hypersimplicial filaments follows easily from Proposition 3.1.

4. GL-Invariant Chains

Let $L_{m,n} = \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$. A point of the space $L_{m,n}$ is determined by a collection $(e_1, \ldots, e_m)$ of m vectors in $\mathbf{R}^n$, where e_j is the image of the j-th basis vector of the space $\mathbf{R}^m$. Let $\hat{C}^*(L_{m,n})$ be the complex of semialgebraic cooriented chains of the space $L_{m,n}$ (see Remark 1.9). The group $GL(n, \mathbf{R})$ acts on $L_{m,n}$ and hence on $\hat{C}^*(L_{m,n})$.

Definition 4.1. The subcomplex of $\hat{C}^*(L_{m,n})$, consisting of chains invariant with respect to the action of the group $GL(n, \mathbf{R})$, is called the complex of GL-invariant cooriented chains and is denoted by $\Xi^*(L_{m,n})$.

Definition 4.2. Let ξ be a cycle from $\Xi^p(L_{m,n})$, X be a smooth manifold, E be a bundle over X with fiber $\mathbf{R}^n$, and $e = (e_1, \ldots, e_m)$ be a collection of sections of the bundle E. If a trivialization of E over the open set $U \subset X$ is chosen, then the collection e determines a map $\varphi_U: E|_U \to L_{m,n}$. Let us assume that ϕ_U is transverse to ξ for all U (this condition does not depend on the choice of trivialization and is satisfied for collections e in general position, i.e., from a certain open, everywhere dense set). Then $\varphi_U^*\xi$ is a cooriented cycle in U which does not depend on the choice of trivialization. If $V \subset U$, then $\varphi_U^*\xi|_V = \varphi_V^*\xi$, so the collection $\{\varphi_U^*\xi\}_{U \subset X}$ determines a cycle from $\hat{C}^p(X)$, which we denote by ξ_e and call the characteristic cycle corresponding to the cycle ξ and the collection of sections e.

Proposition 4.3. Let $\xi, X, E, e = (e_1, \ldots, e_m)$ and ξ_e denote the same things as in Definition 4.2. Then

a) the homology class of the cycle ξ_e is independent of the choice of sections $e_1, \ldots, e_m$;

b) if $f\colon Y \to X$ is a map which is transverse to ξ_e, and f^*e is the collection of sections of the bundle f^*E which is determined by the collection e, then $\xi_{f^*e} = f^*\xi_e$;

c) the homology class of the cycle ξ_e is unchanged if one replaces the cycle ξ by a homologous one.

The proof is trivial.

Example 4.4. Let $k \geqslant 1$, $n \geqslant 2k$, $m \geqslant n-2k+2$ and

$$P_{k,n} = \{e_1, \ldots, e_m \mid \mathrm{rk}(e_1, \ldots, e_{n-2k+2}) \leqslant n-2k\} \subset L_{m,n}.$$

The fiber of the normal bundle of $P_{k,n}$ at a nonsingular point can be identified with $\mathbf{R}^{2k} \oplus \mathbf{R}^{2k}$, where R^{2k} is the supplement in R^n to the space generated by the vectors $e_1, \ldots, e_{n-2k+2}$. Since $\mathbf{R}^{2k} \oplus \mathbf{R}^{2k}$ has a canonical orientation, $P_{k,n}$ is a cooriented chain. It is easy to verify that $P_{k,n}$ is a cycle from $\Xi^{4k}(L_{m,n})$. The corresponding characteristic cycle is called the k-th Pontryagin cycle.

The group S(m) of permutations of basis vectors in R^m acts on $L_{m,n}$ (on the right) and consequently also on $\hat{C}^*(L_{m,n})$. Under this action the complex $\Xi^*(L_{m,n})$ is carried into itself so that S(m) acts on $\Xi^*(L_{m,n})$. Throughout in what follows, in speaking of permutations of vectors we shall have in mind the indicated action of the group S(m).

The following construction is closely connected with the definition of hypersimplices.

We consider the maps $\alpha_j : L_{m,n}^{\prime} \to L_{m-1,n}$ and $\beta_j : L_{m,n} \to L_{m,n}$, defined by the formulas $\alpha_j(e_1, \ldots, e_m) = (e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_m)$, $\beta_j(e_1, \ldots, e_m) = (e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_m, e_j)$.

These maps determine homomorphisms of complexes

$$\alpha_j^*\colon \Xi^*(L_{m-1,n}) \to \Xi^*(L_{m,n}), \quad \beta_j^*\colon \Xi^*(L_{m,n}) \to \Xi^*(L_{m,n}).$$

Definition 4.5. By a hypersimplicial GL-invariant cooriented p-chain of type (q, n) is meant a collection $\tilde{\xi} = (\xi^{k,l})$, where k = l = 0 or $k \geqslant 1$, $1 \leqslant l \leqslant q$, $k+l \leqslant p+1$, $\xi^{0,0} \in \Xi^p(L_{q,n})$, $\xi^{k,l} \in \Xi^{p-k-l+1}(L_{k+q,n})$ for $(k, l) \neq (0, 0)$, the chain $\xi^{k,l}$ is invariant with respect to permutations of the last q − l vectors and skew invariant with respect to permutations of the first k + l vectors. The space of such chains is denoted by $\Xi^p_{q,n}$.

The differential $\tilde{\partial} : \Xi^p_{q,n} \to \Xi^{p+1}_{q,n}$ is defined by the formulas

$$(\tilde{\partial}\tilde{\xi})^{0,0} = \hat{\partial}\xi^{0,0}, \quad (\tilde{\partial}\tilde{\xi})^{1,1} = \hat{\partial}\xi^{1,1} + (-1)^p[\alpha_1^*\xi^{0,0} - \alpha_2^*\xi^{0,0}],$$

$$(\tilde{\partial}\tilde{\xi})^{k,l} = \hat{\partial}\xi^{k,l} + (-1)^{p-k-l}\left[\sum_{\mu=1}^{k+l} (-1)^{\mu-1}\alpha_\mu^*\xi^{k-1,l} - \sum_{\mu=1}^{k+l} (-1)^{\mu-1}\beta_\mu^*\xi^{k,l-1}\right]$$

for $k+l > 2$, while for k = 1 (respectively, for l = 1) the first (respectively, second) sum in the last formula is omitted.

It is not hard to verify that $\tilde{\partial}^2 = 0$. We set $\Xi^*_{q,n} = \{\bigoplus_{p \geqslant 0} \Xi^p_{q,n}, \tilde{\partial}\}$.

Proposition 4.6. a) The cycle $\bar{P}_{k,n}$, obtained from the cycle $P_{k,n}$ (Example 4.4) by averaging over the group of permutations of vectors, is homologous to $P_{k,n}$ in $\Xi^*(L_{m,n})$;

b) for $k \geqslant 1$, $n \geqslant 2k$, and $q \geqslant n-2k+2$, there exists a cycle $\tilde{\xi} = (\xi^{k,l}) \in \Xi^{4k}_{q,n}$ such that $\xi^{0,0} = \bar{P}_{k,n}$.

Proof. The group GL(m, R) acts (on the right) on $L_{m,n}$, and hence also on $\hat{C}^*(L_{m,n})$. Since this action commutes with the action of GL(n, R), the group GL(m, R) also acts on $\Xi^*(L_{m,n})$. We consider the subcomplex $'\Xi^*(L_{m,n}) \subset \Xi^*(L_{m,n})$, consisting of chains, invariant with respect to the action of the subgroup $G \subset GL(m, \mathbf{R})$, generated by reflections in the basic hyperplanes.

LEMMA 4.7. The group S(m) acts trivially on the homology of the complex $'\Xi^*(L_{m,n})$.

Proof. The homology of the complex $'\Xi^*(L_{m,n})$ is the homology of the complex $\Xi^*(L_{m,n})$, invariant with respect to the action of the group G. Since the space GL(m, R)/G is connected, this homology is invariant with respect to the action of the whole group GL(m, R), and in particular, with respect to the action of $S(m) \subset GL(m, \mathbf{R})$, which is what was asserted.

It is easy to verify that the cycle $P_{k,n} \in \Xi^{4k}(L_{m,n})$ belongs to $'\Xi^{4k}(L_{m,n})$, so assertion a) of Proposition 4.6 follows from Lemma 4.7.

To prove assertion b), we consider in the complex $\Xi^*_{q,n}$ the subcomplex $'\Xi^*_{q,n}$, consisting of collections $(\xi^{k,l})$ in which $\xi^{k,l} \in \Xi^*(L_{k+q,n})$. Using Lemma 4.7 and the skew invariance of the chains $\xi^{k,l}$ $((k, l) \neq (0, 0))$ with respect to certain permutations, it is easy to prove that the homology of this complex coincides with the homology of its quotient by the subcomplex of collections $(\xi^{k,l})$: $\xi^{0,0}=0$. Hence, for each S(q)-invariant cycle $\xi \in {}'\Xi^*(L_{q,n})$ (in particular, for $\bar{P}_{k,n}$) there exists a cycle $\tilde{\xi}=(\xi^{k,l}) \in {}'\Xi^*_{q,n}$, $\xi^{0,0}=\xi$, which is what had to be proved.

5. Cycles of the Complex Ξ^* and Combinatorial Formulas

As before, let X be an M-dimensional combinatorial manifold, X_i be a simplex of X, Σ_i the corresponding configuration space, and $\Delta=\Delta^{k,l}_{M+1}(v_1,\ldots,v_{k+M+1})$ a hypersimplex of $I_*(i)$. We define a map $\rho_\Delta: \Sigma_i \to L_{k+M+1,M+1}$ in the following way.

Let R^m be imbedded in $\mathbf{R}^{M+1}$ as the hyperplane $\{x_1+\ldots+x_{M+1}=1\}$ and let $a_1,\ldots,a_{M+1}$ be the points of this hyperplane corresponding to the ends of the basis vectors in $\mathbf{R}^{M+1}$. Let $\alpha_1,\ldots,\alpha_{M+1}$ be vertices of any M-dimensional simplex of SX_i. Then for each point $\sigma\in\Sigma_i$, i.e., class of affine-equivalent smoothings $SX_i \to R^M$, there exists a unique smoothing of this class for which α_ν is carried into a_ν for $\nu = 1, \ldots, M+1$. The images of the vertices $v_1,\ldots,v_{k+M+1}$ under this smoothing determine a point of $L_{k+M+1,M+1}$, which we denote by $\rho_\Delta(\sigma)$. The map ρ_Δ depends on the choice of vertices $\alpha_1,\ldots,\alpha_{M+1}$. However, maps obtained by different choices differ by transformations from $GL(M+1, \mathbf{R})$. Hence, for each chain $\xi\in\Xi^*(L_{k+M+1,M+1})$, transverse to the image of ρ_Δ, there is a uniquely defined chain $\rho^*_\Delta\xi\in\hat{C}^*(\Sigma_i)$. If $\tilde{\xi}$ is a chain from $\Xi^*_{M+1,M+1}$, then the component $\xi^{k,l}$ of this chain lies in $\Xi^*(L_{k+M+1,M+1})$. We set $\rho^*_\Delta\tilde{\xi}=\rho^*_\Delta\xi^{k,l}$.

Proposition 5.1. Let $\tilde{\xi}$ be a cycle from $\Xi^p_{M+1,M+1}$ and $\{b_i\}$ be a hypersimplicial filament of the combinatorial manifold X, $b_i=\sum c_{ij}\Delta_{ij}$, where Δ_{ij} are hypersimplices of $I_*(i)$. We set $Z_i=\Sigma c_{ij}\rho^*_{\Delta_{ij}}\tilde{\xi}$. Then

a) the collection $Z=\{Z_i\}$ is a cycle of $\hat{C}^*(\Sigma)$ (Definition 2.3);

b) the homology Z is independent of the choice of hypersimplicial filament and is unchanged if the cycle $\tilde{\xi}$ is replaced by a homologous one.

The proof proceeds by direct computation.

THEOREM 5.2. Let us assume that the M-dimensional combinatorial manifold X satisfies condition A_{4k} (Definition 2.2). Let $\tilde{\xi}\in\Xi^{4k}_{M+1,M+1}$ be a cycle corresponding to the cycle $P_{k,M+1}$ (Proposition 4.6). Let Z be the cycle of $\hat{C}^{4k}(\Sigma)$ constructed from the cycle $\tilde{\xi}$, and Γ be the cycle of $\hat{C}^{4k}(X)$, constructed from the cycle Z. If the manifold X is smoothable, then the homology class of the cycle Γ coincides with the k-th Pontryagin class of the manifold X.

The proof of this theorem is rather cumbersome and will not be given here.

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