Submitted exclusively to the London Mathematical Society
DOI: 10.1112/S0000000000000000

BETTI NUMBERS OF SEMIALGEBRAIC AND SUB-PFAFFIAN SETS

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Abstract

Let X be a subset in $[-1,1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \dots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\},$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $[-1,1]^{n_0+\dots+n_\nu}$ being a difference between a finite CW-complex and its subcomplex. We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν.

In important particular cases of semialgebraic and semi-Pfaffian sets defined by quantifier-free formulae with polynomials and Pfaffian functions respectively, upper bounds on Betti numbers of X_ν are well known. Our results allow to extend the bounds to sets defined with quantifiers, in particular to sub-Pfaffian sets.

Introduction

Well-known results of Petrovskii, Oleinik [**15**], [**14**], Milnor [**12**], and Thom [**18**] provide an upper bound for the sum of Betti numbers of a semialgebraic set defined by a Boolean combination of polynomial equations and inequalities. A refinement of these results can be found in [**1**]. For semi-Pfaffian sets the analogous bounds were obtained by Khovanskii [**10**] (see also [**22**]). In this paper we describe a reduction of estimating Betti numbers of sets defined by formulae with quantifiers to a similar problem for sets defined by a quantifier-free formulae.

More precisely, let X be a subset in $[-1,1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by a formula

$$X = \{\mathbf{x}_0 |\, Q_1\mathbf{x}_1 Q_2\mathbf{x}_2 \dots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}, \tag{0.1}$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and X_ν be either an open or a closed set in $[-1,1]^{n_0+\dots+n_\nu}$ being a difference between a finite CW-complex and one of its subcomplexes. For instance, if $\nu = 1$ and $Q_1 = \exists$, then X is the projection of X_ν.

We express an upper bound on each Betti number of X via a sum of Betti numbers of some sets defined by quantifier-free formulae involving X_ν. In conjunction with Petrovskii-Oleinik-Thom-Milnor's result this implies a new upper bound for semialgebraic sets defined by formulae with quantifiers, which is significantly better than a bound following from the cylindrical cell decomposition approach. In conjunction with Khovanskii's result our method produces an analogous upper bound for restricted sub-Pfaffian sets defined by formulae with quantifiers. Apparently in this case no general upper bounds were previously known.

2000 *Mathematics Subject Classification* 14P10 (primary), 14P15, 55T99 (secondary).

The first author was supported by NSF grant # DMS-0070666 and by James S. McDonnell Foundation. The second author was supported by the European RTN Network RAAG 2002–2006 (contract no HPRN-CT-0027). A part of this work was done when he was visiting Purdue University in August–September, 2001.

Throughout the paper each topological space is assumed to be a difference between a finite CW-complex and one of its subcomplexes.

EXAMPLE 1. The closure X of the interior of a compact set $Y \subset [-1,1]^n$ is homotopy equivalent to

$$X_{\varepsilon,\delta} = \{\mathbf{x} |\ \exists \mathbf{y} (\|\mathbf{x} - \mathbf{y}\| \le \delta)\ \forall \mathbf{z} (\|\mathbf{y} - \mathbf{z}\| < \varepsilon)\ (\mathbf{z} \in Y)\}$$

for small enough $\delta, \varepsilon > 0$ such that $\delta \gg \varepsilon$. Representing $X_{\varepsilon,\delta}$ in the form (0.1), we conclude that X is homotopy equivalent to $X_{\varepsilon,\delta} = \{\mathbf{x} |\ \exists \mathbf{y}\ \forall \mathbf{z}\ X_2\}$, where

$$X_2 = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) |\ (\|\mathbf{x} - \mathbf{y}\| \le \delta \wedge (\|\mathbf{y} - \mathbf{z}\| \ge \varepsilon \vee \mathbf{z} \in Y))\}$$

is a closed set in $[-1,1]^{3n}$. Our results allow to bound from above Betti numbers of X in terms of Betti numbers of X_2.

1. *A spectral sequence associated with a surjective map*

DEFINITION 1. A continuous map $f:\ X \to Y$ is *locally split* if for any $y \in Y$ there is an open neighbourhood U of y and a section $s:\ U \to X$ of f (i.e., s is continuous and $fs = \mathrm{Id}$). In particular, a projection of an open set in $\mathbb{R}^n$ on a subspace of $\mathbb{R}^n$ is always locally split.

DEFINITION 2. For two maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, the *fibered product* of X_1 and X_2 is defined as

$$X_1 \times_Y X_2 := \{(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2 |\ f_1(\mathbf{x}_1) = f_2(\mathbf{x}_2)\}.$$

THEOREM 1. *Let $f:\ X \to Y$ be a surjective cellular map. Assume that f is either closed or locally split. Then for any Abelian group G, there exists a spectral sequence $E^r_{p,q}$ converging to $H_*(Y, G)$ with*

$$E^1_{p,q} = H_q(W_p, G) \tag{1.1}$$

where

$$W_p = \underbrace{X \times_Y \ldots \times_Y X}_{p+1 \text{ times}} \tag{1.2}$$

In particular,

$$\dim H_k(Y, G) \le \sum_{p+q=k} \dim H_q(W_p, G), \tag{1.3}$$

for all k.

For a locally split map f, this theorem can be derived from [**4**], Corollary 1.3. We present below a proof for a closed map f.

REMARK 1. In the sequel we use Theorem 1 only for projections of either closed or open sets in $\mathbb{R}^n$. If f is a projection of an open set, then (1.3) easily follows from the analogous result for closed maps which will be proved below, without references to [**4**]. Indeed, for an open set Z define its *shrinking* $S(Z)$ as the closed set $Z \backslash N(\partial Z)$ where N denotes an open neighbourhood. For a small enough $N(\partial Z)$, the set Z is homotopy equivalent to $S(Z)$ (recall that Z is a difference between a finite CW– complex and a subcomplex). Let X be open and $S(X)$ be its shrinking with a sufficiently small $N(\partial X)$. It induces shrinkings $S(Y) = f(S(X))$ and $S(W_p) = S(X)\times_{S(Y)}\ldots\times_{S(Y)}S(X)$ which are homotopy equivalent to Y and W_p respectively. The statement for open sets X and Y follows from the statement for closed sets applied to $f:\ S(X)\to S(Y)$.

Definition 3. For a sequence $(P_0,\ldots,P_p)$ of topological spaces, their *join* $P_0 * \ldots * P_p$ can be defined as follows. Let $\Delta^p = \{s_0 \geq 0,\ldots,s_p \geq 0,\ s_0+\ldots+s_p = 1\}$ be the standard p-simplex. Then $P_0 * \ldots * P_p$ is the quotient space of $P_0 \times \ldots \times P_p \times \Delta^p$ over the following relation:

$$(x_0,\ldots,x_p,s) \sim (x'_0,\ldots,x'_p,s) \text{ if } s = (s_0,\ldots,s_p) \text{ and } x_i = x'_i \text{ whenever } s_i \neq 0. \tag{1.4}$$

Given a continuous surjective map $f_i : P_i \to Y$ for each $i = 0,\ldots,p$, the *fibered join* $P_0 *_Y \ldots *_Y P_p$ is defined as the quotient space of $P_0 \times_Y \ldots \times_Y P_p \times \Delta^p$ over the relation (1.4).

Definition 4. For a space Z, 1-*st suspension* of Z is defined as the suspension (see [**11**]) of $(Z \sqcup \{point\})$. For an integer $p > 0$, the p-th iteration of this operation will be called *p-th suspension of Z*.

Lemma 1. *Let $f_i : P_i \to Y$, $i = 0,\ldots,p$, be continuous surjective maps and $P = P_0 *_Y \ldots *_Y P_p$ their fibered join. There is a natural map $F : P \to Y$ induced by the maps $f_0,\ldots,f_p$. For a point $y \in Y$ the fiber $F^{-1}y$ coincides with the join $f_0^{-1}y * \ldots * f_p^{-1}y$ of the fibers of f_i.*

There is a natural map $\pi : P \to \Delta^p$. The fiber of π over an interior point of Δ^p is $P_0 \times_Y \ldots \times_Y P_p$. For each $i = 0,\ldots,p$, there is a natural embedding

$$\phi_i : P(i) = P_0 *_Y \ldots *_Y P_{i-1} *_Y P_{i+1} *_Y \ldots *_Y P_p \to P. \tag{1.5}$$

Its image coincides with $\pi^{-1}(\{s_i = 0\})$, and the space $P/\left(\bigcup_i \phi_i(P(i))\right)$ is homotopy equivalent to the p-th suspension of $P_0 \times_Y \ldots \times_Y P_p$.

Proof. Directly follows from Definitions 3, 4. □

Definition 5. Let $f : X \to Y$ be a surjective continuous map. Its *join space* $J^f(X)$ is the quotient space of the disjoint union of spaces

$$J^f_p(X) = \underbrace{X *_Y \ldots *_Y X}_{p+1 \text{ times}}, \quad p = 0,1,\ldots, \tag{1.6}$$

identifying $J^f_{p-1}(X)$ with each of its images $\phi_i(J^f_{p-1}(X))$ in $J^f_p(X)$ for $i = 0,\ldots,p$, where ϕ_i is defined in (1.5). When Y is a point, we write $J_p(X)$ instead of $J^f_p(X)$ and $J(X)$ instead of $J^f(X)$.

Lemma 2. *Let $\phi : J_p(X) \to J(X)$ be the natural map induced by the maps ϕ_i. Then $\phi(J_{p-1}(X))$ is contractible in $\phi(J_p(X))$.*

Proof. Let x be a point in X. For $t \in [0,1]$, the maps

$$g_t(x,x_1,\ldots,x_p,s) \mapsto (x,x_1,\ldots,x_p,(1-t+ts_0,ts_1,\ldots,ts_p))$$

define a contraction of $\phi_0(J_{p-1}(X))$ to the point $x \in X$ where X is identified with its embedding in $J_p(X)$ as $\pi^{-1}(1, 0, \ldots, 0)$. It is easy to see that the maps g_t are compatible with the equivalence relations in Definition 5 and define a contraction of $\phi(J_{p-1}(X))$ to a point in $\phi(J_p(X))$. □

LEMMA 3. *The join space $J(X)$ is homologically trivial.*

Proof. Any cycle in $J(X)$ belongs to $\phi(J_p(X))$ for some p, while according to Lemma 2 $\phi(J_p(X))$ is contractible in $J(X)$. Hence the cycle is homologous to 0. □

Proof of Theorem 1. Let f be closed. Let $F : J^f(X) \to Y$ be the natural map induced by f. Then F is also closed. Its fiber $F^{-1}y$ over a point $y \in Y$ coincides with the join space $J(f^{-1}y)$ which is homologically trivial according to Lemma 3. It follows that $\tilde{\bar{H}}^*(J(f^{-1}y)) \cong 0$, where $\bar{H}^*$ is the Alexander cohomology ([**17**], p. 308), since $\bar{H}^*(Z)) \cong H^*(Z)$ for any locally contractible space Z ([**17**], p. 340), in particular for a difference between CW-complex and a subcomplex.

Vietoris-Begle theorem ([**17**], p. 344) applied to $F : J^f(X) \to Y$, implies

$$\bar{H}^*(J^f(X), G) \cong \bar{H}^*(Y, G)$$

and therefore

$$H_*(J^f(X), G) \cong H_*(Y, G).$$

By Lemma 1, the space $J_p^f(X)/\left(\bigcup_{q<p} J_q^f(X)\right)$ is homotopy equivalent to the p-th suspension of W_p. Theorem 1 follows now from the spectral sequence associated with filtration of $J^f(X)$ by the spaces $J_p^f(X)$. □

REMARK 2. For a map f with 0-dimensional fibers, a similar spectral sequence, "image computing spectral sequence" was applied to problems in theory of singularities and topology by Vassiliev [**19**], Goryunov-Mond [**7**], Goryunov [**6**], Houston [**9**], and others.

REMARK 3. A continuous map $f : X \to Y$ is called *compact-covering* if any compact set in Y is an image of a compact set in X. This condition includes both the closed and the locally split cases and may be more convenient for applications. For a compact-covering $f : X \to Y$ Theorem 1 is also true. A proof will appear elsewhere.

2. *Alexander's duality and Mayer-Vietoris inequality*

Let

$$I_i^n := \bigcap_{1 \le j \le n} \{-i \le x_j \le i\} \subset \mathbb{R}^n.$$

Define the "thick boundary" $\partial I_i^n := I_{i+1}^n \setminus I_i^n$. The following lemma is a version of Alexander's duality theorem.

LEMMA 4. (Alexander's duality) *If $X \subset I_i^n$ is an open set in I_i^n, then for any $q \in \mathbb{Z}$, $q \le n-1$,*

$$H_q(I_i^n \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \partial I_i^n, \mathbb{R}). \tag{2.1}$$

If $X \subset I_i^n$ is a closed set in I_i^n, then for any $q \in \mathbb{Z}$, $q \le n-1$,

$$H_q(I_i^n \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup closure(\partial I_i^n), \mathbb{R}). \tag{2.2}$$

Proof. For definiteness let X be closed. Compactifying $\mathbb{R}^n$ at infinity as $\mathbb{R}^n \cup \infty \simeq S^n$, we have, by Alexander's duality [**11**],

$$\tilde{H}_q(S^n \setminus (X \cup closure(\partial I_i^n), \mathbb{R}) \cong \tilde{H}_{n-q-1}((X \cup closure(\partial I_i^n), \mathbb{R}).$$

The first group is isomorphic to $H_q(I_i^n \setminus X, \mathbb{R})$ when $q > 0$, and to $\tilde{H}_0(I_i^n \setminus X, \mathbb{R}) + \mathbb{R} \cong H_0(I_i^n \setminus X, \mathbb{R})$ when $q = 0$. Combining these two cases, we obtain (2.2). □

Lemma 5. (Mayer-Vietoris inequality) *Let $X_1, \dots, X_m \subset I_1^n$ be all open or all closed in I_1^n. Then*

$$\mathrm{b}_i\Big(\bigcup_{1 \le j \le n} X_j\Big) \le \sum_{J \subset \{1,\dots,n\}} \mathrm{b}_{i-|J|+1}\Big(\bigcap_{j \in J} X_j\Big)$$

and

$$\mathrm{b}_i\Big(\bigcap_{1 \le j \le n} X_j\Big) \le \sum_{J \subset \{1,\dots,n\}} \mathrm{b}_{i+|J|-1}\Big(\bigcup_{j \in J} X_j\Big),$$

where b_i is the ith Betti number.

Proof. A well-known corollary to Mayer-Vietoris sequence. □

3. *Thom-Milnor's and Khovanskii's bounds*

Necessary definitions regarding semi-Pfaffian and sub-Pfaffian sets can be found in [**10**], [**5**]. In this paper we consider only *restricted* sub-Pfaffian sets.

To apply our results to semialgebraic sets and to restricted sub-Pfaffian sets, defined by formulae with quantifiers, we need the following known upper bounds on Betti numbers for sets defined by quantifier-free formulae.

Let $X = \{\varphi\} \subset I_1^n$ be a semialgebraic set, where φ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$, f being polynomials in n variables with coefficients in $\mathbb{R}$, $\deg(f) < d$. We will refer to the sequence (n, s, d) as to *format* of φ. It follows from [**18**], [**12**], [**1**] that the sum of Betti numbers of X is

$$\mathrm{b}(X) \le O(sd)^n. \tag{3.1}$$

If $X = \{\varphi\}$ is a *compact* semialgebraic set, where φ is a Boolean combination with no negations of s atomic formulae of the kind either $f \ge 0$ or $f > 0$, f being polynomials in n variables, $\deg(f) < d$, then a combination of results from [**18**], [**12**], [**1**] and [**13**], [**21**] implies that that the sum of Betti numbers of X also satisfies (3.1).

Now let $X = \{\varphi\} \subset I_1^n$ be a semi-Pfaffian set, where φ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$, f being Pfaffian functions in an open domain $G \supset I_1^n$ of order ρ, degree (α, β), having a common Pfaffian chain with coefficients in $\mathbb{R}$. The sequence $(n, s, \alpha, \beta, \rho)$ is called *format* of φ. It follows from [**10**], [**22**] that the sum of Betti numbers of X is

$$\mathrm{b}(X) \le s^n 2^{\rho(\rho-1)/2} O(n\beta + \min\{n, \rho\}\alpha)^{n+\rho}. \tag{3.2}$$

Let $X \subset I_1^{n_0}$ be a semialgebraic set defined by a formula

$$Q_1 \mathbf{x}_1 Q_2 \mathbf{x}_2 \dots Q_\nu \mathbf{x}_\nu F(\mathbf{x}_0, \mathbf{x}_1, \dots \mathbf{x}_\nu), \tag{3.3}$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,n_i}) \in I_1^{n_i}$, and F is a quantifier-free Boolean formula with no negations having s atoms of the kind $f > 0$, where f's are polynomials with real coefficients of degrees less than d. The *cylindrical algebraic decomposition* technique from [**3**], [**20**] allows to bound from above the number of cells in a representation of X as a difference between a CW-complex and its subcomplex. In particular,

$$\mathrm{b}(X) \leq (sd)^{2^{O(n)}}. \tag{3.4}$$

A better upper bound can be obtained as follows. According to [**2**] (which refines [**8**], [**16**]), there exists a Boolean combination

$$\psi(\mathbf{x}_0) = \bigvee_{1 \leq i \leq I} \bigwedge_{1 \leq j \leq J_i} (g_{i,j}(\mathbf{x}_0) *_{i,j} 0),$$

such that $X = \{\psi(\mathbf{x}_0)\}$. Here

$$*_{i,j} \in \{=, <, >\}, \quad g_{i,j} \in \mathbb{R}[\mathbf{x}_0], \quad \deg(g_{i,j}) < d^{\prod_{i \geq 1} O(n_i)},$$

$$I < s^{(n_0+1)\prod_{i \geq 1}(n_i+1)} d^{(n_0+1)\prod_{i \geq 1} O(n_i)},$$

$$J_i < s^{\prod_{i \geq 1}(n_i+1)} d^{\prod_{i \geq 1} O(n_i)}.$$

Applying (3.1) to $X = \{\psi(\mathbf{x}_0)\}$, we get

$$\mathrm{b}(X_0) \leq s^{O(n_0^2 \prod_{i \geq 1} n_i)} d^{O(n_0^2) \prod_{i \geq 1} O(n_i)} \leq (sd)^{O(n_0^2) \prod_{i \geq 1} O(n_i)} \tag{3.5}$$

4. *Basic notation*

Let $X = \widetilde{X_0} = I_1^{n_0} \setminus X_0 \subset I_1^{n_0}$ be a set defined by a formula (0.1). For example, X could be a sub-Pfaffian or a semialgebraic set defined by (3.3), where F is a quantifier-free Boolean formula with no negations. For definiteness assume that $Q_1 = \exists$ and X is open in $I_1^{n_0}$.

Define

$$X_i := \{(\mathbf{x}_0, \dots, \mathbf{x}_i) |\, Q_{i+1} \mathbf{x}_{i+1} Q_{i+2} \mathbf{x}_{i+2} \dots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}$$

for odd i and

$$X_i := I_1^{n_0 + \dots + n_i} \setminus \{(\mathbf{x}_0, \dots, \mathbf{x}_i) | Q_{i+1} \mathbf{x}_{i+1} Q_{i+2} \mathbf{x}_{i+2} \dots Q_\nu \mathbf{x}_\nu ((\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu) \in X_\nu)\}$$

for even i. Then $\pi_i(X_i) = \widetilde{X_{i-1}}$, where $\pi_i : \mathbb{R}^{n_0 + \dots + n_i} \to \mathbb{R}^{n_0 + \dots + n_{i-1}}$ and tilde denotes the complement in $I_1^{n_0 + \dots + n_{i-1}}$.

For a set $I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1}$ define $\partial(I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1})$ as

$$(I_{i+1}^{m_i} \times I_i^{m_{i-1}} \times \dots \times I_2^{m_1}) \setminus (I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1})$$

for even i and as the closure of this difference for odd i.

Let $p_1, \dots, p_i$ be some positive integers to be specified later. Define

$$B_i^i := \partial(I_{i-1}^{n_0 + (p_1+1)n_1} \times I_{i-2}^{(p_2+1)n_2} \times \dots \times I_1^{(p_{i-1}+1)n_{i-1}}) \times I_1^{n_i}.$$

For any j, $i < j \leq \nu$ define $B_j^i := \widetilde{B_{j-1}^i} \times I_1^{n_j}$, where tilde denotes the complement in the appropriate cube.

Definition 6.

(i) Let $Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \ldots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\ldots+n_i}$, where $1 \le l \le i$, $v \ge i$, and let $J \subset \{(j_l, \ldots, j_i) | \, 1 \le j_k \le p_k+1, \, l \le k \le i\}$. Then define $\prod_{i,J}^{l} Y$ as an intersection of sets

$$\{(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_1^{(p_1+1)}, \ldots, \mathbf{x}_i^{(1)}, \ldots, \mathbf{x}_i^{(p_i+1)})|$$

$$\mathbf{x}_0 \in I_v^{n_0}, \; \mathbf{x}_k^{(m)} \in I_{v-k+1}^{n_k} \; (1 \le k \le l-1),$$

$$\mathbf{x}_k^{(m)} \in I_1^{n_k} \; (l \le k \le i), \; (\mathbf{x}_0, \mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_{l-1}^{(p_{l-1}+1)}, \mathbf{x}_l^{(j_l)}, \ldots, \mathbf{x}_i^{(j_i)}) \in Y\}$$

over all $(j_l, \ldots, j_i) \in J$.

(ii) Let $Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \ldots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\ldots+n_i+n_{i+1}}$. Define $\prod_{i,J}^{l,i+1} Y$ as an intersection of sets

$$\{(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_1^{(p_1+1)}, \ldots, \mathbf{x}_i^{(1)}, \ldots, \mathbf{x}_i^{(p_i+1)}, \mathbf{x}_{i+1})|$$

$$\mathbf{x}_0 \in I_v^{n_0}, \mathbf{x}_k^{(m)} \in I_{v-k+1}^{n_k} (1 \le k \le l-1), \mathbf{x}_k^{(m)} \in I_1^{n_k} (l \le k \le i), \; \mathbf{x}_{i+1} \in I_1^{n_{i+1}},$$

$$(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_{l-1}^{(p_{l-1}+1)}, \mathbf{x}_l^{(j_l)}, \ldots, \mathbf{x}_i^{(j_i)}, \mathbf{x}_{i+1}) \in Y\}$$

over all $(j_l, \ldots, j_i) \in J$.

(iii) If $l = i$ and $J = \{j | \, 1 \le j \le p_i + 1\}$ we use the notation $\prod_i^i Y$ for $\prod_{i,J}^{i} Y$.

Lemma 6. *Let*

$$Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \ldots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\ldots+n_i+n_{i+1}}.$$

Then for any $J \subset \{j | 1 \le j \le p_{i+1}+1\}$, $J' \subset \{(j_l, \ldots, j_i) | 1 \le j_k \le p_k+1, l \le k \le i\}$ *we have*

$$\prod_{i+1,J}^{i+1} \prod_{i,J'}^{l,i+1} Y = \prod_{i+1,J' \times J}^{l} Y.$$

Proof. Straightforward. □

Definition 7. Let Y, l, i, J be as in Definition 6. Define $\coprod_{i,J}^{l} Y$ and $\coprod_{i,J}^{l,i+1} Y$ similar to $\prod_{i,J}^{l} Y$ and $\prod_{i,J}^{l,i+1} Y$ respectively, replacing in Definition 6 "intersection" by "union".

Lemma 7. (De Morgan law)

$$\coprod_{i,J}^{l} Y = \left(\prod_{i,J}^{l} \widetilde{Y} \right)^{\sim};$$

$$\coprod_{i,J}^{l,i+1} Y = \left(\prod_{i,J}^{l,i+1} \widetilde{Y} \right)^{\sim},$$

where tildes denote complements in the appropriate cubes.

Proof. Straightforward. □

Definition 8. Let $t_i = n_0 + n_1(p_1+1) + \ldots + n_i(p_i+1)$. Define projection maps

$$\pi_i : \mathbb{R}^{n_0+\ldots+n_i} \to \mathbb{R}^{n_0+\ldots+n_{i-1}}$$

$$(\mathbf{x}_0, \dots, \mathbf{x}_i) \mapsto (\mathbf{x}_0, \dots, \mathbf{x}_{i-1}),$$

and for $j < i$,

$$\pi_{i,j} : \mathbb{R}^{t_j + n_{j+1} + \dots + n_i} \to \mathbb{R}^{t_j + n_{j+1} + \dots + n_{i-1}}$$

$$(\mathbf{x}_0, \mathbf{x}_1^{(1)}, \dots, \mathbf{x}_j^{(p_j+1)}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_i) \mapsto (\mathbf{x}_0, \mathbf{x}_1^{(1)}, \dots, \mathbf{x}_j^{(p_j+1)}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_{i-1}).$$

Lemma 8. *Let*

$$Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l + \dots + n_i + n_{i+1}}.$$

Then

$$\bigsqcup{}_{i,J}^{l} \pi_{i+1,l-1}(Y) = \pi_{i+1,i}\Big(\bigsqcup{}_{i,J}^{l,i+1} Y\Big).$$

Proof. Straightforward. □

5. *Case of a single quantifier block*

According to Theorem 1,

$$\mathrm{b}_{q_0}(X) = \mathrm{b}_{q_0}(\widetilde{X_0}) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}\Big(\prod{}_{1,J_1^1}^{1} X_1\Big), \tag{5.1}$$

where $J_1^1 = \{1, \dots, p_1+1\}$.

Let $\nu = 1$, then (3.3) turns into $\exists \mathbf{x}_1 F(\mathbf{x}_0, \mathbf{x}_1)$, where $X_1 = \{F(\mathbf{x}_0, \mathbf{x}_1)\}$ and $F(\mathbf{x}_0, \mathbf{x}_1)$ is a Boolean combination with no negations of s atomic formulae of the kind $f > 0$.

5.1. *Polynomial case*

Suppose that X_1 is semialgebraic, with f's being polynomials of degrees $\deg(f) < d$. For any $k \le \dim(X)$, we bound the Betti number $\mathrm{b}_k(X)$ from above in the following way. Observe that $\prod_{1,J_1^1}^{1} X_1$ is an open set in $I_1^{n_0+(p_1+1)n_1}$ definable by a Boolean combination with no negations of $(p_1+1)s$ atomic formulae of the kind $g > 0$, $\deg(g) < d$ in $t_1 = n_0 + (p_1+1)n_1$ variables.

According to (3.1), for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}\Big(\prod{}_{1,J_1^1}^{1} X_1\Big) \le O(p_1 s d)^{n_0+(p_1+1)n_1}.$$

Then due to (5.1), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} O(p_1 s d)^{n_0+(p_1+1)n_1} \le (ksd)^{O(n_0+kn_1)}.$$

5.2. *Pfaffian case*

Suppose that $X_1 \subset I_1^n$ is sub-Pfaffian, with f's being Pfaffian functions in an open domain $G \supset I_1^n$ of order ρ, degree (α, β), having a common Pfaffian chain. Observe that $\prod_{1,J_1^1}^{1} X_1$ is an open set definable by a Boolean combination with no negations of $(p_1+1)s$ atomic formulae of the kind $g > 0$, where g are Pfaffian functions in an open domain contained in $I_1^{n_0+(p_1+1)n_1}$ of degrees (α, β), order $(p_1+1)\rho$ in $n_0 + (p_1+1)n_1$ variables, having a common Pfaffian chain. According to (3.2), for any $q_1 \le \dim(X)$,

$$\mathrm{b}_{q_1}\Big(\prod{}^{1}_{1,J_1^1} X_1\Big) \le ((p_1+1)s)^{n_0+(p_1+1)n_1} 2^{(p_1+1)\rho((p_1+1)\rho-1)/2}\cdot$$

$$\cdot O((n_0+p_1n_1)\beta+\min\{p_1\rho, n_0+p_1n_1\}\alpha)^{n_0+(p_1+1)(n_1+\rho)}.$$

Then due to (5.1), for any $k \le \dim(X) \le n_0$,

$$\mathrm{b}_k(X) \le \sum_{p_1+q_1=k} \mathrm{b}_{q_1}\Big(\prod{}^{1}_{1,J_1^1} X_1\Big) \le$$

$$k((k+1)s)^{n_0+(k+1)n_1} 2^{(k+1)\rho((k+1)\rho-1)/2}\cdot$$

$$\cdot O((n_0+kn_1)\beta+\min\{k\rho, n_0+kn_1\}\alpha)^{n_0+(k+1)(n_1+\rho)}.$$

Let $d > \alpha+\beta$. Relaxing the obtained bound, we get

$$\mathrm{b}_k(X) \le (ks)^{O(n_0+kn_1)} 2^{(O(k\rho))^2}((n_0+kn_1)d)^{O(n_0+kn_1+k\rho)}.$$

6. *Cases of two and three quantifier blocks*

In this section we obtain a generalization of (5.1) to the case of two and three blocks of quantifiers, as a preparation for cumbersome general formulae in the next section. The case of three quantifier blocks is considered separately also because of a technical difficulty that appears first in that case (see the discussion after (6.1)).

Recall that

$$\pi_i:\ \mathbb{R}^{n_0+\ldots+n_i} \to \mathbb{R}^{n_0+\ldots+n_{i-1}},$$

for $j < i$,

$$\pi_{i,j}:\ \mathbb{R}^{t_j+n_{j+1}+\ldots+n_i} \to \mathbb{R}^{t_j+n_{j+1}+\ldots+n_{i-1}}.$$

Let $\nu = 3$, then the original formula becomes $\exists \mathbf{x}_1 \forall \mathbf{x}_2 \exists \mathbf{x}_3 ((\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \in X_3)$. Thereby,

$$X_1 = \{\forall \mathbf{x}_2 \exists \mathbf{x}_3((\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \in X_3)\},\ \widetilde{X_2} = \{\exists \mathbf{x}_3((\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \in X_3)\},$$

$X = \widetilde{X_0}$ is open in $I_1^{n_0}$.

According to Theorem 1,

$$\mathrm{b}_{q_0}(\widetilde{X_0}) \le \sum_{p_1+q_1=q_0} \mathrm{b}_{q_1}\Big(\prod{}^{1}_{1,J_1^1} X_1\Big).$$

Applying in succession Lemma 7 (De Morgan law), Lemma 4 (Alexander's duality), definitions of π_2 and $\pi_{2,1}$, and Lemma 8 we get

$$\mathrm{b}_{q_1}\Big(\prod{}^{1}_{1,J_1^1} X_1\Big) = \mathrm{b}_{q_1}\Big(\widetilde{\Big(\bigsqcup{}^{1}_{1,J_1^1} \widetilde{X_1}\Big)}\Big) \le$$

$$\le \mathrm{b}_{t_1-q_1-1}\Big(\bigsqcup{}^{1}_{1,J_1^1} \widetilde{X_1} \cup \partial I_1^{t_1}\Big) = \mathrm{b}_{t_1-q_1-1}\Big(\bigsqcup{}^{1}_{1,J_1^1} \pi_2(X_2) \cup \pi_{2,1}(\partial I_1^{t_1} \times I_1^{n_2})\Big) =$$

$$= \mathrm{b}_{t_1-q_1-1}\Big(\pi_{2,1}\Big(\bigsqcup{}^{1,2}_{1,J_1^1} X_2 \cup \partial I_1^{t_1} \times I_1^{n_2}\Big)\Big).$$

Due to Theorem 1, the last expression does not exceed

$$\sum_{p_2+q_2=t_1-q_1-1} \mathrm{b}_{q_2}\Big(\prod{}^{2}_{2}\Big(\bigsqcup{}^{1,2}_{1,J_1^1} X_2 \cup \partial I_1^{t_1} \times I_1^{n_2}\Big)\Big) =$$

$$= \sum_{p_2+q_2=t_1-q_1-1} \mathrm{b}_{q_2}\Big(\prod{}^2_2\Big(\bigsqcup{}^{1,2}_{1,J^1_1} X_2 \cup B^2_2\Big)\Big).$$

In a case of sub-Pfaffian or semialgebraic X it is now possible to estimate

$$\mathrm{b}_{q_2}\Big(\prod{}^2_2\Big(\bigsqcup{}^{1,2}_{1,J^1_1} X_2 \cup B^2_2\Big)\Big)$$

via the format of X_2. This completes the description of the case of two quantifier blocks. We now proceed to the case of three blocks.

Due to Lemma 7 (De Morgan law) and Lemma 4 (Alexander's duality),

$$\mathrm{b}_{q_2}\Big(\prod{}^2_2\Big(\bigsqcup{}^{1,2}_{1,J^1_1} X_2 \cup B^2_2\Big)\Big) = \mathrm{b}_{q_2}\Big(\widetilde{\Big(\bigsqcup{}^2_2\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big)\Big)}\Big) = \tag{6.1}$$

$$= \mathrm{b}_{t_2-q_2-1}\Big(\bigsqcup{}^2_2\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big) \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big).$$

From this point we could have proceeded in a "natural" way similar to the just considered case of two blocks, namely, replacing in the previous expression the set $\widetilde{X_2}$ by $\pi_3(X_3)$, then carrying the projection operator to the left to obtain an expression of the kind $\mathrm{b}_{t_2-q_2-1}(\pi_{3,2}(\ldots))$, and after that applying Theorem 1. However, carrying the projection operator through the symbol $\prod{}^{1,2}_{1,J^1_1}$ (which corresponds to an intersection of some cylindrical sets) would require an introduction of $p_1 n_2$ new variables. This would result in a significantly higher upper bound for $\mathrm{b}_{q_0}(X)$. Instead we reduce intersections to unions, then carrying the projection operator to the left does not require new variables.

More precisely, by Lemma 5 (Mayer-Vietoris inequality) expression (6.1) does not exceed

$$\sum_{1 \le k_2 \le p_2+1} \quad \sum_{\hat{J}^2_2 \subset \{1,\ldots,p_2+1\},\ |\hat{J}^2_2| = k_2}$$

$$\mathrm{b}_{t_2-q_2-k_2}\Big(\prod{}^2_{2,\hat{J}^2_2}\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big) \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big).$$

(We estimate a Betti number of the union of cylindrical sets from the definition of the symbol $\bigsqcup{}^2_2$ by a sum of Betti numbers of intersections of various combinations of these sets.)

By Lemma 6,

$$\mathrm{b}_{t_2-q_2-k_2}\Big(\prod{}^2_{2,\hat{J}^2_2}\Big(\prod{}^{1,2}_{1,J^1_1} \widetilde{X_2} \cap \widetilde{B^2_2}\Big) \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big) =$$

$$= \mathrm{b}_{t_2-q_2-k_2}\Big(\prod{}^1_{2,J^1_1 \times \hat{J}^2_2} \widetilde{X_2} \cap \prod{}^2_{2,J^2_1 \times \hat{J}^2_2} \widetilde{B^2_2} \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big),$$

with $J^2_1 = \{1\}$. By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1 \le s_2 \le q_2+k_2+1} \quad \sum_{J^1_2 \subset J^1_1 \times \hat{J}^2_2,\ J^2_2 \subset J^2_1 \times \hat{J}^2_2,\ |J^1_2|+|J^2_2| = s_2}$$

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^1_{2,J^1_2} \widetilde{X_2} \cup \bigsqcup{}^2_{2,J^2_2} \widetilde{B^2_2} \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big),$$

taking into the account that

$$\dim\Big(\bigsqcup{}^1_{2,J^1_2} \widetilde{X_2} \cup \bigsqcup{}^2_{2,J^2_2} \widetilde{B^2_2} \cup \partial(I_2^{t_1} \times I_1^{n_2(p_2+1)})\Big) \le t_2$$

and therefore

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^{1}_{2,J^1_2}\widetilde{X_2}\cup\bigsqcup{}^{2}_{2,J^2_2}\widetilde{B^2_2}\cup\partial(I^{t_1}_2\times I_1^{n_2(p_2+1)})\Big)=0$$

for $s_2 > q_2 + k_2 + 1$.

We have

$$\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^{1}_{2,J^1_2}\widetilde{X_2}\cup\bigsqcup{}^{2}_{2,J^2_2}\widetilde{B^2_2}\cup\partial(I^{t_1}_2\times I_1^{n_2(p_2+1)})\Big)=$$

$$=\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\bigsqcup{}^{1}_{2,J^1_2}\pi_3(X_3)\cup\bigsqcup{}^{2}_{2,J^2_2}\pi_{3,1}(B^2_3)\cup\pi_{3,2}(\partial(I^{t_1}_2\times I_1^{n_2(p_2+1)})\times I_1^{n_3})\Big)=$$

$$=\mathrm{b}_{t_2-q_2-k_2+s_2-1}\Big(\pi_{3,2}\Big(\bigsqcup{}^{1,3}_{2,J^1_2}X_3\cup\bigsqcup{}^{2,3}_{2,J^2_2}B^2_3\cup B^3_3\Big)\Big).$$

Due to Theorem 1 the last expression does not exceed

$$\sum_{p_3+q_3=t_2-q_2-k_2+s_2-1}\mathrm{b}_{q_3}\Big(\prod{}^{3}_{3}\Big(\bigsqcup{}^{1,3}_{2,J^1_2}X_3\cup\bigsqcup{}^{2,3}_{2,J^2_2}B^2_3\cup B^3_3\Big)\Big).$$

In case of a sub-Pfaffian or a semialgebraic X it is now possible to estimate

$$\mathrm{b}_{q_3}\Big(\prod{}^{3}_{3}\Big(\bigsqcup{}^{1,3}_{2,J^1_2}X_3\cup\bigsqcup{}^{2,3}_{2,J^2_2}B^2_3\cup B^3_3\Big)\Big)$$

via the format of X_3.

7. *Arbitrary number of quantifiers*

THEOREM 2. *For any i the Betti number $\mathrm{b}_{q_0}(X)$ does not exceed*

$$\sum_{p_1+q_1=q_0}\ \sum_{p_2+q_2=t_1-q_1-1}\ \sum_{1\le k_2\le p_2+1}\ \sum_{\hat{J}^2_2\subset\{1,\ldots,p_2+1\},\ |\hat{J}^2_2|=k_2} \tag{7.1}$$

$$\sum_{1\le s_2\le q_2+k_2+1}\ \sum_{J^1_2\subset J^1_1\times\hat{J}^2_2,\ J^2_2\subset J^2_1\times\hat{J}^2_2,\ |J^1_2|+|J^2_2|=s_2}\ \sum_{p_3+q_3=t_2-k_2+s_2-1}\cdots$$

$$\cdots\sum_{1\le k_{i-1}\le p_{i-1}+1}\ \sum_{\hat{J}^{i-1}_{i-1}\subset\{1,\ldots,p_{i-1}+1\},\ |\hat{J}^{i-1}_{i-1}|=k_{i-1}}\ \sum_{1\le s_{i-1}\le q_{i-1}+k_{i-1}+1}$$

$$\sum_{J^1_{i-1}\subset J^1_{i-2}\times\hat{J}^{i-1}_{i-1},\ldots,\ J^{i-1}_{i-1}\subset J^{i-1}_{i-2}\times\hat{J}^{i-1}_{i-1},\ |J^1_{i-1}|+\ldots+|J^{i-1}_{i-1}|=s_{i-1}}\ \sum_{p_i+q_i=t_{i-1}-q_{i-1}-k_{i-1}+s_{i-1}-1}$$

$$\mathrm{b}_{q_i}\Big(\prod{}^{i}_{i}\Big(\bigsqcup{}^{1,i}_{i-1,J^1_{i-1}}X_i\cup\bigcup_{2\le r\le i-1}\bigsqcup{}^{r,i}_{i-1,J^r_{i-1}}B^r_i\cup B^i_i\Big)\Big).$$

Proof. Induction on i. Suppose (7.1) is true. Due to Lemma 7 (De Morgan law) and Lemma 4 (Alexander's duality),

$$\mathrm{b}_{q_i}\Big(\prod{}^{i}_{i}\Big(\bigsqcup{}^{1,i}_{i-1,J^1_{i-1}}X_i\cup\bigcup_{2\le r\le i-1}\bigsqcup{}^{r,i}_{i-1,J^r_{i-1}}B^r_i\cup B^i_i\Big)\Big)=$$

$$=\mathrm{b}_{q_i}\Big(\widetilde{\Big(\bigsqcup{}^{i}_{i}\Big(\prod{}^{1,i}_{i-1,J^1_{i-1}}\widetilde{X_i}\cap\bigcap_{2\le r\le i-1}\prod{}^{r,i}_{i-1,J^r_{i-1}}\widetilde{B^r_i}\cap\widetilde{B^i_i}\Big)\Big)}\Big)\le$$

$$\leq \mathrm{b}_{t_i-q_i-1}\Big(\bigsqcup\nolimits_i^i\Big(\prod\nolimits_{i-1,J_{i-1}^1}^{1,i}\widetilde{X_i}\cap\bigcap_{2\leq r\leq i-1}\prod\nolimits_{i-1,J_{i-1}^r}^{r,i}\widetilde{B_i^r}\cap\widetilde{B_i^i}\Big)\cup$$

$$\cup\,\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\Big).$$

By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\leq k_i\leq p_i+1}\quad\sum_{\hat{J}_i^i\subset\{1,\ldots,p_i+1\},\ |\hat{J}_i^i|=k_i}$$

$$\mathrm{b}_{t_i-q_i-k_i}\Big(\prod\nolimits_{i,\hat{J}_i^i}^{i}\Big(\prod\nolimits_{i-1,J_{i-1}^1}^{1,i}\widetilde{X_i}\cap\bigcap_{2\leq r\leq i-1}\prod\nolimits_{i-1,J_{i-1}^r}^{r,i}\widetilde{B_i^r}\cap\widetilde{B_i^i}\Big)\cup$$

$$\cup\,\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\Big),$$

where, by Lemma 6,

$$\mathrm{b}_{t_i-q_i-k_i}\Big(\prod\nolimits_{i,\hat{J}_i^i}^{i}\Big(\prod\nolimits_{i-1,J_{i-1}^1}^{1,i}\widetilde{X_i}\cap\bigcap_{2\leq r\leq i-1}\prod\nolimits_{i-1,J_{i-1}^r}^{r,i}\widetilde{B_i^r}\cap\widetilde{B_i^i}\Big)\cup$$

$$\cup\,\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\Big)=$$

$$=\mathrm{b}_{t_i-q_i-k_i}\Big(\prod\nolimits_{i,J_{i-1}^1\times\hat{J}_i^i}^{1}\widetilde{X_i}\cap\bigcap_{2\leq r\leq i}\prod\nolimits_{i,J_{i-1}^r\times\hat{J}_i^i}^{r}\widetilde{B_i^r}\cup$$

$$\cup\,\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\Big),$$

where $J_{i-1}^i=\{1\}$. By Lemma 5 (Mayer-Vietoris inequality) the last expression does not exceed

$$\sum_{1\leq s_i\leq q_i+k_i+1}\quad\sum_{J_i^1\subset J_{i-1}^1\times\hat{J}_i^i,\ \ldots,\ J_i^i\subset J_{i-1}^i\times\hat{J}_i^i,\ |J_i^1|+\ldots+|J_i^i|=s_i}$$

$$\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1}\widetilde{X_i}\cup\bigcup_{2\leq r\leq i}\bigsqcup\nolimits_{i,J_i^r}^{r}\widetilde{B_i^r}\cup\,\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\Big).$$

We have

$$\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1}\widetilde{X_i}\cup\bigcup_{2\leq r\leq i}\bigsqcup\nolimits_{i,J_i^r}^{r}\widetilde{B_i^r}\cup\,\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\Big)=$$

$$=\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1}\pi_{i+1}(X_{i+1})\cup\bigcup_{2\leq r\leq i}\bigsqcup\nolimits_{i,J_i^r}^{r}\pi_{i+1,r-1}(B_{i+1}^r)\cup$$

$$\cup\,\pi_{i+1,i}(\partial(I_i^{n_0+(p_1+1)n_1}\times\ldots\times I_1^{(p_i+1)n_i})\times I_1^{n_{i+1}})\Big)=$$

$$=\mathrm{b}_{t_i-q_i-k_i+s_i-1}\Big(\pi_{i+1,i}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1,i+1}X_{i+1}\cup\bigcup_{2\leq r\leq i}\bigsqcup\nolimits_{i,J_i^r}^{r,i+1}B_{i+1}^r\cup B_{i+1}^{i+1}\Big)\Big).$$

Due to Theorem 1 the last expression does not exceed

$$\sum_{p_{i+1}+q_{i+1}=t_i-q_i-k_i+s_i-1}$$

$$\mathrm{b}_{q_{i+1}}\Big(\prod\nolimits_{i+1}^{i+1}\Big(\bigsqcup\nolimits_{i,J_i^1}^{1,i+1} X_{i+1} \cup \bigcup_{2\le r\le i} \bigsqcup\nolimits_{i,J_i^r}^{r,i+1} B_{i+1}^r \cup B_{i+1}^{i+1}\Big)\Big).$$

□

8. *Upper bounds for sub-Pfaffian sets*

We first estimate from above the number of additive terms in (7.1). These terms can be partitioned into $i-1$ groups of the kind

$$\sum_{1\le k_j\le p_j+1} \quad \sum_{\hat{J}_j^j\subset\{1,\ldots,p_j+1\},\ |\hat{J}_j^j|=k_j}$$

$$\sum_{1\le s_j\le q_j+k_j+1} \quad \sum_{J_j^1\subset J_{j-1}^1\times \hat{J}_j^j,\ldots,\ J_j^j\subset J_{j-1}^j\times \hat{J}_j^j,\ |J_j^1|+\ldots+|J_j^j|=s_j}$$

$$\sum_{p_{j+1}+q_{j+1}=t_j-q_j-k_j+s_j-1},$$

where $1\le j\le i-1$.

The number of terms in

$$\sum_{1\le k_j\le p_j+1} \quad \sum_{\hat{J}_j^j\subset\{1,\ldots,p_j+1\},\ |\hat{J}_j^j|=k_j}$$

is 2^{p_j+1}. The number of terms in

$$\sum_{1\le s_j\le q_j+k_j+1} \quad \sum_{J_j^1\subset J_{j-1}^1\times \hat{J}_j^j,\ldots,\ J_j^j\subset J_{j-1}^j\times \hat{J}_j^j,\ |J_j^1|+\ldots+|J_j^j|=s_j}$$

does not exceed $2^{j(q_j+k_j+1)}$. The number of terms in

$$\sum_{p_{j+1}+q_{j+1}=t_j-q_j-k_j+s_j-1}$$

does not exceed t_j+1.

It follows that the total number of terms in jth group does not exceed

$$2^{p_j+1+j(q_j+k_j+1)}(t_j+1)\le 2^{O(jt_{j-1})}.$$

Since $t_j=n_0+n_1(p_1+1)+\ldots+n_j(p_j+1)$, $p_l\le t_{l-1}$, and therefore $t_j\le 2^j n_0n_1\ldots n_j$, the number of terms in jth group does not exceed $2^{O(j2^j n_0n_1\ldots n_{j-1})}$. It follows that the total number of terms in (7.1) does not exceed $2^{O(i^2 2^i n_0 n_1\ldots n_{i-2})}$.

We now find an upper bound for

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits_\nu^\nu\Big(\bigsqcup\nolimits_{\nu-1,J_{\nu-1}^1}^{1,\nu} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \cup B_\nu^\nu\Big)\Big).$$

Assume that $X_\nu=\{F(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_\nu)\}$, where F is a quantifier-free Boolean formula with no negations having s atoms of the kind $f>0$, f's are polynomials or Pfaffian functions of degrees less than d or (α,β) respectively. In Pfaffian case, let functions f be defined in an open domain G by the same Pfaffian chain of order ρ. We assume without loss of generality that $I_\nu^{n_0+\ldots+n_\nu}\subset G$.

The set $\bigsqcup\nolimits_{\nu-1,J_{\nu-1}^1}^{1,\nu} X_\nu\subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having $|J_{\nu-1}^1|s\le s_{\nu-1}s\le (2t_{\nu-2}+1)s$ atoms of degrees less than d (for polynomials) or less than (α, β) (for Pfaffian functions) and at most $2t_{\nu-1} + 2n_\nu$ linear atoms (defining $I_1^{t_{\nu-1}+n_\nu}$).

For any $2 \le r \le \nu$ the set $B_r^r \subset \mathbb{R}^{t_{r-1}+n_r}$ is defined by a Boolean formula with no negations having $4t_{r-1} + 2n_r$ linear atomic inequalities. Therefore, all sets of the kind B_j^r for $j \ge r$ are defined by Boolean formulae with no negations having $4t_{r-1}+2(n_r+\ldots+n_j)$ linear inequalities. In particular, the set $B_\nu^r \subset \mathbb{R}^{t_{r-1}+n_r+\ldots+n_\nu}$ is defined by $4t_{r-1} + 2(n_r + \ldots + n_\nu)$ linear atomic inequalities.

For any $2 \le r \le \nu - 1$ the set $\bigsqcup_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having at most

$$(4t_{r-1} + 2(n_r + \ldots + n_\nu))|J_{\nu-1}^r| + 2t_{\nu-1} + 2n_\nu \le$$

$$\le (4t_{r-1} + 2(n_r + \ldots + n_\nu))s_{\nu-1} + 2t_{\nu-1} + 2n_\nu \le$$

$$\le (4t_{r-1} + 2(n_r + \ldots + n_\nu))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_\nu$$

linear atoms.

It follows that the set $\bigcup_{2\le r\le \nu-1} \bigsqcup_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \subset \mathbb{R}^{t_{\nu-1}+n_\nu}$ is defined by a Boolean formula with no negations having at most

$$((4t_{\nu-1} + 2(n_2 + \ldots + n_\nu))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_\nu)(\nu - 2)$$

linear atoms.

The set

$$\prod\nolimits_\nu^\nu \Big(\bigsqcup\nolimits_{\nu-1,J_{\nu-1}^1}^{1,\nu} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \cup B_\nu^\nu\Big) \subset \mathbb{R}^{t_\nu} \tag{8.1}$$

is defined by a Boolean formula with no negations having at most

$$((2t_{\nu-2}+1)s+2t_{\nu-1}+2n_\nu+((4t_{\nu-1}+2(n_2+\ldots+n_\nu))(2t_{\nu-2}+1)+2t_{\nu-1}+2n_\nu)(\nu-2))\cdot$$

$$\cdot(t_{\nu-1} + 1) \le s t_{\nu-1}^{O(1)}$$

atoms of degrees less than d for polynomials or less than (α, β) for Pfaffian functions.

Similar calculation shows that, in the Pfaffian case, the set (8.1) is defined by Pfaffian functions having the order at most $\rho(2t_{\nu-2}+1)(t_{\nu-1}+1) \le O(\rho t_{\nu-2} t_{\nu-1})$.

8.1. *Polynomial case*

Let functions f in formula F be polynomials of degrees $\deg(f) < d$. Then, according to (3.1),

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits_\nu^\nu \Big(\bigsqcup\nolimits_{\nu-1,J_{\nu-1}^1}^{1,\nu} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits_{\nu-1,J_{\nu-1}^r}^{r,\nu} B_\nu^r \cup B_\nu^\nu\Big)\Big) \le$$

$$\le O(ds)^{t_\nu} t_{\nu-1}^{O(t_\nu)} \le (2^\nu d s n_0 n_1 \ldots n_{\nu-1})^{O(2^\nu n_0 n_1 \ldots n_\nu)}.$$

Using (7.1) in case $i = \nu$, we get

$$\mathrm{b}_{q_0}(X) \le (2^{\nu^2} d s n_0 n_1 \ldots n_{\nu-1})^{O(2^\nu n_0 n_1 \ldots n_\nu)}$$

(compare with (3.4) and (3.5)).

8.2. *Pfaffian case*

Let f be Pfaffian functions in an open domain $G \supset I_1^n$ of order ρ, degree (α, β), having a common Pfaffian chain. Then, according to (3.2),

$$\mathrm{b}_{q_\nu}\Big(\prod\nolimits_\nu^\nu\Big(\bigsqcup\nolimits_{\nu-1,J^1_{\nu-1}}^{1,\nu} X_\nu \cup \bigcup_{2\le r\le \nu-1} \bigsqcup\nolimits_{\nu-1,J^r_{\nu-1}}^{r,\nu} B^r_\nu \cup B^\nu_\nu\Big)\Big) \le$$

$$\le 2^{O(\rho^2 t_{\nu-2}^2 t_{\nu-1}^2)} (s t_{\nu-1})^{O(t_\nu)} O(t_\nu \beta + \min\{t_\nu, \rho\}\alpha)^{O(t_\nu + \rho t_{\nu-2} t_{\nu-1})} \le$$

$$\le 2^{O(\rho^2 2^{4\nu} n_0^4 n_1^4 \dots n_{\nu-2}^4 n_{\nu-1}^2)} s^{O(2^\nu n_0 n_1 \dots n_\nu)} \cdot$$

$$\cdot (2^\nu n_0 n_1 \dots n_\nu (\alpha+\beta))^{O(2^\nu n_0 n_1 \dots n_\nu + \rho 2^{2\nu} n_0^2 n_1^2 \dots n_{\nu-2}^2 n_{\nu-1})}.$$

Using (7.1) in case $i = \nu$, we get

$$\mathrm{b}_{q_0}(X) \le 2^{O(\nu 2^\nu n_0 n_1 \dots n_\nu + \rho^2 2^{4\nu} n_0^4 n_1^4 \dots n_{\nu-2}^4 n_{\nu-1}^2)} s^{O(2^\nu n_0 n_1 \dots n_\nu)} \cdot$$

$$\cdot (n_0 n_1 \dots n_\nu (\alpha+\beta))^{O(2^\nu n_0 n_1 \dots n_\nu + \rho 2^{2\nu} n_0^2 n_1^2 \dots n_{\nu-2}^2 n_{\nu-1})}.$$

Introducing the notations:

$$u_\nu := 2^\nu n_0 n_1 \dots n_\nu, \quad v_\nu := 2^{2\nu} n_0^2 n_1^2 \dots n_{\nu-2}^2 n_{\nu-1},$$

we can rewrite this bound in a more compact form

$$\mathrm{b}_{q_0}(X) \le 2^{O(\nu u_\nu + \rho^2 v_\nu^2)} s^{O(u_\nu)} (u_\nu(\alpha+\beta))^{O(u_\nu + \rho v_\nu)}.$$

Acknowledgements. Authors are very grateful to D. Dugger, D.B. Fuchs, and V.A. Vassiliev for useful discussions.

References

1. S. Basu, 'On bounding Betti numbers and computing Euler characteristic of semi-algebraic sets', *Discrete and Comput. Geom.* 22 (1999) 1–18.

2. S. Basu, R. Pollack and M.-F. Roy, 'On the combinatorial and algebraic complexity of quantifier elimination', *Journal of the ACM* 43 (1996) 1002–1045.

3. G.E. Collins, 'Quantifier elimination for real closed fields by cylindrical algebraic decomposition', *Lecture Notes in Computer Science* 33 (1975) 134–183.

4. D. Dugger, D. C. Isaksen, 'Hypercovers in topology', Preprint, 2001.

5. A. Gabrielov, N. Vorobjov, 'Complexity of cylindrical decompositions of sub-Pfaffian sets', *J. Pure and Appl. Algebra* 164 (2001) 179–197.

6. V. V. Goryunov, 'Semi-simplicial resolutions and homology of images and discriminants of mappings', *Proc. London Math. Soc.* 70 (1995) 363–385.

7. V. V. Goryunov, D. M. Q. Mond, 'Vanishing cohomology of singularities of mappings', *Compositio Math.* 89 (1993) 45–80.

8. J. Heintz, M.-F. Roy and P. Solernó, 'Sur la complexité du principe de Tarski-Seidenberg', *Bull. Soc. Math. France* 118 (1990) 101–126.

9. K. Houston, 'An introduction to the image computing spectral sequence', in: *Singularity Theory* 305–324, London Math. Soc. Lecture Notes Ser., 263, Combridge Univ. Press, Cambridge, 1999.

10. A. G. Khovanskii, *Fewnomials*, Translations of Mathematical Monographs 88 (AMS, Providence, RI, 1991).

11. W. S. Massey, *A Basic Course in Algebraic Topology* (Springer-Verlag, New York, 1991).

12. J. Milnor, 'On the Betti numbers of real varieties', *Proc. AMS* 15 (1964) 275–280.

13. J. L. Montaña, J. E. Morais and L. M. Pardo, 'Lower bounds for arithmetic networks II: sum of Betti numbers', *Applic. Algebra Eng. Commun. Comp.* 7 (1996) 41–51.

14. O. A. Oleinik, 'Estimates of the Betti numbers of real algebraic hypersurfaces' (Russian), *Mat. Sbornik* 28 (1951) 635–640.

15. O. A. Oleinik, I. G. Petrovskii, 'On the topology of real algebraic hypersurfaces' (Russian), *Izv. Acad. Nauk SSSR* 13 (1949) 389–402. English transl.: *Amer. Math. Soc. Transl.* 7 (1962) 399–417.

16. J. Renegar, 'On the computetional complexity and geometry of the first order theory of reals, I–III', *J. Symbolic Comput.* 13 (1992) 255–352.

17. E. Spanier, *Algebraic Topology* (Springer-Verlag, New York-Berlin, 1981).

18. R. Thom, 'Sur l'homologie des variétés algebriques réelles', in: *Differential and Combinatorial Topology* 255–265 (Princeton University Press, Princeton, 1965).

19. V. A. Vassiliev, *Complements of Discriminants of Smooth Maps: Topology and Applications*, Translations of Mathematical Monographs, 98 (AMS, Providence, RI, 1992).

20. H. R. Wüthrich, 'Ein Entschedungsvefahren für die Theorie der reell-abgeschlossenen Körper', *Lecture Notes in Computer Science* 43 (1976) 138–162.

21. A. C. C. Yao, 'Decision tree complexity and Betti numbers', in: *Proceedings of 26th ACM Symp. on Theory of Computing, Montreal, Canada* 615–624 (ACM Press, New York, 1994).

22. T. Zell, 'Betti numbers of semi-Pfaffian sets', *J. Pure Appl. Algebra* 139 (1999) 323–338.

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