INTERSECTION MATRICES FOR CERTAIN SINGULARITIES

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In this paper a method is determined for computing intersection matrices in homologies of a noncritical level manifold of a complex analytic function of the form P(x) + Q(y) in a neighborhood of an isolated singular point. As a corollary, the intersection matrices are computed and the generators of the monodromy group are determined for the singularities $\sum x_i^{a_i}$.

Introduction

As is well-known (see [1-3]), a noncritical level manifold of a complex analytic function in a neighborhood of an isolated singular point is homotopically equivalent to a bouquet of spheres of mean dimension ("vanishing cycles"). The homology group of mean dimension of this manifold is generated by these vanishing cycles. The intersection index of these cycles defines a bilinear form on this homology group. Closely connected with this form is the monodromy group of the singularity, which acts in the homologies of the noncritical level manifold (the image of the representation of the local fundamental group of the complement of the bifurcation diagram of the singularity).

This paper uses special bases consisting of vanishing cycles in the homology group (distinguished and weakly distinguished bases) to investigate the bilinear form and the monodromy group of the singularity.

If distinguished bases in the homologies are known for the singularities of P(x) and Q(y), then from these bases a distinguished basis can be constructed for P(x) + Q(y), whose intersection matrix is expressed by a simple formula in terms of the intersection matrices of the distinguished bases for the singularities of P(x) and Q(y).

The monodromy group of a singularity is uniquely defined by the intersection matrix of a weakly distinguished basis: it is generated by the reflections in the hyperplanes that are orthogonal (in the sense of the bilinear form of the intersections) to the elements of the basis.

As a corollary, distinguished bases are constructed and the intersection matrices are computed for the singularities $\sum x_i^{a_i}$.

For the "parabolic" singularities $x^3 + y^3 + z^3$, $x^4 + y^4 + z^2$, and $x^6 + y^3 + z^2$ the monodromy groups are computed.

The method of constructing a basis for P(x) + Q(y) given in this paper resembles the method of M. Sebastiani and R. Thom [4] for computing the Picard–Lefschetz monodromy operator for a singularity of the form P(x) + Q(y).

Another method of computing intersection matrices and monodromies was given by E. Brieskorn in a report to the conference in Tbilisi in October 1972. His method is based on first investigating the generators and relations of the fundamental group of the complement of the bifurcation diagram, and then obtaining, with the help of the Picard–Lefschetz theorem, information about the intersections of the vanishing cycles. Brieskorn pointed out that Lazzeri computed the intersection matrix for $x^a + y^3$ by this method.

F. Pham [6] found a basis for the singularity $\sum x_i^{a_i}$ and computed the intersection matrix of this basis. The basis for this singularity constructed in the present paper can be deformed to the basis given by

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Pham, and the intersection matrix of this basis can be obtained from his formulas. V. I. Arnol'd gave a direct proof of the fact that Pham's basis is distinguished.

The structure of the monodromy group of "parabolic" singularities was determined by J. Milnor in a letter to Arnol'd on October 31, 1972.

I would like to take this opportunity to express my deep thanks to V. I. Arnol'd for numerous useful discussions.

1. Distinguished Bases

Let P_0 be an analytic function with an isolated singularity at the origin in C^p. The number $\mu = \dim_C C\{x\} \Big/ \left(\frac{\partial P_0(x)}{\partial x_i}\right)$ will be called the multiplicity of the singularity of P_0. Let $P_\lambda(\lambda \in C^\mu)$ be a versal deformation of P_0 (see [2]) and $\Sigma \subset C^\mu$ the bifurcation diagram of P_λ, i.e., the set of λ such that P_λ has zero as a critical value.

For sufficiently small $r > 0$ and sufficiently small (less than some function of r) λ not belonging to Σ $\{x: |x| < r, P_\lambda(x) = 0\}$ is a real $(2p-2)$-dimensional manifold homotopically equivalent to a bouquet μ of spheres of dimension $p-1$, and its $(p-1)$-dimensional homologies are generated by Picard–Lefschetz vanishing cycles.*

In the sequel we shall not indicate the number r explicitly nor the related constraints on x and λ.

Let $\lambda = (\lambda', u)$ $(\lambda' \in C^{\mu-1}, u \in C)$ be a decomposition of the parameter space of the versal deformation such that $P_\lambda = P_{(\lambda', 0)} - u$. If λ' does not belong to some proper analytic subset Δ of $C^{\mu-1}$, then P_λ has μ nondegenerate critical points, and all its critical values are distinct. A function P_λ that satisfies this condition will be called a morsovization of P_0 and denoted simply by P.

Definition 1. Let $P(x)$ $(x \in C^p)$ be an analytic function, $\check{x}$ a nondegenerate critical point of $P(x)$, $\check{u} = P(\check{x})$ the corresponding critical value, and u^0 some noncritical value of $P(x)$. A cycle $e \in \tilde{H}_{p-1}(\{P(x) = u^0\}, Z)$ is called vanishing (along the path u(s)) if the following conditions are satisfied:

1) in the plane of values of P there is defined a smooth path u(s) $(s \in [0, 1])$ that is equal to u^0 for $s = 0$, $\check{u}$ for $s = 1$, and does not pass through a critical value of P for $s \neq 1$;

2) for $s < 1$ smooth spheres $e(s) \subset \{x: P(x) = u(s)\}$ are defined such that $e(0) = e$, the mapping $\bigcup_{0 \leqslant s < 1} e(s) \to [0, 1)$ is a smooth fibration, and, if $1 - s$ is sufficiently small, there exists a change of coordinates $x \to x'$ in a neighborhood of the point $\check{x}$ ($\check{x}$ is the origin of x') such that $P(x') = \check{u} + \sum x_i'^2$ and $e(s) = \sqrt{\check{u} - u(s)}S^{p-1}$, where $S^{p-1} = \{x' \in R^p, \sum x_i'^2 = 1\}$ is the standard sphere.

Definition 2. Let $P_0(x)$ be a function with an isolated singularity of multiplicity μ, $P(x)$ a morsovization of $P_0(x)$, $u_m (m = 1, \ldots, \mu)$ the critical values of P(x), and u^0 a noncritical value of P(x).

A basis (e_m) in $\tilde{H}_{p-1}(\{P(x) = u^0\}, Z)$ is said to be distinguished if:

1) the e_m are vanishing cycles and the corresponding paths $u_m(s)$ are equal to u_m for $s = 1$;

2) the $u_m(s)$ are non-self-intersecting paths, and for $m' \neq m$ the paths $u_m(s)$ and $u_{m'}(s)$ intersect only at u^0;

3) $\arg \frac{du_{m+1}}{ds}(0) < \arg \frac{du_m}{ds}(0)$.

Remark 1. The existence of a distinguished basis was essentially proved in the addendum to Brieskorn's article [3]. Moreover, it follows from Brieskorn's arguments that any system of cycles that satisfies conditions 1) and 2) of Definition 2 is a basis in $\tilde{H}_{p-1}(\{P(x) = u^0\}, Z)$.

Remark 2. By deforming u^0 and the system of paths $u_m(s)$, we can always guarantee that, in addition to conditions 1)–3) of Definition 2, the following conditions will be satisfied:

*Henceforth, by the homologies of a space X we mean (and denote by $\tilde{H}(X, Z)$) integral reduced homologies, i.e., the kernel of the natural mapping $H_*(X, Z) \to H_*$ (a point of Z).

4) $\mathrm{Re}\, u^0 < \mathrm{Re}\, u_m$ for all m;

5) $\mathrm{Re}\, u_m(s) > \mathrm{Re}\, u^0$ for $s > 0$.

2. Intersection Matrices

THEOREM 1. Let $P_0(x)$ ($x \in \mathbf{C}^p$) and $Q_0(y)$ ($y \in \mathbf{C}^q$) be analytic functions with isolated singularities of multiplicities μ and ν, respectively, and let P(x) and Q(y) be morsovizations of them. Let (e_m) [respectively, (h_n)] be a distinguished basis in $\tilde{H}_{p-1}(\{P(x) = u^0\}, \mathbf{Z})$ (respectively, in $\tilde{H}_{q-1}(\{Q(y) = v^0\}, \mathbf{Z})$). Consider the function $P_0(x) + Q_0(y)$. It has an isolated singularity of multiplicity $\mu\nu$. We can choose P(x) and Q(y) so that $P(x) + Q(y)$ is a morsovization of $P_0(x) + Q_0(y)$ and $u^0 + v^0$ is a noncritical value of $P(x) + Q(y)$.

There exists a basis (γ_{mn}) in $\tilde{H}_{p+q-1}(\{P(x) + Q(y) = u^0 + v^0\}, \mathbf{Z})$ whose intersection matrix is defined by the formulas

$$(\gamma_{mn}, \gamma_{mn}) = \begin{cases} 2(-1)^{\frac{p+q-1}{2}}, & \text{if } p+q-1 \text{ is even} \\ 0, & \text{if } p+q-1 \text{ is odd} \end{cases} \tag{1}$$

$$(\gamma_{mn}, \gamma_{mn'}) = \mathrm{sgn}(n'-n)^p (-1)^{pq + \frac{p(p-1)}{2}} (h_n, h_{n'}), \text{ if} \tag{2}$$

$$(\gamma_{mn}, \gamma_{m'n}) = \mathrm{sgn}(m'-m)^q (-1)^{pq + \frac{q(q-1)}{2}} (e_m, e_{m'}), \text{ if } m \neq m';\ n \neq n'; \tag{3}$$

$$(\gamma_{mn}, \gamma_{m'n'}) = 0, \quad \text{if} \quad \mathrm{sgn}(m'-m)\,\mathrm{sgn}(n'-n) = -1;$$

$$(\gamma_{mn}, \gamma_{m'n'}) = \mathrm{sgn}(m'-m) \quad (-1)^{pq} (e_m, e_{m'}) (h_n, h_{n'}), \tag{5}$$

if $\mathrm{sgn}(m'-m)\ \mathrm{sgn}(n'-n) = 1$.

If the pairs (m, n) are ordered lexicographically (i.e., $(m', n') > (m, n)$ if $m' > m$, or $m' = m$ and $n' > n$), then (γ_{mn}) is a distinguished basis.

Proof. 1) Construction of the Cycles γ_{mn}. Let C_u (respectively C_v) be the range of values of P(x) (respectively Q(y)). Let u_m ($m = 1, \ldots, \mu$) and v_n ($n = 1, \ldots, \nu$) be critical values of P(x) and Q(y) respectively; let $u_m(s)$ and $v_n(s)$ ($s \in [0, 1]$) be paths in C_u and C_v corresponding to the distinguished bases e_m and h_n. We shall assume that the conditions of Remark 2 are satisfied for u^0, v^0, $u_m(s)$, and $v_n(s)$.

Define a mapping ρ: $C_v \to C_u$ by the formula $\rho(v) = u^0 + v^0 - v$. The paths $\rho v_n(s)$ join u^0 and ρv_n and lie (for $s > 0$) in the half-plane $\mathrm{Re}\, u < \mathrm{Re}\, u^0$. The paths $u_m(s)$ join u^0 and u_m and lie in the half-plane $\mathrm{Re}\, u > \mathrm{Re}\, u^0$ (Fig. 1). Set

$$\pi'_{mn}(t) = \begin{cases} \rho v_n(1-2t) & \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ u_m(2t-1) & \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

and let $\pi_{mn}(t)$ be smooth paths obtained from $\pi'_{mn}(t)$ by a homotopy in a neighborhood of u^0 and equal to u^0 for $t = 1/2$.

Let $e_m(s) \subset \{P(x) = u_m(s)\}$ and $h_n(s) \subset \{Q(y) = v_n(s)\}$ be families of spheres corresponding to the vanishing cycles e_m and h_n. Set $e_m(1) = x_{(m)}$ and $h_n(1) = y_{(n)}$, where $x_{(m)}$ and $y_{(n)}$ are points to which the spheres $e_m(s)$ and $h_n(s)$ shrink as $s \to 1$.

For each n we can define a smooth deformation $e_{mn}(t)$ of e_m along the path $\pi_{mn}(t)$, such that $e_{mn}(t) \subset \{P(x) = \pi_{mn}(t)\}$, $e_{mn}(1/2) = e_m$, $e_{mn}(t) = e_m(2t-1)$, if $1-t$ is sufficiently small.

Moreover, $\bigcup_{0 \leqslant t < 1} e_{mn}(t)$ is diffeomorphic to $e_m \times [0, 1)$ and $\bigcup_{0 \leqslant t \leqslant 1} e_{mn}(t)$ is a smooth disk with boundary $e_{mn}(0)$. Similarly, for each m we define a smooth deformation $h_{mn}(t)$ of h_n along the path $\pi_{mn}(t)$, such that $h_{mn}(t) \subset \{Q(y) = \rho^{-1}\pi_{mn}(t)\}$, $h_{mn}(1/2) = h_n$, $h_{mn}(t) = h_n(1-2t)$, if t is sufficiently small. $\bigcup_{0 < t \leqslant 1} h_{mn}(t)$ is diffeomorphic to $h_n \times (0, 1]$, and $\bigcup_{0 \leqslant t \leqslant 1} h_{mn}(t)$ is a smooth disk with boundary $h_{mn}(1)$.

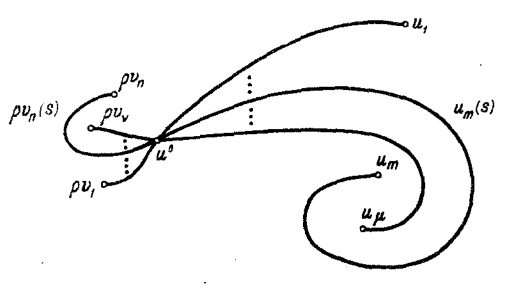

Fig. 1

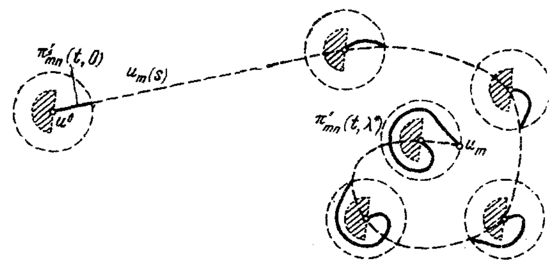

Fig. 2

Set $\gamma_{mn} = \bigcup_{0\leqslant t\leqslant 1} e_{mn}(t) \times h_{mn}(t)$. The set γ_{mn} is contained in $\{P(x) + Q(y) = u^0 + v^0\}$ and is diffeomorphic to a sphere. An orientation on γ_{mn} is introduced so that the orientation of the direct product is induced on $\bigcup_{0<t<1} e_{mn}(t) \times h_{mn}(t) \subset \gamma_{mn}$, which is diffeomorphic to $e_m \times h_n \times (0, 1)$ (we assume that the interval $(0, 1)$ is positively oriented).

2. (γ_{mn}) is a Distinguished Basis. Suppose that the function $Q(y)$, the point v^0, and the paths $v_n(s)$ are chosen so that $|v_n(s) - v^0| < \varepsilon$, where ε is sufficiently small. Then the function $P(x) + Q(y)$ has $\mu\nu$ distinct critical values $u_m + v_n$, and so it is a morsovization of $P_0(x) + Q_0(y)$.

Let C_W be the range of values of $P(x) + Q(y)$. We shall construct smooth paths $w_{mn}(\lambda)$ ($\lambda \in [0,1]$, $w_{mn}(0) = u^0 + v^0$, $w_{mn}(1) = u_m + v_n$), in C_W that satisfy the conditions of Definition 2, and a smooth homotopy $\pi_{mn}(t, \lambda)$ of the paths $\pi_{mn}(t)$ in C_u such that:

1*) $\pi_{mn}(t, 0) = \pi_{mn}(t)$;

2*) $\pi_{mn}(0, \lambda) = w_{mn}(\lambda) - v_n$, $\pi_{mn}(1, \lambda) = u_m$;

3*) for fixed λ the path $\pi_{mn}(t, \lambda)$ does not contain points $u_{m'}$ and $w_{mn}(\lambda) - v_{n'}$ for $m' \neq n$, $n' \neq n$ or the points u_m and $w_{mn}(\lambda) - v_n$ for $t \neq 0, 1$;

4*) $\pi_{mn}(t, \lambda) = (1 - t)u_m + t(w_{mn}(\lambda) - v_n)$, if $(1 - \lambda)$ is sufficiently small.

By means of the homotopy $\pi_{mn}(t, \lambda)$ we can construct a smooth deformation $e_{mn}(t, \lambda)$ of the family of spheres $e_{mn}(t)$, such that:

1) $e_{mn}(t, \lambda) \subset \{P(x) = \pi_{mn}(t, \lambda)\}$;

2) $e_{mn}(t, 0) = e_{mn}(t)$;

3) if we let $1 - \lambda$ be sufficiently small and if we set $w_{mn}(\lambda) - u_m - v_n = r(\lambda)$ ($r(\lambda) \to 0$ as $\lambda \to 1$), then $\pi_{mn}(t, \lambda) = u_m + tr(\lambda)$, and there exists a change of variables $x \leftrightarrow x'$ in a neighborhood of $x_{(m)}$ such that $P(x') = u_m + \sum x_i'^2$ and $e_{mn}(t, \lambda) = \sqrt{tr(\lambda)}\, S^{p-1}$.

Similarly, we construct a smooth deformation $h_{mn}(t, \lambda)$ of the family of spheres $h_{mn}(t)$, such that:

1) $h_{mn}(t, \lambda) \subset \{Q(y) = w_{mn}(\lambda) - \pi_{mn}(t, \lambda)\}$;

2) $h_{mn}(t, 0) = h_{mn}(t)$;

3) if we let $1 - \lambda$ be sufficiently small, then $w_{mn}(\lambda) - \pi_{mn}(t, \lambda) = v_n + (1 - t)r(\lambda)$, and there exists a change of variables $y \leftrightarrow y'$ in a neighborhood of $y_{(n)}$ such that $Q(y') = v_n + \sum y_i'^2$ and $h_{mn}(t, \lambda) = \sqrt{(1 - t)r(\lambda)}\, S^{q-1}$.

Set $\gamma_{mn}(\lambda) = \bigcup_{0\leqslant t\leqslant 1} e_{mn}(t, \lambda) \times h_{mn}(t, \lambda)$. Obviously, $\gamma_{mn}(\lambda) \subset \{P(x) + Q(y) = w_{mn}(\lambda)\}$, $\gamma_{mn}(0) = \gamma_{mn}$ and, if $1 - \lambda$ is sufficiently small, then in a neighborhood of $(x_{(m)}, y_{(n)})$

$$P(x') + Q(y') = u_m + v_n + \sum x_i'^2 + \sum y_i'^2,$$

$$\gamma_{mn}(\lambda) = \sqrt{r(\lambda)} \bigcup_{0\leqslant t\leqslant 1} (\sqrt{t}\, S^{p-1} \times \sqrt{1-t}\, S^{q-1}) = \sqrt{r(\lambda)}\, S^{p+q-1},$$

where $S^{p+q-1} = \{(x', y') \subset \mathbf{R}^{p+q}, \sum x_i'^2 + \sum y_i'^2 = 1\}$ is the standard sphere. Therefore the cycles γ_{mn} vanish along the paths $w_{mn}(\lambda)$, and from Remark 1 it follows that (γ_{mn}) is a distinguished basis.

Let us turn to the construction of the paths $w_{mn}(\lambda)$ and $\pi_{mn}(t, \lambda)$. More precisely, we shall construct (nonsmooth) paths $w'_{mn}(\lambda)$ and homotopies $\pi'_{mn}(t, \lambda)$, which can then be replaced by homotopic smooth paths $w_{mn}(\lambda)$ that satisfy the conditions of Definition 2 and smooth homotopies $\pi_{mn}(\lambda)$ that satisfy conditions 1*)-4*).

The construction will be in three stages.

1) If ε is sufficiently small, then the path $u_m(s)$ satisfies the following conditions:

the disk $|u - u_m(s)| \leqslant 4\varepsilon$ does not contain points $u_{m'}(m' \neq m)$;†

there exists an s^* such that

$|u_m - u_m(s)| \leqslant 2\varepsilon$ for $s^* \leqslant s \leqslant 1$, and $|u_m - u_m(s)| > 2\varepsilon$ for $0 \leqslant s < s^*$,

for $0 \leqslant \lambda \leqslant s^*/2$ the path $u_m(s)$ $(2\lambda \leqslant s \leqslant 1)$ intersects the circle $\Gamma_\lambda = \{u : |u - u_m(2\lambda)| = 2\varepsilon\}$ at precisely one point $u_m(s_\lambda)$.

Let $\lambda^* = s^*/2$. For $0 \leqslant \lambda \leqslant \lambda^*$ set $w'_{mn}(\lambda) = v^0 + u_m(2\lambda)$. Set $\pi'_{mn}(t, \lambda) = u_m((2t-1)(1-2\lambda) + 2\lambda)$ if $s_\lambda \leqslant (2t-1)(1-2\lambda) + 2\lambda \leqslant 1$, i.e., $\frac{1}{2} + \frac{s_\lambda - 2\lambda}{2(1-2\lambda)} \leqslant t \leqslant 1$. If $\frac{1}{2} \leqslant t \leqslant \frac{1}{2} + \frac{s_\lambda - 2\lambda}{2(1-2\lambda)}$ the path $\pi'_{mn}(t, \lambda)$ is constructed so that $\pi'_{mn}(t, 0) = u_m(2t-1)$, $\pi'_{mn}(\frac{1}{2}, \lambda) = u_m(2\lambda)$, $\pi'_{mn}(t, \lambda) = u_m(s_\lambda)$ for $t = \frac{1}{2} + \frac{s_\lambda - 2\lambda}{2(1-2\lambda)}$, and $|\pi'_{mn}(t, \lambda) - u_m(2\lambda)| \leqslant 2\varepsilon$; for $t > 1/2$ the path $\pi'_{mn}(t, \lambda)$ does not intersect the half-disk $D_\lambda = \{\mathrm{Re}\, u \leq \mathrm{Re}\, u_m(2\lambda),\ |u - u_m(2\lambda)| \leqslant \varepsilon\}$ (see Fig. 2, where the solid lines denote the path $\pi_{mn}(t, \lambda)$ for $\frac{1}{2} \leqslant t \leqslant \frac{1}{2} + \frac{s_\lambda - 2\lambda}{2(1-2\lambda)}$ for six different values of λ from 0 to λ^*; the shading denotes the half-disk D_λ; and the dotted line denotes the circle Γ_λ).

For $0 \leq t \leq 1/2$ set $\pi'_{mn}(t, \lambda) = u_m(2\lambda) + v^0 - v_n(1-2t)$.

2) Let $\lambda^* \leqslant \lambda \leqslant 1/2$. Set

$$w'_{mn}(\lambda) = w'_{mn}(\lambda^*) + u_m - \pi'_{mn}\left(1 - \frac{\lambda - \lambda^*}{1 - 2\lambda^*}, \lambda^*\right)$$

(see Fig. 3, where the solid line denotes the path $w'_{mn}(\lambda)$ for $\lambda^* \leqslant \lambda \leqslant 1/2$; the shading denotes the half-disk $\{\mathrm{Re}\, w \geqslant u_m + v^0,\ |u_m + v^0 - w| \leqslant \varepsilon\}$; the dotted line denotes the circle $\{|u_m + v^0 - w| = 2\varepsilon\}$).

For $\frac{1}{2} + \frac{\lambda - \lambda^*}{1 - 2\lambda^*} \leqslant t \leqslant 1$ set

$$\pi'_{mn}(t, \lambda) = \pi'_{mn}\left(t - \frac{\lambda - \lambda^*}{1 - 2\lambda^*}, \lambda^*\right) + u_m - \pi'_{mn}\left(1 - \frac{\lambda - \lambda^*}{1 - 2\lambda^*}, \lambda^*\right),$$

and for $0 \leqslant t \leqslant \frac{1}{2} + \frac{\lambda - \lambda^*}{1 - 2\lambda^*}$ set

$$\pi'_{mn}(t, \lambda) = w'_{mn}(\lambda) - v_n\left(1 - \frac{2t(1 - 2\lambda^*)}{1 + 2\lambda - 4\lambda^*}\right)$$

(see Fig. 4, where the solid lines 1, 2, and 3 denote the path $\pi'_{mn}(t, \lambda)$ for $\lambda_1 = \lambda_1 = \lambda^*, \lambda^* < \lambda_2 < 1/2$ and $\lambda_3 = 1/2$).

3) Let $1/2 \leq \lambda \leq 1$. Set

$$w'_{mn}(\lambda) = u_m + v_n(2\lambda - 1),$$
$$\pi'_{mn}(t, \lambda) = w'_{mn}(\lambda) - v_n(1 - 2t(1 - \lambda)).$$

3. Computation of the Intersection Indices. Formula (1) follows from the fact that the γ_{mn} are vanishing cycles (see [5]).

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†If this condition is satisfied, then the paths $w'_{mn}(\lambda)$ constructed below for various m intersect only at $u^0 + v^0$, and the paths $\pi'_{mn}(t, \lambda)$ do not contain points $u_{m'}$ for $m' \neq m$.

Let $n' < n$. Then, replacing the paths $\pi_{mn}(t)$ and $\pi_{mn'}(t)$ by homotopic ones, we can assume that:

1) $\pi_{mn}(t)$ and $\pi_{mn'}(t)$ intersect only for $t = 1$ at the point u_m;

2) in a neighborhood of u_m

$$\pi_{mn}(t) = u_m + (1-t)e^{i\psi}, \quad \pi_{mn'}(t) = u_m + (1-t)e^{i(\varphi+\alpha_0)}, \\ 0 < \alpha_0 < \pi,$$

where the path $\pi_{mn'}(t)$ is obtained from $\pi_{mn}(t)$ by means of the deformation

$$u \mapsto u_m + e^{i\alpha}(u - u_m), \quad \alpha \in [0, \alpha_0]. \tag{6}$$

Let $x_{(m)}$ be a critical point of $P(x)$ with critical value u_m. Then as $t \to 1$ the spheres $e_{mn}(t)$ and $e_{mn'}(t)$ shrink to the point $x_{(m)}$. Therefore all the intersection points of the cycles γ_{mn} and $\gamma_{mn'}$ lie in the set $\{x_{(m)}\} \times \{Q(y) = \rho^{-1}u_m\}$. Let $x \mapsto x'$ be a change of coordinates in a neighborhood of $x_{(m)}$ such that $P(x') = u_m + \sum x_i'^2$, and let $B^p = \{x' \in \mathbf{R}^p, \sum x_i'^2 \leqslant 1\}$ be a p-dimensional disk, $S^{p-1} = \partial B^p$. If $\delta > 0$ is sufficiently small, we can assume that for $1 - \delta \le t \le 1$

$$e_{mn}(t) = \sqrt{1-t}\, e^{i\frac{\varphi}{2}} S^{p-1}, \qquad D = \bigcup_{1-\delta \leqslant t \leqslant 1} e_{mn}(t) = \sqrt{\delta}\, e^{i\frac{\varphi}{2}} B^p.$$

It follows from (6) that

$$e_{mn'}(t) = \sqrt{1-t}\, e^{i\frac{\varphi+\alpha_0}{2}} S^{p-1}, \qquad D' = \bigcup_{1-\delta \leqslant t \leqslant 1} e_{mn'}(t) = \sqrt{\delta}\, e^{i\frac{\varphi+\alpha_0}{2}} B^p,$$

where $0 < \alpha_0 < \pi$, and D' is obtained from D by the deformation

$$x' \mapsto e^{i\frac{\alpha}{2}} x', \quad \alpha \in [0, \alpha_0]. \tag{7}$$

Therefore D and D' intersect transversally at zero, and $(D, D') = (-1)^{\frac{p(p-1)}{2}}$. In a neighborhood of the set $\{x_{(m)}\} \times \{Q(y) = \rho^{-1}u_m\}$ γ_{mn} is diffeomorphic to $D \times h_{mn}(1)$ (the orientation in D is induced by this diffeomorphism); $\gamma_{mn'}$ is diffeomorphic to $D' \times h_{mn'}(1)$ [the orientation in D' is determined by the orientation in D from (7), and the diffeomorphism preserves the orientation]; and $\{P(x) + Q(y) = u^0 + v^0\}$ is diffeomorphic to $\mathbf{C}^p \times \{Q(y) = \rho^{-1}u_m\}$ (with preservation of orientation). Therefore the intersection index of γ_{mn} and $\gamma_{mn'}$ in $\{P(x) + Q(y) = u^0 + v^0\}$ is equal to the intersection index of $D \times h_{mn}(1)$ and $D' \times h_{mn'}(1)$ in $\mathbf{C}^p \times \{Q(y) = \rho^{-1}u_m\}$, which equals

$$(-1)^{p(q-1)}(D, D')(h_{mn}(1), h_{mn'}(1)) = (-1)^{p(q-1)}(D, D')(h_n, h_{n'}) = (-1)^{p(q-1)+\frac{p(p-1)}{2}}(h_n, h_{n'}).$$

Formula (2) now follows from the fact that

$$(\gamma_{mn'}, \gamma_{mn}) = (-1)^{p+q-1}(\gamma_{mn}, \gamma_{mn'}), \qquad (h_{n'}, h_n) = (-1)^{q-1}(h_n, h_{n'}).$$

Formula (3) is proved just as formula (2).

Formula (4) follows from the fact that for $\operatorname{sgn}(m' - m)\operatorname{sgn}(n' - n) = -1$ we can choose paths that are homotopic to $\pi_{mn}(t)$ and $\pi_{m'n'}(t)$ and do not intersect at any point.

Let $m' < m$, $n' < n$. Then we can assume that the paths $\pi_{mn}(t)$ and $\pi_{m'n'}(t)$ intersect only at u^0, and $\left(\frac{d\pi_{mn}}{dt}(u^0), \frac{d\pi_{m'n'}}{dt}(u^0)\right)$ is a positively oriented basis in $\mathbf{C}_u$.

Therefore γ_{mn} and $\gamma_{m'n'}$ intersect only at points of the set $\{P(x) = u^0\} \times \{Q(y) = v^0\}$. In a neighborhood of this set $\{P(x) + Q(y) = u^0 + v^0\}$ is diffeomorphic to $\{P(x) = u^0\} \times \{Q(y) = v^0\} \times \mathbf{C}_u$ (with preservation of orientation), γ_{mn} is diffoemorphic to $e_m \times h_n \times \pi_{mn}(t)$, and $\gamma_{m'n'}$ is diffeomorphic to $e_{m'} \times h_{n'} \times \pi_{m'n'}(t)$.

Therefore

$$(\gamma_{mn}, \gamma_{m'n'}) = (-1)^{(p-1)+(q-1)+(p-1)(q-1)}(e_m, e_{m'})(h_n, h_{n'}) = (-1)^{pq+1}(e_m, e_{m'})(h_n, h_{n'}).$$

Formula (5) follows from the fact that

$$(\gamma_{m'n'}, \gamma_{mn}) = (-1)^{p+q-1}(\gamma_{mn}, \gamma_{m'n'}), \ (e_{m'}, e_m) = (-1)^{p-1}(e_m, e_{m'}), \ (h_{n'}, h_n) = (-1)^{q-1}(h_n, h_{n'}).$$

This proves the theorem.

COROLLARY 1. Let the conditions of Theorem 1 be satisfied, where $Q_0(y) = y_1^2 + \dots + y_q^2$. Then $\nu = 1$, and if we set $e_m = \gamma_{m1}$, then the basis (e_m) has intersection matrix

$$(e_m, e_m) = \begin{cases} 2(-1)^{\frac{p+q-1}{2}}, & \text{if } p+q-1 \text{ is even} \\ 0, & \text{if } p+q-1 \text{ is odd} \end{cases}$$

$$(e_m, e_{m'}) = \operatorname{sgn}(m'-m)^q (-1)^{pq+\frac{q(q-1)}{2}} (e_m, e_{m'}),$$

if $m \neq m'$. In particular, if $q = 2k$, then $(e_m, e_{m'}) = (-1)^k (e_m, e_{m'})$ for all m, m'.

COROLLARY 2. Let e_m be a distinguished basis in $\tilde{H}_{p-1}(\{P(x) = u^0\}, Z)$, and e_m the corresponding basis (see Corollary 1) in $\tilde{H}_{p+q-1}(\{P(x) + y_1^2 + \dots + y_q^2 = u^0\}, Z)$. Let h_n be a distinguished basis in $\tilde{H}_{q-1}(\{Q(y) = v^0\}, Z)$ and h_n the corresponding basis in $\tilde{H}_{p+q-1}(\{x_1^2 + \dots + x_p^2 + Q(y) = v^0\}, Z)$. Then there exists a distinguished basis γ_{mn} in $\tilde{H}_{p+q-1}(\{P(x) + Q(y) = u^0 + v^0\}, Z)$ with intersection matrix

$$(\gamma_{mn}, \gamma_{mn'}) = (h_n, h_{n'}),$$

$$(\gamma_{mn}, \gamma_{m'n}) = (e_m, e_{m'}).$$

$$(\gamma_{mn}, \gamma_{m'n'}) = 0, \quad \text{if} \quad \operatorname{sgn}(m'-m)\operatorname{sgn}(n'-n) = -1,$$

$$(\gamma_{mn}, \gamma_{m'n'}) = \operatorname{sgn}(m'-m)^{p+q-1}(-1)^{\frac{(p+q)(p+q-1)}{2}} (e_m, e_{m'})(h_n, h_{n'}), \quad \text{if} \quad \operatorname{sgn}(m'-m)\operatorname{sgn}(n'-n) = 1.$$

PROPOSITION 1. The function $y^a - \lambda y$ for $\lambda \neq 0$ is a morsovization of the function y^a, and if v^0 is a noncritical value of $y^a - \lambda y$, then there exists a distinguished basis $h_n (n = 1, \dots, a-1)$ in $\tilde{H}_0(\{y^a - \lambda y = v^0\}, Z)$ with intersection matrix

$$(h_n, h_n) = 2, \ (h_n, h_{n+1}) = -1, \ (h_n, h_{n'}) = 0, \text{ if } |n'-n| > 1. \tag{8}$$

Proof. We can assume that λ is a positive real number. The function $y^a - \lambda y$ has $a-1$ nondegenerate critical points $y_{(n)} = \sqrt[a-1]{\lambda/a}\, e^{\frac{2\pi i(n-1)}{a-1}}$ $(1 \leqslant n \leqslant a-1)$ with critical values $v_n = -\frac{(a-1)}{a} \lambda \sqrt[a-1]{\lambda/a}\, e^{\frac{2\pi i(n-1)}{a-1}}$. If v^0 is a noncritical value of $y^a - \lambda y$, then the set $\{y : y^a - \lambda y = v^0\}$ consists of a points z_k $(0 \leqslant k \leqslant a-1)$. (If v^0 is a negative real number and $|v^0| \gg \lambda^{a/(a-1)}$, then $z_k \approx \sqrt[a]{|v^0|}\, e^{\frac{\pi i(2k-1)}{a}}$.

On the rays $\{y = te^{\frac{\pi i(2l-1)}{a-1}}, t > 0\}$ the function $y^a - \lambda y$ equals $e^{\frac{\pi i(2l-1)}{a-1}}(t^a + \lambda t)$, and so it cannot assume a negative real value. Therefore, while moving along the path $v_1(s) = (1-s)v^0 - s\frac{a-1}{a}\sqrt[a-1]{\lambda/a}$, the roots $z_i(s)$ of the equation $y^a - \lambda y = v_1(s)$ do not intersect the boundaries of the sector $-\frac{\pi i}{a-1} < \arg y < \frac{\pi i}{a-1}$. Since precisely two points that satisfy the equation $y^a - \lambda y = v^0$ (namely, z_0 and z_1) are contained in this sector for $s = 0$, and $y_{(1)}$ is a repeated root of the equation $y^a - \lambda y = v_1$, for $s = 1$, it follows that the points $z_0(s)$ and $z_1(s)$ merge at $y_{(1)}$; i.e., the cycle $h_1 = (\{z_1\} - \{z_0\})$ vanishes along $v_1(s)$.

Similarly, we can show that for $n = 2, \dots, a-1$ the cycles $h_n = (\{z_n\} - \{z_{n-1}\})$ vanish along the paths

$$v_n(s) = e^{\frac{2\pi is(n-1)}{a-1}}\left((1-s)v^0 - s\frac{a-1}{a}\lambda \sqrt[a-1]{\lambda/a}\right)$$

at the points $y_{(n)}$. Obviously, (h_n) is a basis in $\tilde{H}_0(\{y^a - \lambda y = v^0\}, Z)$ with intersection matrix (8), and the system of paths $v_n(s)$ satisfies the conditions of Definition 2.

COROLLARY 3. Let the conditions of Theorem 1 be satisfied, where $q = 1$ and $Q_0(y) = y^a$. Let e_m be a distinguished basis in $\tilde{H}_{p-1}(\{P(x) = u^0\}, Z)$ and e_m the corresponding distinguished basis in $\tilde{H}_p(\{P(x) + y^2 = u^0\}, Z)$. Then for sufficiently small λ there exists a distinguished basis $\gamma_{m,n} (m = 1, \dots, \mu; n = 1, \dots, a-1)$ in $\tilde{H}_p(\{P(x) + y^{\kappa} - \lambda y = u^0\}, Z)$ with intersection matrix

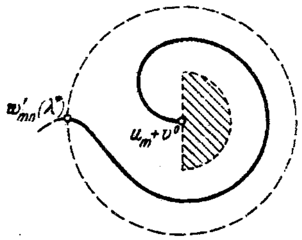


Fig. 3

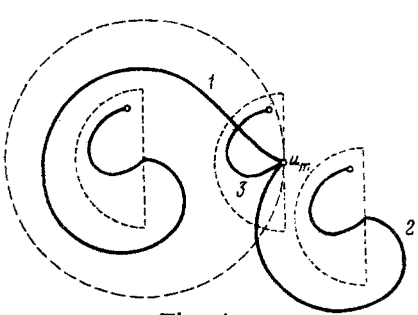


Fig. 4

$$(\gamma_{m,n}, \gamma_{m,n+1}) = (-1)^{\frac{(n-1)(p-2)}{2}}, \quad (\gamma_{m,n}\gamma_{m',n}) = (e_m, e_{m'}),$$
$$(\gamma_{m,n}, \gamma_{m',n+1}) = -(e_m, e_{m'}), \text{ if } m' \neq m,$$
$$(\gamma_{m,n}, \gamma_{m',n'}) = 0, \text{ if } |n'-n| > 1 \text{ or } \operatorname{sgn}(m'-m)\operatorname{sgn}(n'-n) = -1.$$

COROLLARY 4. Let $P_0 = x_1^{a_1} + \ldots + x_p^{a_p}$, P a morsovization of P_0, and u^0 a noncritical value of P. Then there exists a distinguished basis e_m $(m=(m_1, \ldots, m_p)$, $1 \leqslant m_i \leqslant a_i - 1$, and the order is lexicographic) in $H_{p-1}(\{P(x) = u^0\}, Z)$ with intersection matrix

$$(e_m, e_m) = \begin{cases} 2(-1)^{\frac{p-1}{2}}, & \text{if } p-1 \text{ is even} \\ 0, & \text{if } p-1 \text{ is odd} \end{cases}$$
$$(e_m, e_{m'}) = (-1)^{\frac{p(p-1)}{2} + \sum (m_i' - m_i)}, \quad \text{if } m' \neq m,\ 0 \leqslant m_i' - m_i \leqslant 1,$$

$(e_m, e_{m'}) = 0$ if $|m_i' - m_i| > 1$ for some i or $\operatorname{sgn}(m_i' - m_i)\operatorname{sgn}(m_j' - m_j) = -1$ for some i, j.

3. Dynkin Diagrams

Let P_0 be an analytic function in C^p with an isolated singularity of multiplicity μ, P a morsovization of P_0, and $e_m (m = 1, \ldots, \mu)$ a basis of vanishing cycles in $\tilde{H}_{p-1}(\{P(x) = u^0\}, Z)$. If $p-1$ is even, then $(e_m, e_{m'}) = (e_{m'}, e_m)$ and $(e_m, e_m) = 2(-1)^{\frac{p-1}{2}}$.

For the intersection matrix of the cycles e_m we can construct a graph as follows:

1) the number of vertices of the graph is equal to μ;

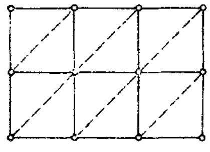
2) the vertices with the numbers m and m' are joined by k solid edges if $(e_m, e_{m'}) = k(-1)^{\frac{p+1}{2}}$, and k dotted edges if $(e_m, e_{m'}) = k(-1)^{\frac{p-1}{2}}$.

Conversely, such a graph determines an intersection matrix of the basis (e_m). We shall call this graph the Dynkin diagram of the basis (e_m). (In case all the interaction indices $(e_m, e_{m'})$ for $m \neq m'$ equal 0 or $(-1)^{\frac{p-1}{2}}$, the corresponding graph is an ordinary Dynkin diagram with the associated form $(-1)^{\frac{p-1}{2}} \times (e_m, e_{m'})/2$ (see [7]).

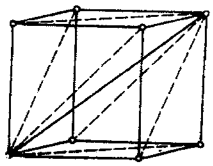
Corollary 4 can now be stated as follows:

To the singularity $x_1^{a_1} + \ldots + x_p^{a_p}$, where $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k > a_{k+1} = \ldots = a_p = 2$, there corresponds a distinguished basis whose Dynkin diagram has the form of a parallelepiped consisting of $(a_1 - 2) \ldots (a_k - 2)$ k-dimensional cubes, in each of which all the positive diagonals are drawn. Moreover, the diagonal is drawn with a dotted line or a solid line, depending on whether an even or odd number of coordinates increases as we move along this diagonal.

Examples. To the singularity $x^5 + y^4 + z^2$ corresponds the Dynkin diagram



To the singularity $x^3 + y^3 + z^3$ corresponds the Dynkin diagram



See also § 6.

4. The Monodromy Group

As in §1, let P_0 be a function in CP with an isolated singularity of multiplicity μ, a versal deformation of P_0, Σ its bifurcation diagram, $\lambda = (\lambda', u)$, and $P_\lambda = P_{(\lambda',0)} - u$.

Since over $C^\mu \backslash \Sigma$ the mapping $\{x : P_\lambda(x) = 0\} \to \lambda$ is a smooth fibration, the group $\pi_1(C^\mu \backslash \Sigma)$ acts in $\tilde{H}_{p-1}(\{P_\lambda(x) = 0\}, Z)$. This representation is called a monodromy, and its image is the monodromy group of the singularity of P_0.

For a description of the generators of $\pi_1(C \backslash \Sigma)$, we need the concept of a "simple loop."

Let $\lambda^0 = (\lambda'_0, u^0)$ be chosen so that $\lambda^0 \notin \Sigma$, $\lambda'_0 \notin \Delta$ (see §1). Then $\Sigma^0 = \Sigma \cap (\{\lambda'_0\} \times C_u)$ consists of μ points $u_1, \ldots, u_\mu$. Let $u_m \in \Sigma^0$ and let $u_m(s)$ be a smooth path in C_u, $u_m(0) = u^0$, $u_m(1) = u_m$, $u_m(s) \notin \Sigma^0$, for $s < 1$. By a simple loop corresponding to the path $u_m(s)$ we mean a closed path $\tau_m \in \pi_1(C_u \backslash \Sigma_0, u^0)$, which is obtained as follows: starting at the point u^0, arrive at the point u_m along the path $u_m(s)$, then go around u_m in the positive direction and return to u^0 along $u_m(s)$ (Fig. 5).

Definition 3. Let (e_m) be a basis of vanishing cycles in $\tilde{H}_{p-1}(\{P_{(\lambda_0',0)}(x) = u^0\}, Z)$ and $u_m(s)$ corresponding paths in C_u. The basis (e_m) is said to be weakly distinguished if the simple loops that correspond to the paths $u_m(s)$ generate the group $\pi_1(C_u \backslash \Sigma^0, u^0)$.

Remark. A distinguished basis is obviously weakly distinguished.

PROPOSITION 2. Let (e_m) be a weakly distinguished basis. Then the monodromy group of the singularity of P_0 is generated by the operators T_m by the formula

$$T_m(e) = e + (-1)^{\frac{p(p+1)}{2}} (e, e_m) e_m. \tag{9}$$

Proof. It can be shown (see [8]) that the natural mapping $\rho : \pi_1(C_u \setminus \Sigma^0, u^0) \to \pi_1(C^\mu \setminus \Sigma, \lambda^0)$ is an epimorphism. Since the simple loops τ_m corresponding to the paths $u_m(s)$ generate $\pi_1(C_u \backslash \Sigma^0, u^0)$, the paths $\rho\tau_m$ generate $\pi_1(C^\mu \backslash \Sigma, \lambda^0)$. Therefore the operators T_m corresponding to these paths generate the monodromy group. Formula (9) is simply the Picard-Lefschetz formula (see [5]).

Remark 1. It follows from Proposition 2 that the monodromy group of a singularity is determined by the intersection matrix of a weakly distinguished basis. Thus Theorem 1 allows one to compute the monodromy group of the singularity of $P_0(x) + Q_0(y)$ if the monodromy groups of the singularities of $P_0(x)$ and $Q_0(y)$ are known (more precisely, special systems of generators of these groups).

Remark 2. For each singularity of P_0 Milnor defined a Picard–Lefschetz local monodromy operator h_{P_0}. In our notation this is an element of the monodromy group that corresponds to the path $\{\lambda = (0, u^0 e^{2\pi i t}), 0 \leqslant t \leqslant 1\}$ in $C^\mu \backslash \Sigma$. If (e_m) is a distinguished basis, then $h_{P_0} = T_1 \circ \ldots \circ T_\mu$.

Sebastiani and Thom [4] proved that the operator $h_{P_0+Q_0}$ of a singularity of $P_0(x) + Q_0(y)$ can be computed by the formula $h_{P_0+Q_0} = h_{P_0} \otimes h_{Q_0}$.

5. Change of Basis

Let (e_m) be a basis of vanishing cycles, $u_m(s)$ corresponding paths, and τ_m simple loops corresponding to the $u_m(s)$. For any pair (m, m') we can construct a new basis of vanishing cycles as follows:

a) Replace $u_{m'}(s)$ by $\tau_m^{-1} u_{m'}(s)$. Moreover, according to the Picard-Lefschetz formula $e_{m'}$ is replaced by $e_{m'} + (-1)^{\frac{p(p+1)}{2}} (e_{m'}, e_m) e_m$. We denote this operation by $\alpha_m(m')$.

b) Replace $u_{m'}(s)$ by $\tau_m u_{m'}(s)$. Moreover, $e_{m'}$ is replaced by $e_{m'} + (-1)^{\frac{p(p+1)}{2}} (e_m, e_{m'}) e_m$ (the inverse Picard–Lefschetz transformation). We denote this operation by $\beta_m(m')$.

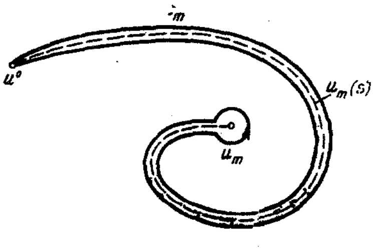


Fig. 5

PROPOSITION 3. The operations $\alpha_m(m')$ and $\beta_m(m')$ send a weakly distinguished basis to a weakly distinguished basis.

The proof is trivial.

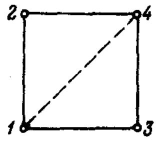
If (e_m) is a distinguished basis, we denote by α_m the operation $\alpha_m(m+1)$ with successive transposition of the numbers m and m + 1, and by β_m the operation $\beta_m(m-1)$ with successive transposition of the numbers m and m − 1.

PROPOSITION 4. The operations α_m and β_m send a distinguished basis to a distinguished basis.

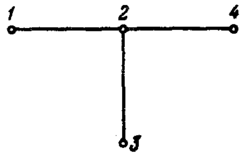
Remark. It can be shown that any two distinguished bases can be sent into each other by a sequence of operations α_m, β_m, and a change of orientation of the cycles.

6. Examples

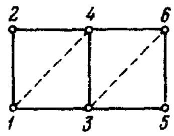
Example 1. Let $F(x, y, z) = x^3 + y^3 + z^2$. According to Corollary 4, to this singularity corresponds a distinguished basis with Dynkin diagram



(the numbers on the vertices indicate the order of the elements of the distinguished basis). The sequence of operations $\alpha_1, \beta_3, e_3 \mapsto -e_3$ sends this basis to a distinguished basis with Dynkin diagram D_4:



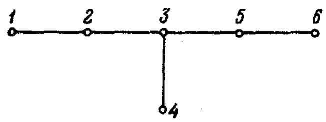
Example 2. Let $F(x, y, z) = x^4 + y^3 + z^2$. According to Corollary 4, to this singularity corresponds a distinguished basis with Dynkin diagram



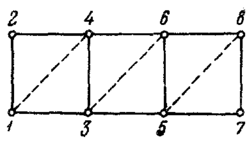
The sequence of operations

$$\alpha_1, \beta_3, \beta_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1, e_1 \mapsto -e_1, e_4 \mapsto -e_4$$

sends this basis to a distinguished basis with Dynkin diagram E_6:



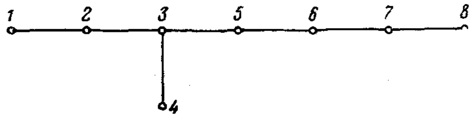
Example 3. Let $F(x, y, z) = x^5 + y^3 + z^2$. According to Corollary 4, to this singularity corresponds a distinguished basis with Dynkin diagram



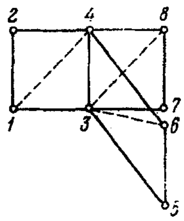
The sequence of operations

$$\alpha_1,\ \beta_3,\ \beta_6,\ \alpha_4,\ \alpha_3,\ \alpha_2,\ \alpha_1,\ \alpha_7,\ \alpha_6,\ \alpha_5,\ \beta_4,\ \alpha_4,\ \alpha_3,\ \beta_3,\ \beta_4,\ e_1 \mapsto -e_1,\ e_2 \mapsto -e_2$$

sends this basis to a distinguished basis with Dynkin diagram E_8:



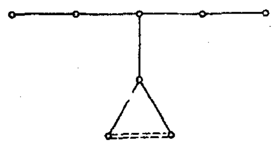
Example 4. Let $F(x, y, z) = x^3 + y^3 + z^3$. For $x^3 + y^3 + z^2$, taking the distinguished basis of D_4 from Example 1 and applying Corollary 3, for $F(x, y, z)$ we obtain a distinguished basis with Dynkin diagram



The sequence of operations

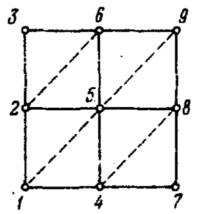
$$\alpha_3(1),\ \alpha_8(3),\ \alpha_6(3),\ \alpha_3(4),\ \alpha_4(1)$$

sends this basis to a weakly distinguished basis with Dynkin diagram



(10)

Example 5. Let $F(x, y, z) = x^4 + y^4 + z^2$. According to Corollary 4, to this singularity corresponds a distinguished basis with Dynkin diagram



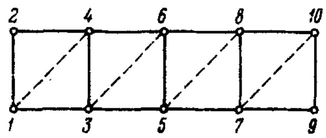
The sequence of operations

$$\alpha_2(6),\ \alpha_4(8),\ \alpha_9(5),\ \alpha_5(1),\ \alpha_6(5),\ \alpha_8(5),\ \alpha_5(9)$$

sends this basis to a weakly distinguished basis with Dynkin diagram

(11)

Example 6. Let $F(x, y, z) = x^6 + y^3 + z^2$. According to Corollary 4, to this singularity corresponds a distinguished basis with Dynkin diagram



The sequence of operations

$$\alpha_4(1),\ \alpha_6(4),\ \alpha_5(6),\ \alpha_{10}(7),\ \alpha_8(5),\ \alpha_{10}(5),\ \alpha_5(9),\ \alpha_6(5),\ \alpha_7(5),\ \alpha_3(5),$$
$$\alpha_9(5),\ \alpha_5(6),\ e_3 \mapsto -e_3$$

sends this basis to a weakly distinguished basis with Dynkin diagram

(12)

Remark. Let (e_m) be a basis with Dynkin diagram (10) (respectively (11), (12)), where $(e_1, e_2) = -2$. Then $(e_1 - e_2, e_m) = 0$ for all m, and if e_1 is replaced by $e_1 - e_2$, then the intersection matrix reduces to the form $\tilde{E}_6 \oplus (0)$ (respectively, $\tilde{E}_7 \oplus (0)$, $\tilde{E}_8 \oplus (0)$), where (0) is the zero form on Z. By using the well-known properties of $\tilde{E}_k$, it is not difficult to obtain the following assertion:

PROPOSITION 5. The intersection matrix of the singularity $x^3 + y^3 + z^3$ (respectively, $x^4 + y^4 + z^2$, $x^6 + y^3 + z^2$) is negative and singular. The corresponding bilinear form has corank 2 and splits into the direct sum $E_6 \oplus 0^2$ (respectively, $E_7 \oplus 0^2$, $E_8 \oplus 0^2$). (Here 0^2 denotes the zero form on Z^2.) The monodromy group contains as a normal subgroup a free abelian subgroup of rank 12 (respectively, 14, 16) the factor group of which is equal to E_6 (respectively, E_7, E_8).

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